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DESCRIPTION OF SOME FAMILIES OF FILIFORM LIE ALGEBRAS

F. J. ECHARTE, J. NÚÑEZ, AND F. RAMÍREZ

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Abstract. In this paper we describe some families of filiform Lie algebras by giving a method which allows to obtain them in any arbitrary dimension n starting from the triple (p, q, m), where $m = n$ and p and q are, respectively, invariants z_1 and z_2 of those algebras. After obtaining the general law of complex filiform Lie algebras corresponding to triples (p, q, m), some concrete examples of this method are shown.

1. Introduction

Sometimes, people who work on Lie Theory need dispose of lots of examples of Lie algebras of high dimensions to check if some property verified by Lie algebras of low dimensions is also satisfied in them. These examples are also needed, for instance, to verify if a previous conjecture, which has been observed in Lie algebras of low dimensions, continues being true when the dimension increases.

However, at present, it is difficult to have those examples of Lie algebras of dimension, say, non-excessively low. Indeed, if we do not take into consideration simple and semisimple Lie algebras, whose classification was obtained by Killing, Cartan and others, in the last decade of the last next century, only the classifications of solvable, nilpotent and filiform complex Lie algebras of dimension n, with $n \leq 5$, $n \leq 7$ and $n \leq 12$, respectively, are actually known (see [1], [10] and [4], for instance).

Moreover, this problem worsens if we note that not only the classification problem of Lie algebras is still unsolved, but, in fact, it seems to be accepted by experts (like Shalev and Zelmanov [13]) that one will not totally classify Lie

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algebras of finite dimension (see also Goze-Khakimdjanov [11]). Indeed, it is reasonable for research workers in this theory to impose further conditions to these algebras to obtain, at least, partial results.

In this sense, the main goal of this paper is to completely describe a family of complex filiform Lie algebras in any dimension n defined by some invariants (introduced by us in [7]) computed on centralizer of some characteristic ideal of its descending central sequence.

Now, we will briefly explain why we deal with complex filiform Lie algebras. These algebras were introduced by M. Vergne in the late 60's of the past century [14]. However, before that, Blackburn studied the analogous class of finite p-groups and used the term *maximal class* to call them, which is also now used for Lie algebras [2]. In fact, both terms *filiform* and *maximal class* are synonymous.

Vergne showed that, within the variety of nilpotent Lie multiplications on a fixed vector space, non-filiforms can be relegated to small-dimensional components; thus, from some intuitive point of view, it is possible to consider that *quite a lot* nilpotent Lie algebras are filiform, in spite of this last subset not being dense in the space of nilpotent Lie algebras. Apart from that, complex filiform Lie algebras are the most structured subset of nilpotent Lie algebras, with respect to an adapted basis (see next section). In this sense, we can study and classify them easier than the set of nilpotent Lie algebras.

So, in earlier papers, some of us have deeply studied these algebras and obtained quite a lot results about them. Indeed, we already got the classification of those having dimensions $10, 11$ and 12 (see [3], [10] and [4]), for instance). Moreover, we think that this study can be also considered a little step forward in the problem of the classification of these algebras, although it is not the principal objective of this work.

2. Definitions and notations

A nilpotent Lie algebra $\mathbf{g}$ is said to be *filiform*, if it is verified that:

$$dim\,\mathbf{g}^2 = n-2; \;\; \ldots \;\; dim\,\mathbf{g}^k = n-k; \;\; \ldots \;\; dim\mathbf{g}^n = 0,$$

where $dim\,\mathbf{g} = n$, and $\mathbf{g}^k = [\mathbf{g}, \mathbf{g}^{k-1}], \quad 2 \leq k \leq n.$

A basis $\{e_1, \ldots, e_n\}$ of $\mathbf{g}$ is called an *adapted basis* if:

$$[e_1, e_2] = 0; \quad [e_1, e_h] = e_{h-1} \quad (h = 3, \ldots, n), \quad [e_3, e_n] = 0.$$

Note that the definition of filiformity assures that every filiform Lie algebra has an adapted basis.

It is easy to deduce that, with respect to this basis, it is verified:

$$\mathbf{g}^2 \equiv \{e_2, \ldots, e_{n-1}\},\ \mathbf{g}^3 \equiv \{e_2, \ldots, e_{n-2}\}, \ldots, \mathbf{g}^{n-1} \equiv \{e_2\},\ \mathbf{g}^n \equiv \{0\}.$$

A filiform Lie algebra $\mathbf{g}$ is said to be *model*, if the only non-null brackets between the elements of an adapted basis are: $[e_1, e_h] = e_{h-1} \quad (h = 3, \ldots, n)$. When the dimension is given, this model is unique up an isomorphism and every filiform Lie algebra is a deformation of the model (see Goze-Khakimdjanov [11]).

We note by $\mathcal{C}_{\mathbf{g}}\langle$ the centralizer of a subalgebra $\langle$ in $\mathbf{g}$.

In [6] (although by using a different notation for denoting it, which was later improved), two of the authors of this paper defined the integer:

$$z_1 = z_1(\mathbf{g}) = max\{k \in \mathbb{N} \mid \mathcal{C}_{\mathbf{g}}(\mathbf{g}^{n-k+2}) \supset \mathbf{g}^2\}.$$

Note that this definition means that the ideal $\mathbf{g}^{n-z_1+2}$ is the greatest ideal whose centralizer contains $\mathbf{g}^2$, that is, the ideal whose centralizer is the ideal $\overline{\mathbf{g}}$, generated by $\{e_2, \ldots, e_{n-1}, e_n\}$.

Note as well that, according to this last definition, $z_1(\mathbf{g})$ is an invariant of filiform Lie algebras. In terms of adapted basis, it is deduced in [6] that:

$$z_1 = min\{k \in \mathbb{N} - \{1\} \mid [e_k, e_n] \neq 0\}.$$

However, note that $z_1(\mathbf{g})$ does not always exist. In this case, it is also easy to see that $\mathbf{g}$ is a model algebra. It implies that the algebra with basis $\{e_2, \ldots, e_n\}$ is commutative. It allows us to give a new definition of model algebra, which is also independent of any adapted basis: we say that a filiform Lie algebra $\mathbf{g}$ is a model algebra if $C_{\mathbf{g}}(\mathbf{g}^{n-2})$ is commutative. Both definitions are equivalent, since $\mathbf{g}^{n-2}$ is the ideal with basis $\{e_2, e_3\}$ whose centralizer is the ideal with basis $\{e_2, \ldots, e_n\}$.

In [7] (a different notation for denoting it was also used) the integer $z_2(\mathbf{g})$ was introduced by us as follows:

$$z_2 = z_2(\mathbf{g}) = max\{k \in \mathbb{N} \,|\, \mathbf{g}^{n-k+1} \text{ is commutative}\}$$

Note that this definition means that the ideal $\mathbf{g}^{n-z_2+1} \equiv \{e_2, \ldots, e_{z_2}\}$ is the greatest commutative subalgebra in the nilpotency sequence.

In that paper we also proved the three following asserts: a) that $z_2(\mathbf{g})$ is an invariant of complex filiform Lie algebras. b): that there exists at least some bracket $[e_k, e_{k+1}] \neq 0$, for some $k < n$, in every non-model complex filiform Lie algebra of dimension n and c): that

$$z_2(\mathbf{g}) = min\ \{k \in \mathbb{N} \mid [e_k, e_{k+1}] \neq 0\}.$$

Similarly as the case of $z_1(\mathbf{g})$, if the set of this definition is empty, then $\mathbf{g}$ is a model algebra. In the other case, the smallest value of $z_2(\mathbf{g})$ is 4, because of being $[e_1, e_2] = [e_2, e_3] = [e_3, e_4] = 0$ in any adapted basis.

With respect to both invariants, the following two results are verified [9]:

Lemma 1. *$[e_{z_1+k-1}, e_{z_2+1}] = \alpha_1\, e_{k+1} + \alpha_2\, e_k + \ldots + \alpha_{k-1}\, e_3 + \alpha_k\, e_2$, with $1 \leq k \leq z_2 - z_1 + 1$. Moreover, $\alpha_p \neq 0$ for some p such that $1 \leq p \leq z_2 - z_1 + 1$.*

Lemma 2. *$[e_{z_1}, e_{z_2+k}] = \beta_1\, e_{k+1} + \beta_2\, e_k + \ldots + \beta_{k-1}\, e_3 + \beta_k\, e_2$, with $1 \leq k \leq n - z_2$ (where $\beta_1 = \alpha_1$ from Lemma 1). Moreover, $\beta_q \neq 0$ for some q such that $1 \leq q \leq n - z_2$.*

Finally, it is also proved in [7] the following relation among invariants z_1, z_2 and n of complex filiform Lie algebras:

$$4 \leq z_1 \leq z_2 < n \leq 2z_2 - 2$$

which will be used later.

From now on, we will suppose that all the Lie algebras appearing in this paper are complex filiform and that all bases are adapted. We denote by $J(a, b, c) = 0$ Jacobi identity associated with vectors a, b and c.

3. Filiform Lie algebras verifying $z_2(\mathbf{g}) = n - 2$

In this section we study n-dimensional filiform Lie algebras satisfying $z_2(\mathbf{g}) = n - 2$. Note that in these algebras, $n \geq 6$, since $z_2 \geq 4$ is always verified.

Due to the filiformity and to definitions of invariants z_1 and z_2, the only non-null brackets are $[e_1, e_h] = e_{h-1}$, for all $3 \leq h \leq n$, and $[e_h, e_k]$, with $z_1 \leq h < k \leq n - 1$ (obviously, $[e_k, e_h] = -[e_h, e_k]$, if $h < k$).

Remember that, according Lemma 1, we have the following expressions, where α_1 could be equal to 0.

$$\begin{array}{lcl}
[e_{z_1}, e_{n-1}] & = & \alpha_1\, e_2 \\
[e_{z_1+1}, e_{n-1}] & = & \alpha_1\, e_3 + \alpha_2\, e_2 \\
\vdots & & \vdots \\
[e_{n-3}, e_{n-1}] & = & \alpha_1\, e_{n-z_1-1} + \alpha_2\, e_{n-z_1-2} + \ldots + \alpha_{n-z_1-3}\, e_3 + \alpha_{n-z_1-2}\, e_2 \\
[e_{n-2}, e_{n-1}] & = & \alpha_1\, e_{n-z_1} + \alpha_2\, e_{n-z_1-1} + \ldots + \alpha_{n-z_1-2}\, e_3 + \alpha_{n-z_1-1}\, e_2
\end{array}$$

in which at least, some α_p (with $1 \leq p \leq n - z_1 - 1$) is necessarily non-null (see Lemma 1).

Then, by taking into consideration Lemma 2, if we denote $\beta_2 = \gamma_2$, we have $[e_{z_1}, e_{n-1}] = \alpha_1\, e_2$ and $[e_{z_1}, e_n] = \alpha_1\, e_3 + \gamma_2\, e_2$ and thus, at least one of both, α_1 or γ_2 is non-null.

Now, from Jacobi identity $J(e_1, e_{z_1+1}, e_n) = 0$, we deduce:

$$[e_{z_1+1}, e_n] = 2\,\alpha_1\,e_4 + (\alpha_2 + \gamma_2)\,e_3 + \gamma_3\,e_2$$

In the same way, from Jacobi identities $J(e_1, e_{z_1+2}, e_n) = 0$ and $J(e_1, e_{z_1+3}, e_n) = 0$, we have, respectively:

$$[e_{z_1+2}, e_n] = 3\,\alpha_1\,e_5 + (2\,\alpha_2 + \gamma_2)\,e_4 + (\alpha_3 + \gamma_3)\,e_3 + \gamma_4\,e_2, \quad \text{and}$$

$$[e_{z_1+3}, e_n] = 4\,\alpha_1\,e_6 + (3\,\alpha_2 + \gamma_2)\,e_5 + (2\,\alpha_3 + \gamma_3)\,e_4 + (\alpha_4 + \gamma_4)\,e_3 + \gamma_5\,e_2.$$

So, by proceeding in the same way, we can deduce (when the first subindex is $z_1 + (n - z_1 + 3) \equiv n - 3$) that

$$\begin{gathered}[e_{n-3}, e_n] = (n - z_1 - 2)\,\alpha_1\,e_{n-z_1} + ((n - z_1 - 3)\,\alpha_2 + \gamma_2)\,e_{n-z_1-1} + \\ ((n - z_1 - 4)\,\alpha_3 + \gamma_3)\,e_{n-z_1-2} + \ldots + (3\,\alpha_{n-z_1-4} + \gamma_{n-z_1-4})\,e_5 + \\ (2\,\alpha_{n-z_1-3} + \gamma_{n-z_1-3})\,e_4 + (\alpha_{n-z_1-2} + \gamma_{n-z_1-2})\,e_3 + \gamma_{n-z_1-1}\,e_2\end{gathered}$$

By replacing now these two results previously obtained in Jacobi identity $J(e_1, e_{n-2}, e_n) = 0$, we have that $[e_1, [e_{n-2}, e_n] = (n - z_1 - 1)\,\alpha_1\,e_{n-z_1} + ((n - z_1 - 2)\,\alpha_2 + \gamma_2)\,e_{n-z_1-1} + ((n - z_1 - 3)\,\alpha_3 + \gamma_3)\,e_{n-z_1-2} + \ldots + (3\,\alpha_{n-z_1-3} + \gamma_{n-z_1-3})\,e_4 + (2\,\alpha_{n-z_1-2} + \gamma_{n-z_1-2})\,e_3 + (\alpha_{n-z_1-1} + \gamma_{n-z_1-1})\,e_2$ and thus:

$$\begin{gathered}[e_{n-2}, e_n] = (n - z_1 - 1)\,\alpha_1\,e_{n-z_1+1} + ((n - z_1 - 2)\,\alpha_2 + \gamma_2)\,e_{n-z_1} + \\ ((n - z_1 - 3)\,\alpha_3 + \gamma_3)\,e_{n-z_1-1} + \ldots + (3\,\alpha_{n-z_1-3} + \gamma_{n-z_1-3})\,e_5 + \\ (2\,\alpha_{n-z_1-2} + \gamma_{n-z_1-2})\,e_4 + (\alpha_{n-z_1-1} + \gamma_{n-z_1-1})\,e_3 + \gamma_{n-z_1}\,e_2.\end{gathered}$$

Finally, by using Jacobi identity $J(e_1, e_{n-1}, e_n) = 0$ and by proceeding in the same way we deduce that:

$$\begin{gathered}[e_{n-1}, e_n] = (n - z_1 - 1)\,\alpha_1\,e_{n-z_1+2} + ((n - z_1 - 2)\,\alpha_2 + \gamma_2)\,e_{n-z_1+1} + \\ ((n - z_1 - 3)\,\alpha_3 + \gamma_3)\,e_{n-z_1} + \ldots + (3\,\alpha_{n-z_1-3} + \gamma_{n-z_1-3})\,e_6 + \\ (2\,\alpha_{n-z_1-2} + \gamma_{n-z_1-2})\,e_5 + (\alpha_{n-z_1-1} + \gamma_{n-z_1-1})\,e_4 + \gamma_{n-z_1}\,e_3 + \gamma_{n-z_1+1}\,e_2.\end{gathered}$$

Therefore, the following result has been proved:

Theorem 3.1. *(Main Theorem) The laws of filiform Lie algebras verifying* $z_2(\mathbf{g}) = n-2$*, with respect to an adapted basis* $\{e_1, \dots, e_n\}$*, are the following:*

$$\begin{array}{lcl}
[e_1, e_h] & = & e_{h-1} \qquad (3 \leq h \leq n) \\
[e_{z_1}, e_{n-1}] & = & \alpha_1\, e_2 \\
[e_{z_1}, e_n] & = & \alpha_1\, e_3 + \gamma_2\, e_2 \qquad (\textit{with either } \alpha_1 \neq 0 \textit{ or } \gamma_2 \neq 0) \\
[e_{z_1+1}, e_{n-1}] & = & \alpha_1\, e_3 + \alpha_2\, e_2 \\
[e_{z_1+1}, e_n] & = & 2\,\alpha_1\, e_4 + (\alpha_2 + \gamma_2)\, e_3 + \gamma_3\, e_2 \\
[e_{z_1+2}, e_{n-1}] & = & \alpha_1\, e_4 + \alpha_2\, e_3 + \alpha_3\, e_2 \\
[e_{z_1+2}, e_n] & = & 3\,\alpha_1\, e_5 + (2\,\alpha_2 + \gamma_2)\, e_4 + (\alpha_3 + \gamma_3)\, e_3 + \gamma_4\, e_2 \\
[e_{z_1+3}, e_{n-1}] & = & \alpha_1\, e_5 + \alpha_2\, e_4 + \alpha_3\, e_3 + \alpha_4\, e_2 \\
[e_{z_1+3}, e_n] & = & 4\,\alpha_1\, e_6 + (3\,\alpha_2 + \gamma_2)\, e_5 + (2\,\alpha_3 + \gamma_3)\, e_4 + (\alpha_4 + \gamma_4)\, e_3 \\
 & & +\gamma_5\, e_2 \\
[e_{z_1+4}, e_{n-1}] & = & \alpha_1\, e_6 + \alpha_2\, e_5 + \alpha_3\, e_4 + \alpha_4\, e_3 + \alpha_5\, e_2 \\
[e_{z_1+4}, e_n] & = & 5\,\alpha_1\, e_7 + (4\,\alpha_2 + \gamma_2)\, e_6 + (3\,\alpha_3 + \gamma_3)\, e_5 + (2\,\alpha_4 + \gamma_4)\, e_4 + \\
 & & (\alpha_5 + \gamma_5)\, e_3 + \gamma_6\, e_2 \\
\vdots & & \vdots \\
[e_{n-3}, e_n] & = & (n - z_1 - 2)\,\alpha_1\, e_{n-z_1} + ((n - z_1 - 3)\,\alpha_2 + \gamma_2)\, e_{n-z_1-1} + \\
 & & ((n - z_1 - 4)\,\alpha_3 + \gamma_3)\, e_{n-z_1-2} + \ldots + \\
 & & (3\,\alpha_{n-z_1-4} + \gamma_{n-z_1-4})\, e_5 + (2\,\alpha_{n-z_1-3} + \gamma_{n-z_1-3})\, e_4 + \\
 & & (\alpha_{n-z_1-2} + \gamma_{n-z_1-2})\, e_3 \gamma_{n-z_1-1}\, e_2 \\
[e_{n-2}, e_{n-1}] & = & \alpha_1\, e_{n-z_1} + \alpha_2 e_{n-z_1-1} + \alpha_3\, e_{n-z_1-2} + \ldots + \alpha_{n-z_1-3}\, e_4 + \\
 & & \alpha_{n-z_1-2}\, e_3 + \alpha_{n-z_1-1}\, e_2 \quad (\textit{with } \alpha_p \neq 0, \\
 & & \ \textit{for at least some } p, \textit{ with } 1 \leq p \leq n - z_1 - 1) \\
[e_{n-2}, e_n] & = & (n - z_1 - 1)\,\alpha_1\, e_{n-z_1+1} + ((n - z_1 - 2)\,\alpha_2 + \gamma_2)\, e_{n-z_1} + \ldots + \\
 & & (4\,\alpha_{n-z_1-4} + \gamma_{n-z_1-4})\, e_6 + (3\,\alpha_{n-z_1-3} + \gamma_{n-z_1-3})\, e_5 + \\
 & & (2\,\alpha_{n-z_1-2} + \gamma_{n-z_1-2})\, e_4 + (\alpha_{n-z_1-1} + \gamma_{n-z_1-1})\, e_3 + \gamma_{n-z_1}\, e_2 \\
[e_{n-1}, e_n] & = & (n - z_1 - 1)\,\alpha_1\, e_{n-z_1+2} + ((n - z_1 - 2)\,\alpha_2 + \gamma_2)\, e_{n-z_1+1} + \\
 & & ((n - z_1 - 3)\,\alpha_3 + \gamma_3)\, e_{n-z_1} \\
 & & + \ldots + (4\,\alpha_{n-z_1-4} + \gamma_{n-z_1-4})\, e_7 + \\
 & & (3\,\alpha_{n-z_1-3} + \gamma_{n-z_1-3})\, e_6 + (2\,\alpha_{n-z_1-2} + \gamma_{n-z_1-2})\, e_5 + \\
 & & (\alpha_{n-z_1-1} + \gamma_{n-z_1-1})\, e_4 + \alpha_{n-z_1}\, e_3 + \gamma_{n-z_1+1}\, e_2
\end{array}$$

where the coefficients α_p *and* γ_q*, with* $1 \leq p \leq n - z_1$ *and* $2 \leq q \leq n - z_1 + 1$ *are abided by the restrictions derived from Jacobi identities* $J(e_h, e_k, e_l) = 0$*, with* $z_1 \leq h < k < l \leq n - 2$. □

So, as either $\alpha_1 \neq 0$ or $\gamma_2 \neq 0$, we now distinguish two cases in the study of these algebras:

- The case $\alpha_1 = 0$.

In this case, $\gamma_2 \neq 0$. Moreover, a suitable adapted base change involves $\gamma_2 = 1$.

Besides, as the bracket $[e_{n-2}, e_{n-1}] \neq 0$, we deduce that there exists at least a coefficient α_p different from 0, with $2 \leq p \leq n - z_1 - 1$.

Example 1. *To illustrate this case, let us suppose that we would like to find an example of a filiform Lie algebra corresponding to the triple* $(4, 5, 7)$*. To do this, by replacing* z_1*,* z_2 *and* k *with* $4, 5$ *and* 7*, respectively, in Theorem 3.1, we obtain the following expressions:*

$$\begin{array}{lcl}
[e_1, e_h] & = & e_{h-1} \qquad (3 \leq h \leq 7) \\
[e_4, e_6] & = & \alpha_1 \, e_2 \\
[e_4, e_7] & = & \alpha_1 \, e_3 + \gamma_2 \, e_2 \\
[e_5, e_6] & = & \alpha_1 \, e_3 + \alpha_2 \, e_2 \\
[e_5, e_7] & = & 2 \, \alpha_1 \, e_4 + (\alpha_2 + \gamma_2) \, e_3 + \gamma_3 \, e_2 \\
[e_6, e_7] & = & \alpha_1 \, e_4 + \alpha_2 \, e_3 + \alpha_3 \, e_2 \\
[e_6, e_7] & = & 3 \, \alpha_1 \, e_5 + (2 \, \alpha_2 + \gamma_2) \, e_4 + (\alpha_3 + \gamma_3) \, e_3 + \gamma_4 \, e_2.
\end{array}$$

So, as $\gamma_2 \neq 0$ *in this case, we can suppose* $\alpha_1 = 0$ *and* $\gamma_2 = 1$ *(remember that a suitable adapted base change involves* $\gamma_2 = 1$ *and that the subject of possible isomorphisms between Lie algebras which are obtained is not being dealt in this paper). After, we can take, for instance, the following values for the rest of coefficients:* $\gamma_3 = -\,\alpha_2 = 1$ *and* $\gamma_4 = 0$*. Then, in this way, we would obtain the following filiform Lie algebra (with respect to the adapted basis* $\{e_1, \ldots, e_7\}$*):*

$$\begin{array}{llll}
[e_1, e_h] = e_{h-1} \qquad (3 \leq h \leq 7) & [e_4, e_7] = e_2, & & \\
[e_5, e_6] = -\,e_2, & [e_5, e_7] = e_2, & [e_6, e_7] = e_3, &
\end{array}$$

So, it can be checked that every filiform Lie algebra satisfying $n = 7, z_1 = 4, z_2 = 5$ *is isomorphic to the one named* μ_7^4 *by Ancochea and Goze's classification of nilpotent Lie algebras of dimension* 7 *(see* [1]*). However, since this classification there exist many other true classifications. For example, the reader can consult the one by Goze and Remm, which is available en the web site* [12].

- The case $\alpha_1 \neq 0$.

In this case, it is always possible to suppose $\alpha_1 = 1$. It can be obtained by using a suitable adapted basis change.

Example 2. *To show an example of this case, let us suppose that we would like to find an example of a filiform Lie algebra corresponding to the triple* $(6, 7, 9)$*. By proceeding in the same way as in previous example, we have the following* *expressions:*

$$\begin{array}{rcl} [e_1, e_h] & = & e_{h-1} \qquad (3 \leq h \leq 9) \\ [e_6, e_8] & = & \alpha_1\, e_2 \\ [e_6, e_9] & = & \alpha_1\, e_3 + \gamma_2\, e_2 \\ [e_7, e_8] & = & \alpha_1\, e_3 + \alpha_2\, e_2 \\ [e_7, e_9] & = & 2\,\alpha_1\, e_4 + (\alpha_2 + \gamma_2)\, e_3 + \gamma_3\, e_2 \\ [e_8, e_9] & = & 3\,\alpha_1\, e_5 + (2\,\alpha_2 + \gamma_2)\, e_4 + (\alpha_3 + \gamma_3)\, e_3 + \gamma_4\, e_2. \end{array}$$

So, as it is possible to get $\alpha_1 = 1$ in this case, we can suppose $\gamma_2 = 1$, for instance. Next, by taking $\alpha_2 = \lambda$ (undetermined) and $\gamma_3 = \gamma_4 = \alpha_3 = 0$, we obtain the following filiform Lie algebra, with respect to the adapted basis $\{e_1, \ldots, e_9\}$:

$$\begin{array}{llll} [e_1, e_h] = & e_{h-1} \qquad (3 \leq h \leq 9) & & [e_6, e_8] = e_2, \\ [e_6, e_9] = & e_3 \,+\, e_2, & & [e_7, e_8] = e_3 \,+\, \lambda\, e_2, \\ [e_7, e_9] = & 2\, e_4 \,+\, (1+\lambda)\, e_3, & & [e_8, e_9] = 2\, e_5 \,+\, (1+\lambda)\, e_4 \end{array}$$

As it can be also checked, this family of algebras is isomorphic with the one named $\mu_9^{16,\lambda}$ by Echarte and Gomez's classification of filiform Lie algebras of dimension 9 (see [5]).

Note that this procedure can be implemented in any symbolic computation package, which allows to obtain concrete examples of filiform Lie algebras in any dimension in an easy way. However, note also that a filiform Lie algebra can be obtained by two cases, as we can observe in the following table (see the triple (5, 6, 8)):

triple	*bracket*	*case*	*triple*	*bracket*	*case*
$(4,4,6)$	$[e_4, e_5] = e_2$	2	$(5,6,8)$	$[e_5, e_7] = 0$	1
$(4,5,7)$	$[e_4, e_6] = 0$	1	$(5,6,8)$	$[e_5, e_7] = e_2$	2
$(5,5,7)$	$[e_5, e_6] = e_2$	2	$(6,6,8)$	$[e_6, e_7] = e_2$	2
$(4,6,8)$	$[e_4, e_7] = 0$	1			

Now, we are going to study some particular cases of these algebras corresponding to the triple $(z_1, n-2, n)$. From this study, we will obtain some consequences which will allow us to establish some conjectures. These cases are the following:

3.1. **The case $z_1 = n-2$.** In this case, there is an unique coefficient α_1, which is $\alpha_1 \neq 0$ (in fact, we can get $\alpha_1 = 1$, as we said before). Then, these algebras are defined, with respect to an adapted basis $\{e_1, \dots, e_n\}$, by brackets:

$$\begin{array}{lcll} [e_1, e_h] & = & e_{h-1} & (3 \leq h \leq n) \\ [e_{n-2}, e_{n-1}] & = & e_2 & \\ [e_{n-2}, e_n] & = & e_3 + \gamma_2\, e_2 & \\ [e_{n-1}, e_n] & = & e_4 + \gamma_2\, e_3 + \gamma_3\, e_2 & \end{array}$$

It is easy to check that each addend of Jacobi identity $J(e_{n-2}, e_{n-1}, e_n) = 0$ is null. Besides, it can be also observed that the adapted base change:

$$e'_k = e_k \quad (1 \leq k \leq 3) \quad ; \quad e'_h = e_h + \frac{\gamma_3}{2}\, e_{h-2} \quad (4 \leq h \leq n)$$

involves $\gamma'_3 = 0$. These results allow to set the following:

Proposition 1. *With respect to an adapted basis $\{e_1, \dots, e_n\}$, filiform Lie algebras belonging to the triple $(n-2, n-2, n)$ are defined by:*

$$\begin{array}{lcll} [e_1, e_h] & = & e_{h-1} & (3 \leq h \leq n) \\ [e_{n-2}, e_{n-1}] & = & e_2 & \\ [e_{n-2}, e_n] & = & e_3 + \gamma_2\, e_2 & \\ [e_{n-1}, e_n] & = & e_4 + \gamma_2\, e_3 & \end{array}$$

with $\gamma_2 \in \mathbb{C}$. □

Note that in the particular cases $n = 6, 7, 8$, this result had been already obtained in [8]. Moreover, if we use a suitable basis change in these cases, it can be observed that if n is 7, that is, odd, then $\gamma_2 = 0$, whereas if n is even (6 or 8), then that coefficient can be or not be equal to zero. We conjecture that it also occurs for any n arbitrary.

Now, we are going to consider a new case of these algebras:

3.2. **The case $z_1 = n-3$.** The filiform Lie algebras corresponding to this case are defined, with respect to an adapted basis $\{e_1, \dots, e_n\}$, by the brackets:

$$\begin{array}{lcll} [e_1, e_h] & = & e_{h-1} & (3 \leq h \leq n) \\ [e_{n-3}, e_{n-1}] & = & \alpha_1\, e_2 & \\ [e_{n-3}, e_n] & = & \alpha_1\, e_3 + \gamma_2\, e_2 & (\alpha_1 \neq 0 \text{ or } \gamma_2 \neq 0) \\ [e_{n-2}, e_{n-1}] & = & \alpha_1\, e_3 + \alpha_2\, e_2 & (\alpha_1 \neq 0 \text{ or } \alpha_2 \neq 0) \\ [e_{n-2}, e_n] & = & 2\,\alpha_1\, e_4 + (\alpha_2 + \gamma_2)\, e_3 + \gamma_3\, e_2 & \\ [e_{n-1}, e_n] & = & 2\,\alpha_1\, e_5 + (\alpha_2 + \gamma_2)\, e_4 + \gamma_3\, e_3 + \gamma_4\, e_2 & \end{array}$$

where coefficients are abided to restrictions resulting from Jacobi identity $J(e_{n-2}, e_{n-1}, e_n) = 0$.

Then, by checking that identity, we find that $-2\,\alpha_1\,[e_4, e_{n-1}] = 0$. So, if $n > 7$, that condition is trivially satisfied and if $n = 7$ (remember that in this case, $n \geq 7$), then that condition is true if and only if $\alpha_1 = 0$. Therefore, in this case, we deduce the following:

Proposition 2. *With respect to an adapted basis $\{e_1, \ldots, e_7\}$, complex filiform Lie algebras corresponding to the triple $(n-3, n-2, n)$ are defined by:*

$$\begin{array}{lcll}
[e_1, e_h] & = & e_{h-1} & (3 \leq h \leq 7) \\
[e_4, e_7] & = & \gamma_2\, e_2 & (\gamma_2 \neq 0) \\
[e_5, e_6] & = & \alpha_2\, e_2 & (\alpha_2 \neq 0) \\
[e_5, e_7] & = & (\alpha_2 + \gamma_2)\, e_3 + \gamma_3\, e_2 & \\
[e_6, e_7] & = & (\alpha_2 + \gamma_2)\, e_4 + \gamma_3\, e_3 + \gamma_4\, e_2 &
\end{array}$$

if $n = 7$ or by the brackets:

$$\begin{array}{lcll}
[e_1, e_h] & = & e_{h-1} & (3 \leq h \leq n) \\
[e_{n-3}, e_{n-1}] & = & \alpha_1\, e_2 & \\
[e_{n-3}, e_n] & = & \alpha_1\, e_3 + \gamma_2\, e_2 & (\alpha_1 \neq 0 \text{ or } \gamma_2 \neq 0) \\
[e_{n-2}, e_{n-1}] & = & \alpha_1\, e_3 + \alpha_2\, e_2 & (\alpha_1 \neq 0 \text{ or } \alpha_2 \neq 0) \\
[e_{n-2}, e_n] & = & 2\,\alpha_1\, e_4 + (\alpha_2 + \gamma_2)\, e_3 + \gamma_3\, e_2 & \\
[e_{n-1}, e_n] & = & 2\,\alpha_1\, e_5 + (\alpha_2 + \gamma_2)\, e_4 + \gamma_3\, e_3 + \gamma_4\, e_2 &
\end{array}$$

if $n \geq 7$. $\square$

3.3. **The case $z_1 = n - 4$.** In this case, it is easily checked that all restrictions resulting from Jacobi identity $J(e_{n-2}, e_{n-1}, e_n) = 0$ have been already obtained from Jacobi identities previous (in the lexicographical order) to it.

So, by computing this last identity we obtain:

$$\alpha_1\,([e_4, e_n] - 3\,[e_5, e_{n-1}]) - \,(2\,\alpha_2 + \gamma_2)\,[e_4, e_{n-1}] = 0. \qquad (*)$$

and, by computing again, we observe that:

a) If $n = 8$, then $(*)$ reduces to $\alpha_1\,(-2\,\alpha_1\, e_3 - 5\,\alpha_2\, e_2) = 0$, and thus, Jacobi identity is satisfied if and only if $\alpha_1 = 0$.

b) If $n = 9$, then $(*)$ reduces to $-3\,\alpha_1^2 e_2 = 0$, and thus, we obtain the same condition as above.

c) If $n \geq 10$, then $z_1 \geq 6$ and thus $(*)$ is the identity $0 = 0$.

So, it is proved the following:

Proposition 3. *With respect to an adapted basis $\{e_1, \ldots, e_n\}$, filiform Lie algebras belonging to the triple $(n-4, n-2, n)$ are defined by:*

$$
\begin{array}{lcl}
[e_1, e_h] & = & e_{h-1} \qquad (3 \leq h \leq n) \\
[e_{n-4}, e_{n-1}] & = & \alpha_1\, e_2 \\
[e_{n-4}, e_n] & = & \alpha_1\, e_3 + \gamma_2\, e_2 \qquad (\alpha_1 \neq 0 \textit{ or } \gamma_2 \neq 0) \\
[e_{n-3}, e_{n-1}] & = & \alpha_1\, e_3 + \alpha_2\, e_2 \\
[e_{n-3}, e_n] & = & 2\,\alpha_1\, e_4 + (\alpha_2 + \gamma_2)\, e_3 + \gamma_3\, e_2 \\
[e_{n-2}, e_{n-1}] & = & \alpha_1\, e_4 + \alpha_2\, e_3 + \alpha_3\, e_2 \quad (\alpha_1 \neq 0 \textit{ or } \alpha_2 \neq 0 \textit{ or } \alpha_3 \neq 0) \\
[e_{n-2}, e_n] & = & 3\,\alpha_1\, e_5 + (2\,\alpha_2 + \gamma_2)\, e_4 + (\alpha_3 + \gamma_3)\, e_3 + \gamma_4\, e_2 \\
[e_{n-1}, e_n] & = & 3\,\alpha_1\, e_6 + (2\,\alpha_2 + \gamma_2)\, e_5 + (\alpha_3 + \gamma_3)\, e_4 + \gamma_4\, e_3 + \gamma_5\, e_2
\end{array}
$$

where the condition $\alpha_1 = 0$ has to be also added if and only if $n = 8$ or $n = 9$. □

Now, we could continue this procedure by taking $z_1 = n-5$, $z_1 = n-6, \ldots$ until to consider the case $z_1 = 4$, that is, filiform Lie algebras corresponding to the triple $(4, n-2, n)$.

3.4. **The case $z_1 = 4$.** In this case, by proceeding in the same way as previous cases it can easily checked the following result:

Proposition 4. *With respect to an adapted basis $\{e_1, \ldots, e_n\}$, filiform Lie algebras belonging to the triple $(4, n-2, n)$ are defined by:*

$$
\begin{array}{lcl}
[e_1, e_h] & = & e_{h-1} \qquad (3 \geq h \geq n) \\
[e_4, e_n] & = & \alpha_1\, e_3 + \gamma_2\, e_2 \\
[e_5, e_{n-1}] & = & \alpha_1\, e_3 + \alpha_2\, e_2 \\
[e_5, e_n] & = & 2\,\alpha_1\, e_4 + (\alpha_2 + \gamma_2)\, e_3 + \gamma_3\, e_2 \\
[e_6, e_{n-1}] & = & \alpha_1\, e_4 + \alpha_2\, e_3 + \alpha_3\, e_2 \\
[e_6, e_n] & = & 3\,\alpha_1\, e_5 + (2\,\alpha_2 + \gamma_2)\, e_4 + (\alpha_3 + \gamma_3)\, e_3 + \gamma_4\, e_2 \\
\vdots & & \vdots \\
[e_{n-4}, e_{n-1}] & = & \alpha_1\, e_{n-6} + \alpha_2\, e_{n-7} + \alpha_3\, e_{n-8} + \ldots + \alpha_{n-9}\, e_4 + \\
 & & \alpha_{n-8}\, e_3 + \alpha_{n-7}\, e_2 \\
[e_{n-4}, e_n] & = & (n-7)\,\alpha_1\, e_{n-5} + ((n-8)\,\alpha_2 + \gamma_2)\, e_{n-6} + \\
 & & ((n-9)\,\alpha_3 + \gamma_3)\, e_{n-7} + \ldots + (3\,\alpha_{n-9} + \gamma_{n-9})\, e_5 + \\
 & & (2\,\alpha_{n-8} + \gamma_{n-8})\, e_4 + (\alpha_{n-7} + \gamma_{n-7}\, e_3 + \gamma_{n-6}\, e_2 \\
[e_{n-3}, e_{n-1}] & = & \alpha_1\, e_{n-5} + \alpha_2\, e_{n-6} + \alpha_3\, e_{n-7} + \ldots + \alpha_{n-8}\, e_4 + \\
 & & \alpha_{n-7}\, e_3 + \alpha_{n-6}\, e_2 \\
[e_{n-3}, e_n] & = & (n-6)\,\alpha_1\, e_{n-4} + ((n-7)\,\alpha_2 + \gamma_2)\, e_{n-5} + \\
 & & ((n-8)\,\alpha_3 + \gamma_3)\, e_{n-6} + \ldots + (3\,\alpha_{n-8} + \gamma_{n-8}\, e_5 + \\
 & & (2\,\alpha_{n-7} + \gamma_{n-7})\, e_4 + (\alpha_{n-6} + \gamma_{n-6}\, e_3 + \gamma_{n-5}\, e_2 \\
[e_{n-2}, e_{n-1}] & = & \alpha_1\, e_{n-4} + \alpha_2\, e_{n-5} + \alpha_3\, e_{n-6} + \ldots + \alpha_{n-7} e_4 + \\
 & & \alpha_{n-6}\, e_3 + \alpha_{n-5}\, e_2 \qquad (\alpha_p \neq 0,\ for\ some\ p) \\
[e_{n-2}, e_n] & = & (n-5)\,\alpha_1\, e_{n-3} + ((n-6)\,\alpha_2 + \gamma_2)\, e_{n-4} + \\
 & & ((n-7)\,\alpha_3 + \gamma_3)\, e_{n-5} + \ldots + (3\,\alpha_{n-7} + \gamma_{n-7}\, e_5 + \\
 & & (2\,\alpha_{n-6} + \gamma_{n-6})\, e_4 + (\alpha_{n-5} + \gamma_{n-5})\, e_3 + \gamma_{n-4}\, e_2 \\
[e_{n-1}, e_n] & = & (n-5)\,\alpha_1\, e_{n-2} + ((n-6)\,\alpha_2 + \gamma_2)\, e_{n-3} + \\
 & & ((n-7)\,\alpha_3 + \gamma_3)\, e_{n-4} + \ldots + (3\,\alpha_{n-7} + \gamma_{n-7}\, e_6 + \\
 & & (2\,\alpha_{n-6} + \gamma_{n-6})\, e_5 + (\alpha_{n-5} + \gamma_{n-5})\, e_4 + \gamma_{n-4}\, e_3 + \gamma_{n-3}\, e_2
\end{array}
$$

where the coefficients α_p and γ_q, with $1 \leq p \leq n-5$ and $2 \leq q \leq n-3$ are abided by the restrictions derived from Jacobi identity concerning to $J(e_{n-2}, e_{n-1}, e_n) = 0$. □

In fact, as in this case we have $n \geq 6$, these restrictions can be easily obtained for low values of n. Indeed, we can observe that Jacobi identity $J(e_{n-2}, e_{n-1}, e_n) = 0$ is satisfied:

(1) always, if $n = 6$.
(2) if and only if $\alpha_1 = 0$, for $n = 7, 8$.
(3) if and only if $\alpha_1 = \alpha_2 = 0$, for $n = 9, 10$.

(4) if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$, for $n = 11, 12$.

Then, with respect to $n \geq 13$, if we call K_l the coefficient of the basic vector e_l in the first member of (JI), it is easy to compute the following table:

dimension	K_l	*dim*	K_l
6	$K_l = 0, \quad \forall l$	7	$K_2 = \alpha_1$
8	$K_3 = \alpha_1$	9	$K_4 = \alpha_1, \quad K_2 = \alpha_2$
10	$K_5 = \alpha_1, \quad K_3 = \alpha_2$		
11	$K_6 = K_7 = \alpha_1, \quad K_4 = K_5 = \alpha_2$		
	$K_2 = K_3 = \alpha_3$		

and, in general, for a given and fixed subindex n, we think that $\alpha_1 = \alpha_2 = \ldots = \alpha_{[\frac{n-5}{2}]} = 0$, where $[\frac{n-5}{2}]$ represents the integer part of $\frac{n-5}{2}$. This is our second conjecture.

In this way, general laws (with respect to an adapted basis $\{e_1, \ldots, e_n\}$) of the filiform Lie algebras corresponding, respectively, to triples $(4,4,6)$, $(4,5,7)$, $(4,6,8)$, $(4,7,9)$, $(4,8,10)$, $(4,9,11)$ and $(4,10,12)$ can be obtained with no difficulty. Note that some of them, those of lower dimensions, already appear in [8].

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(F.J. Echarte, J. Núñez, and F. Ramírez) Dpto. Geometría y Topología, Facultad de Matemáticas, Universidad de Sevilla, Apdo. 1160, 41080-Sevilla (SPAIN)

E-mail address: `[of J.Núñez] jnvaldes@us.es`