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SIMPLY CONNECTED LIE SUBGROUPS OF THE UNIPOTENT LIE GROUP OF ORDER 4

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Abstract. In this paper we obtain simply connected Lie subgroups (up to isomorphism) of the Lie group G_4 of upper triangular matrices of order 4 having "1" in its main diagonal. To do it, we determine Lie subalgebras of the Lie algebra $\mathfrak{g}_4$ associated with G_4. We find that there exist 4 simply connected subgroups of G_4 for dimension less than 4, 3 for dimension 4 and 4 for dimension 5. We also prove that the rest of groups associated with Lie algebras of these dimensions are not representable as subgroups of G_4.

0. Introduction

First of all, we would like to explain why we are dealing with simply connected Lie groups associated with nilpotent Lie algebras. It is because, in our opinion, an adequate knowledge about such groups could allow to get some advances in the study of those algebras, due to the special relation existing between Lie groups and Lie algebras.

In this way, we will study in this paper all the Lie subgroups, up to isomorphism, of the Lie group G_4, whose elements are unipotent matrices of order 4 (see Section 2). With respect to the relation between Lie algebras and Lie groups, Lie's Third Theorem states that there exists a bijective correspondence between Lie algebras and simply connected Lie groups (up to isomorphism). In this way, when considering isomorphism classes, two isomorphic simply connected

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Lie groups can be considered the same since they belong to the same isomorphism class.

In any case, as it can be easily checked, it is equivalent to study all Lie subalgebras of the Lie algebra $\mathfrak{g}_4$, associated with the Lie group G_4.

Nevertheless, there exists another classification of the groups of G_4. Such a classification depends on the conjugacy classes of G_4. Let us observe that if two subgroups are conjugated then they are isomorphic. But the converse condition is not true, in general. As a consequence, when the classification of the subgroups of G_4 according to this point of view is studied, more distinct classes than in the isomorphic case would be obtained.

In this paper we search for unitary matrix representations for every non-isomorphic simply connected Lie subgroup. To get it, we only need to compute a representative of each nilpotent Lie subalgebra of $\mathfrak{g}_4$. Then, we obtain all the nilpotent Lie algebras whose associated simply connected Lie group admits a 4-dimensional matrix representation.

The problem studied here could be interesting in connection with global differential geometric method, as the Fröbenius' Theorem, and its relation with the theory of Pfaffian forms and it could be also seen as the first step towards a conjugacy class classification of the subgroups of G_4, once the possible related Lie algebras have been identified.

The structure of this paper is as follows: In the first section we show some preliminary concepts on Lie groups and Lie algebras. The second section deals with the unipotent Lie group G_4 and with its associated Lie algebra $\mathfrak{g}_4$. Finally, the third section is devoted to determining the simply connected Lie subgroups of the Lie group G_4, showing each of them together their associated Lie algebra (which is a subalgebra of $\mathfrak{g}_4$). The main result of the paper is Theorem 3.15.

1. Preliminaries

For a general overview on Lie groups and Lie algebras, the reader can consult [3]. Other results about these concepts, mainly about the connection between one-parameter subgroups and left-invariant fields and about nilpotent Lie algebras, are the following:

Let G be a Lie group. It is easy to prove that the set of left-invariant fields of G is a Lie algebra, which is denoted by $\mathcal{L}(G)$ or $\mathfrak{g}$, and it is called *Lie algebra associated with* G. Note that dimensions of G and $\mathfrak{g}$ are the same.

The converse states that every Lie algebra is associated with some Lie group. This last result was locally proved by Lie, in his Third Theorem, and globally by Ado. Ado's Theorem (see [3]) says that any finite-dimensional complex Lie algebra is isomorphic to some matrix Lie algebra (in fact, this theorem is not merely valid for complex Lie algebras, but states moreover that any real Lie algebra is actually isomorphic to the Lie algebra of some analytic group of complex matrices). Note, however, that the uniqueness of the converse is only satisfied, up to isomorphism, if Lie groups considered are simply connected. Let us recall that a Lie group is said to be *simply connected* if its associated topological structure is simply connected.

A *representation* of a Lie group of dimension n is a Lie group homomorphism $\phi\colon G \to GL(n,\mathbb{C})$.

A relation between Lie subgroups of a fixed and given Lie group and their respective associated Lie algebras is the following:

Proposition 1.1. *Let G be a Lie group and let $\mathfrak{g}$ be its associated Lie algebra. If H is a Lie subgroup of G, then the Lie algebra $\mathfrak{h}$ associated with H is a Lie subalgebra of $\mathfrak{g}$.*

A Lie algebra $\mathfrak{g}$ is called *abelian* if $[X,Y]=0$, for all $X,Y \in \mathfrak{g}$, where $[X,Y]$ denotes the law of $\mathfrak{g}$.

The central descending sequence of $\mathfrak{g}$ is defined by:

$$\mathfrak{g} = \mathcal{C}^1(\mathfrak{g}) \supseteq \mathcal{C}^2(\mathfrak{g}) \supseteq \cdots \supseteq \mathcal{C}^k(\mathfrak{g}) \supseteq \cdots$$

where the ideal $\mathcal{C}^k(\mathfrak{g})$ of $\mathfrak{g}$ is defined by $\mathcal{C}^k(\mathfrak{g}) = [\mathfrak{g}, \mathcal{C}^{k-1}(\mathfrak{g})]$, $\forall k \in \mathbb{N}$.

The Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if there exists $n \in \mathbb{N}$ such that $\mathcal{C}^n(\mathfrak{g}) = \{0\}$. The least integer n verifying this condition is called *nilindex* of $\mathfrak{g}$ (i.e. $\mathcal{C}^{n-1}(\mathfrak{g}) \neq \{0\}$).

Nilpotent Lie algebras verify the following:

Proposition 1.2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $\mathfrak{g}'$ be a Lie subalgebra of $\mathfrak{g}$. Then, it is verified $\mathcal{C}^k(\mathfrak{g}') \subseteq \mathcal{C}^k(\mathfrak{g})$, for all $k \in \mathbb{N}$.*

Those nilpotent Lie algebras of dimension n whose central descending sequence is $(n-2, n-3, \dots, 2, 1, 0)$ are called *filiform*. These algebras constitute the most structured subset of nilpotent Lie algebras.

To finish this section we recall Fröbenius' Theorem, which will be used to obtain simply connected Lie subgroups of the unipotent Lie group G_4 of order 4: *If M is a complex analytic manifold of dimension n and D is an r-dimensional involutive analytic distribution over M (with $r < m$), then the set $\mathcal{I}(D)$ of all differential forms on M annulling D is an ideal of the module $\Lambda(M)$ of all differential forms over M. Moreover, this ideal is locally generated by $m-r$ independent differential 1-forms over M and it is a differential ideal.*

Then, Fröbenius' Local Theorem sets that if the ideal $\mathcal{I}(D) \subset \Lambda(M)$ is both differential and locally generated by $m-r$ independent differential 1-forms, then there exists a submanifold L of dimension r which is integral of $\mathcal{I}(D)$ containing p, for all $p \in M$ (remember that a submanifold L is said integral of $\mathcal{I}(D)$ if $T_q(L) = D_q$, for a given $q \in L$). Moreover, Fröbenius' Global Theorem both affirms the uniqueness of such a submanifold L and sets that L is connected and maximal. For the reader interested in the details of this theorem (as its local version as its global one), see Section 1.3 in [3].

2. The Lie Group G_4 and Its Associated Lie Algebra $\mathfrak{g}_4$

The Lie group G_4 is that whose elements $g_4(x_{i,j})$ are unipotent matrices of order 4, having the following expression:

$$g_4(x_{i,j}) = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

If this group is considered as an analytic complex manifold, then it has dimension 6. So, the dimension of its associated Lie algebra $\mathfrak{g}_4$ is also 6. Since the proper Lie subalgebras of $\mathfrak{g}_4$ have dimension less than 6, simply connected Lie subgroups of G_4 will have also dimension less than 6, in virtue of Ado's Theorem.

Now, to determine these proper Lie subalgebras of $\mathfrak{g}_4$, we will give, in the first place the law of $\mathfrak{g}_4$ with respect to an adequate basis. By using previous expression (1) of the elements of G_4, we already proved in [2] the following result:

Theorem 2.1. *A basis of the Lie algebra* $\mathfrak{g}_4$ *is given by the vector fields:*

$$\begin{aligned} X_{1,2} &= e_{1,2}, & X_{2,3} &= e_{2,3} + x_{1,2}e_{1,3}, \\ X_{1,3} &= e_{1,3}, & X_{2,4} &= e_{2,4} + x_{1,2}e_{1,4}, \\ X_{1,4} &= e_{1,4}, & X_{3,4} &= e_{3,4} + x_{2,3}e_{2,4} + x_{1,3}e_{1,4}, \end{aligned}$$

where $e_{i,j}$ *denotes* $\frac{\partial}{\partial x_{i,j}}$. *The brackets of the algebra with respect to this basis is:*

$$\begin{aligned} [X_{1,2}, X_{2,3}] &= X_{1,3}, & [X_{1,3}, X_{3,4}] &= X_{1,4}, \\ [X_{1,2}, X_{2,4}] &= X_{1,4}, & [X_{2,3}, X_{3,4}] &= X_{2,4}. \end{aligned} \tag{2}$$

Now, as an immediate consequence of the previous theorem, the following result holds:

Corollary 2.2. *The Lie algebra* $\mathfrak{g}_4$ *is nilpotent and its central descending sequence is* $(3, 1, 0)$.

Therefore, subalgebras of $\mathfrak{g}_4$ must be nilpotent Lie algebras of dimension less than 6. The classification of these algebras was reached by Morozov in [1]. However, he only classified nilpotent Lie algebras which are indecomposable (i.e., non-isomorphic to a direct sum of lower dimensional ideals). We exhibit here the full list of nilpotent Lie algebras of dimension less or equal than 5, independently on the fact whether they are decomposable or not, which will be denoted by (d) or (i), respectively:

Dimension 1:	$\mathfrak{n}_1^1$	abelian.	(i)
Dimension 2:	$\mathfrak{n}_2^1$	abelian.	(d)
Dimension 3:	$\mathfrak{n}_3^1$ $[Y_1, Y_3] = Y_2$	model filiform	(i)
	$\mathfrak{n}_3^2$	abelian.	(d)
Dimension 4:	$\mathfrak{n}_4^1$ $[Y_1, Y_4] = Y_3,\ [Y_1, Y_3] = Y_2$	model filiform	(i)
	$\mathfrak{n}_4^2$ $[Y_1, Y_4] = Y_3$	direct sum $\mathfrak{n}_3^1 \oplus \mathfrak{n}_1^1$	(d)
	$\mathfrak{n}_4^3$	abelian.	(d)

Dimension 5:	$\mathfrak{n}_5^1$ $[Y_1, Y_5] = Y_4,\ [Y_1, Y_4] = Y_3,$ $[Y_1, Y_3] = Y_2$	model filiform	(i)
	$\mathfrak{n}_5^2$ $[Y_1, Y_5] = Y_4,\ [Y_1, Y_4] = Y_3,$ $[Y_1, Y_3] = Y_2, [Y_4, Y_5] = Y_2$	filiform	(i)
	$\mathfrak{n}_5^3$ $[Y_1, Y_5] = Y_4,\ [Y_1, Y_4] = Y_3,$ $[Y_4, Y_5] = Y_2$		(i)
	$\mathfrak{n}_5^4$ $[Y_1, Y_5] = Y_4,\ [Y_1, Y_3] = Y_2,$ $[Y_2, Y_3] = Y_4$		(i)
	$\mathfrak{n}_5^5$ $[Y_1, Y_5] = Y_4,\ [Y_1, Y_3] = Y_2$		(i)
	$\mathfrak{n}_5^6$ $[Y_1, Y_5] = Y_4,\ [Y_2, Y_3] = Y_4$		(i)
	$\mathfrak{n}_5^7$ $[Y_1, Y_5] = Y_4$	direct sum $\mathfrak{n}_3^1 \oplus \mathfrak{n}_2^1$	(d)
	$\mathfrak{n}_5^8$ $[Y_1, Y_5] = Y_4,\ [Y_1, Y_4] = Y_3$	direct sum $\mathfrak{n}_4^1 \oplus \mathfrak{n}_1^1$	(d)
	$\mathfrak{n}_5^9$	abelian.	(d)

Related to their respective central descending sequences, the following result is easily obtained:

Proposition 2.3. *The central descending sequence of each nilpotent Lie algebra of dimension less or equal than* 5 *is:*

algebras $\mathfrak{n}_5^1$ *and* $\mathfrak{n}_5^2$	$(3, 2, 1, 0)$
algebra $\mathfrak{n}_5^3$	$(3, 2, 0)$
algebras $\mathfrak{n}_4^1$, $\mathfrak{n}_5^4$ *and* $\mathfrak{n}_5^8$	$(2, 1, 0)$
algebra $\mathfrak{n}_5^5$	$(2, 0)$
algebras $\mathfrak{n}_3^1$, $\mathfrak{n}_4^2$, $\mathfrak{n}_5^6$ *and* $\mathfrak{n}_5^7$	$(1, 0)$
algebras $\mathfrak{n}_1^1$, $\mathfrak{n}_2^1$, $\mathfrak{n}_3^2$, $\mathfrak{n}_4^3$ *and* $\mathfrak{n}_5^9$	(0).

3. Nilpotent Lie Algebras of Dimension < 6 Which Are Subalgebras of $\mathfrak{g}_4$

The main goal of this section is to obtain nilpotent Lie algebras of dimension less than 6 which are subalgebras of $\mathfrak{g}_4$, or equivalently, in virtue of Ado's Theorem, to obtain the simply connected Lie subgroups of the Lie group G_4.

To do this, we will prove in first place that some nilpotent Lie algebras of dimension less than 6 cannot be subalgebras of $\mathfrak{g}_4$.

Proposition 3.1. *The nilpotent Lie algebras* $\mathfrak{n}_5^1$, $\mathfrak{n}_5^2$ *and* $\mathfrak{n}_5^3$ *are not subalgebras of* $\mathfrak{g}_4$.

Proof. According to Corollary 2.2, it is verified that $\dim \mathcal{C}^3(\mathfrak{g}_4) = 1$. However, Proposition 2.3 implies that $\dim \ \mathcal{C}^3(\mathfrak{n}_5^k) = 2$, for $k = 1, 2, 3$. Therefore, $\dim \mathcal{C}^3(\mathfrak{n}_5^k) > \dim \mathcal{C}^3(\mathfrak{g}_4)$ for $k = 1, 2, 3$, which is contradictory with Proposition 1.2.

As an immediate consequence of this result, it is also proved the following:

Corollary 3.2. *The simply connected Lie groups associated with filiform Lie algebras* $\mathfrak{n}_5^1$ *and* $\mathfrak{n}_5^2$ *and with nilpotent Lie algebra* $\mathfrak{n}_5^3$ *cannot be represented as a subgroup of* G_4.

Now, by taking into consideration that this inconvenience does not appear with the rest of nilpotent Lie algebras of dimension less or equal than 5, we are going to determine which of such algebras are subalgebras of $\mathfrak{g}_4$. We firstly begin with abelian Lie algebras.

Proposition 3.3. *The subalgebras* $\langle X_{1,2}\rangle$, $\langle X_{1,2}, X_{1,3}\rangle$, $\langle X_{1,2}, X_{1,3}, X_{1,4}\rangle$ *and* $\langle X_{1,3}, X_{1,4}, X_{2,3}, X_{2,4}\rangle$ *of* $\mathfrak{g}_4$ *are isomorphic to the abelian Lie algebras* $\mathfrak{n}_1^1$, $\mathfrak{n}_2^1$, $\mathfrak{n}_3^2$ *and* $\mathfrak{n}_4^3$, *respectively.*

Proof. It is an immediate consequence of Theorem 2.1.

Now, as we will see, the obtained expressions of abelian Lie algebras as subalgebras of $\mathfrak{g}_4$ will allow to determine the respective associated simply connected Lie groups as subgroups of G_4.

To do this, we will find in the first place, all the Lie subalgebras of Lie algebra $\mathfrak{g}_4$, associated with G_4. Once a Lie subalgebra of dimension r (with

$r < 6 = \dim(G_4)$) and a basis of its are obtained, we will consider this subalgebra as a r-dimensional involutive differential distribution D over G_4. Secondly, we will extend the basis of D to a basis of $\mathfrak{g}_4$ and we will consider its dual basis. The dual 1-forms corresponding to extended fields annul D and generate the ideal $\mathcal{I}(D)$ defined before. Therefore, $\mathcal{I}(D)$ is a differential ideal generated by $6 - r$ independent differential 1-forms. In consequence, as the assumption of Fröbenius' Theorem are satisfied, a submanifold of G_4 (which is an integral submanifold of $\mathcal{I}(D)$ and a Lie subgroup of G_4) is obtained. Note that to obtain a integral manifold L of G_4, verifying $T_q(L) = D_q$, for all $q \in L$, we have to integrate the $6 - r$ 1-forms which generate $\mathcal{I}(D)$ (see Section 1).

So, by proceeding in this way, we show next several results, whose proofs will be constructed in three steps, as follows:

- **Step 1**: Observe which is the subalgebra of $\mathfrak{g}_4$ which is isomorphic to the Lie algebra previously fixed.
- **Step 2**: Extend the basis of that subalgebra to a basis of $\mathfrak{g}_4$, by using other necessary fields.
- **Step 3**: Consider the corresponding dual basis $\{\omega_i\}_{i=1}^{6}$ of the extended basis and obtain the equations of the corresponding simply connected Lie group associated with such a subalgebra, by integrating the differential equations system obtained from Fröbenius' Theorem.

Corollary 3.4. *The simply connected Lie groups associated with abelian Lie algebras $\mathfrak{n}_1^1$, $\mathfrak{n}_2^1$, $\mathfrak{n}_3^2$ and $\mathfrak{n}_4^3$ admit, respectively, the following matrix representations:*

$$\mathfrak{n}_1^1 \longrightarrow \begin{pmatrix} 1 & x_{1,2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathfrak{n}_2^1 \longrightarrow \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathfrak{n}_3^2 \longrightarrow \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathfrak{n}_4^3 \longrightarrow \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof.

(a) Lie algebra $\mathfrak{n}_1^1$.

Step 1. This abelian Lie algebra is the subalgebra generated by $Y_1 = X_{1,2}$.

Step 2. We extend this basis with $Y_2 = X_{1,3}$, $Y_3 = X_{1,4}$, $Y_4 = X_{2,3}$, $Y_5 = X_{2,4}$ and $Y_6 = X_{3,4}$.

Step 3. We consider the corresponding dual basis $\{\omega_i\}_{i=1}^6$ and we obtain the equations of the associated simply connected Lie group, by using Fröbenius' Theorem, after integrating the following differential equation system:

$$\begin{aligned}
\omega_2 &= dx_{1,3} - x_{1,2}dx_{2,3} && = 0,\\
\omega_3 &= dx_{1,4} - x_{1,3}dx_{2,4} + (x_{1,3}x_{2,3} - x_{1,3})dx_{3,4} && = 0,\\
\omega_4 &= dx_{2,3} && = 0,\\
\omega_5 &= dx_{2,4} - x_{2,3}dx_{3,4} && = 0,\\
\omega_6 &= dx_{3,4} && = 0,
\end{aligned}$$

whose solution is $x_{1,3} = x_{1,4} = x_{2,3} = x_{2,4} = x_{3,4} = 0$.

(b) Lie algebra $\mathfrak{n}_2^1$.

Step 1. We consider the subalgebra $\langle Y_1 = X_{1,2}, Y_2 = X_{1,3}\rangle$.

Step 2. We extend this basis with: $Y_3 = X_{1,4}$, $Y_4 = X_{2,3}$, $Y_5 = X_{2,4}$ and $Y_6 = X_{3,4}$.

Step 3. We solve the differential equations system $\{\omega_i = 0,\ i = 3,4,5,6\}$, whose solution is $x_{1,4} = x_{2,3} = x_{2,4} = x_{3,4} = 0$.

(c) Lie algebra $\mathfrak{n}_3^2$.

Step 1. We consider the subalgebra $\langle Y_1 = X_{1,2}, Y_2 = X_{1,3}, Y_3 = X_{1,4}\rangle$.

Step 2. We extend this basis with: $Y_4 = X_{2,3}$, $Y_5 = X_{2,4}$ and $Y_6 = X_{3,4}$.

Step 3. We solve the differential equations system $\{\omega_i = 0,\ i = 4,5,6\}$, whose solution is $x_{2,3} = x_{2,4} = x_{3,4} = 0$.

(d) Lie algebra $\mathfrak{n}_4^3$.

Step 1. We consider the subalgebra $\langle Y_1 = X_{1,3}, Y_2 = X_{1,4}, Y_3 = X_{2,3}, Y_4 = X_{2,4}\rangle$.

Step 2. We extend this basis of with: $Y_5 = X_{1,2}$ and $Y_6 = X_{3,4}$.

Step 3. We solve the differential equations system $\{\omega_5 = 0, \omega_6 = 0\}$, whose solution is $x_{1,2} = x_{3,4} = 0$.

With respect to these abelian Lie algebras, we must still find a matrix representation for the one of dimension 5, but we will prove later the impossibility to find a representation of its associated Lie group as a Lie subgroup of G_4.

Now, we will consider simply connected Lie groups associated with non abelian nilpotent Lie algebras of dimension less or equal than 5. Propositions from 3.5 to 3.11 show the matrix representations of these algebras. The scheme of the proof of each proposition is the same as previous propositions.

Proposition 3.5. *The simply connected Lie group associated with the filiform Lie algebra* $\mathfrak{n}_3^1$ *admits the following representation:*

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & 0 \\ 0 & 1 & x_{2,3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof.

Step 1. The subalgebra $\langle Y_1 = X_{1,2}, Y_2 = X_{1,3}, Y_3 = X_{2,3}\rangle$ is isomorphic to $\mathfrak{n}_3^1$.

Step 2. We extend this basis with $Y_4 = X_{1,4}$, $Y_5 = X_{2,4}$ and $Y_6 = X_{3,4}$.

Step 3. We solve the differential equations system $\{\omega_i = 0 |\ i = 4, 5, 6\}$, whose solution is $x_{1,4} = x_{2,4} = x_{3,4} = 0$.

Proposition 3.6. *The simply connected Lie group associated with the filiform Lie algebra* $\mathfrak{n}_4^1$ *admits the following representation:*

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & \frac{1}{2}x_{2,3}^2 \\ 0 & 0 & 1 & x_{2,3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof.

Step 1. We consider the subalgebra $\langle Y_1 = -(X_{2,3} + X_{3,4}), Y_2 = X_{1,4}, Y_3 = X_{1,3}, Y_4 = X_{1,2}\rangle$.

Step 2. We extend this basis with the fields $Y_5 = X_{2,4}$ and $Y_6 = X_{3,4}$.

Step 3. We solve the differential equations system $\{\omega_5 = 0, \omega_6 = 0\}$, whose solution is $x_{3,4} = x_{2,3}$ and $x_{2,4} = \frac{x_{2,3}^2}{2}$.

Proposition 3.7. *The simply connected Lie group associated with the nilpotent Lie algebra $\mathfrak{n}_4^2$ admits the following representation:*

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof.

Step 1. We consider the subalgebra $\langle Y_1 = X_{1,2}, Y_2 = X_{1,3}, Y_3 = X_{1,4}, Y_4 = X_{2,4}\rangle$.

Step 2. We extend this basis with $Y_5 = X_{2,3}$ and $Y_6 = X_{3,4}$.

Step 3. We solve the differential equations system $\{\omega_5 = 0, \omega_6 = 0\}$, whose solution is $x_{2,3} = x_{3,4} = 0$.

Next, we will obtain the simply connected Lie subgroups of G_4 of dimension 5.

Proposition 3.8. *The simply connected Lie group associated with the nilpotent Lie algebra $\mathfrak{n}_5^4$ admits the following representation:*

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{1,2} & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof.

Step 1. We consider the subalgebra $\langle Y_1 = X_{3,4}, Y_2 = X_{2,4}, Y_3 = -(X_{2,3} + X_{1,2}), Y_4 = X_{1,4}, Y_5 = -X_{1,3}\rangle$.

Step 2. We extend this basis with $Y_6 = X_{2,3}$.

Step 3. We solve the differential equation $\omega_6 = 0$, whose solution is $x_{2,3} = x_{1,2}$.

Proposition 3.9. *The simply connected Lie group associated with the nilpotent Lie algebra $\mathfrak{n}_5^5$ admits the following representation:*

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof.

Step 1. We consider the subalgebra $\langle Y_1 = X_{1,2}, Y_2 = X_{1,3}, Y_3 = X_{2,3}, Y_4 = X_{1,4}, Y_5 = X_{2,4}\rangle$.

Step 2. We extend this basis with $Y_6 = X_{3,4}$.

Step 3. We solve the differential equation $\omega_6 = 0$, whose solution is $x_{3,4} = 0$.

Proposition 3.10. *The simply connected Lie group associated with the nilpotent Lie algebra $\mathfrak{n}_5^6$ admits the following representation:*

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof.

Step 1. We consider the subalgebra $\langle Y_1 = X_{1,3}, Y_2 = X_{1,2}, Y_3 = X_{2,4}, Y_4 = X_{1,4}, Y_5 = X_{3,4}\rangle$.

Step 2. We extend this basis with $Y_6 = X_{2,3}$.

Step 3. We solve the differential equation $\omega_6 = 0$, whose solution is $x_{2,3} = 0$.

Proposition 3.11. *The simply connected Lie group associated with the Lie algebra $\mathfrak{n}_5^8$ admits the matrix representation:*

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & x_{1,2} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof.

Step 1. We consider the subalgebra $\langle Y_1 = -(X_{1,2}+X_{3,4}), Y_2 = X_{1,3}+X_{2,4}, Y_3 = -2X_{1,4}, Y_4 = -X_{1,3} + X_{2,4}, Y_5 = X_{2,3}\rangle$.

Step 2. We extend this basis with $Y_6 = X_{1,2}$.

Step 3. We solve the differential equation $\omega_6 = 0$, whose solution is $x_{1,2} = x_{3,4}$.

It was already proved in Corollary 3.2 that simply connected Lie groups associated with the filiform Lie algebras $\mathfrak{n}_5^1$ and $\mathfrak{n}_5^2$ and with the nilpotent Lie algebra $\mathfrak{n}_5^3$ are not Lie subgroups of the Lie group G_4. So, we have now to study if simply connected Lie groups associated with the nilpotent Lie algebra $\mathfrak{n}_5^7$ and with the abelian algebra $\mathfrak{n}_5^9$ can be obtained as subgroups of G_4. To do it, we previously proved the following:

Lemma 3.12. *Every subalgebra of $\mathfrak{g}_4$ of dimension 5 has a basis $\{Y_i\}_{i=1}^5$ whose elements are a linear combination of two fields of the basis $\{X_{j,k}\}_{j,k}$ of $\mathfrak{g}_4$, in such a way that one of those two fields is common to all fields Y_i $(i = 1, \dots, 5)$.*

Proof. Each one of elements of an arbitrary set $\{Y_i\}_{i=1}^5$ of fields of $\mathfrak{g}_4$ can be expressed by:

$$Y_i = \sum_{\substack{k=j+1 \\ j=1}}^{\substack{j=3 \\ k=4}} a_{i,j,k} X_{j,k} \ , \quad (a_{i,j,k} \in \mathbb{C}) \ , \ (i = 1, \dots, 5).$$

Since the set $\{Y_i\}_{i=1}^5$ is linearly independent, the matrix of their coefficients must have rank 5, that is:

$$\operatorname{rank} \begin{pmatrix} a_{1,1,2} & a_{1,1,3} & a_{1,1,4} & a_{1,2,3} & a_{1,2,4} & a_{1,3,4} \\ a_{2,1,2} & a_{2,1,3} & a_{2,1,4} & a_{2,2,3} & a_{2,2,4} & a_{2,3,4} \\ a_{3,1,2} & a_{3,1,3} & a_{3,1,4} & a_{3,2,3} & a_{3,2,4} & a_{3,3,4} \\ a_{4,1,2} & a_{4,1,3} & a_{4,1,4} & a_{4,2,3} & a_{4,2,4} & a_{4,3,4} \\ a_{5,1,2} & a_{5,1,3} & a_{5,1,4} & a_{5,2,3} & a_{5,2,4} & a_{5,3,4} \end{pmatrix} = 5.$$

In consequence, this matrix is equivalent to the following:

$$\begin{pmatrix} b_{1,1} & 0 & 0 & 0 & 0 & b_{1,6} \\ 0 & b_{2,2} & 0 & 0 & 0 & b_{2,6} \\ 0 & 0 & b_{3,3} & 0 & 0 & b_{3,6} \\ 0 & 0 & 0 & b_{4,4} & 0 & b_{4,6} \\ 0 & 0 & 0 & 0 & b_{5,5} & b_{5,6} \end{pmatrix}$$

where $b_{i,i} \neq 0$, for $1 \leq i \leq 5$. It completes the proof.

We continue now our study by considering Lie algebras $\mathfrak{n}_5^7$ and $\mathfrak{n}_5^9$.

Proposition 3.13. *The simply connected Lie group associated with the algebra $\mathfrak{n}_5^7$ does not admit a representation as a Lie subgroup of G_4.*

Proof. Starting from Lemma 3.12, we consider the following six possibilities. We will completely detail the first and the last of them. In the rest of them a contradiction similar to the obtained in the first possibility appears:

1. $\langle X_{1,2} + \lambda_1 X_{3,4}, X_{1,3} + \lambda_2 X_{3,4}, X_{1,4} + \lambda_3 X_{3,4}, X_{2,3} + \lambda_4 X_{3,4}, X_{2,4} + \lambda_5 X_{3,4}\rangle$ $(\lambda_i \in \mathbb{C})$.
   We obtain the nonzero brackets $[X_{2,3}+\lambda_4 X_{3,4}, X_{1,2}+\lambda_1 X_{3,4}] = -X_{1,3}+\lambda_1 X_{2,4}$ and $[X_{1,2} + \lambda_1 X_{3,4}, X_{2,4} + \lambda_5 X_{3,4}] = X_{1,4}$. Therefore, the dimension of its derived algebra is greater or equal than 2, which is contradictory with the fact of the derived algebra of $\mathfrak{n}_5^7$ having dimension 1. $\rightarrow\leftarrow$
2. $\langle X_{1,2} + \lambda_1 X_{2,4}, X_{1,3} + \lambda_2 X_{2,4}, X_{1,4} + \lambda_3 X_{2,4}, X_{2,3} + \lambda_4 X_{2,4}, X_{3,4} + \lambda_5 X_{2,4}\rangle$ $(\lambda_i \in \mathbb{C})$.
3. $\langle X_{1,3} + \lambda_1 X_{1,2}, X_{1,4} + \lambda_2 X_{1,2}, X_{2,3} + \lambda_3 X_{1,2}, X_{2,4} + \lambda_4 X_{1,2}, X_{3,4} + \lambda_5 X_{1,2}\rangle$ $(\lambda_i \in \mathbb{C})$.
4. $\langle X_{1,2} + \lambda_1 X_{1,3}, X_{1,4} + \lambda_2 X_{1,3}, X_{2,3} + \lambda_3 X_{1,3}, X_{2,4} + \lambda_4 X_{1,3}, X_{3,4} + \lambda_5 X_{1,3}\rangle$ $(\lambda_i \in \mathbb{C})$.
5. $\langle X_{1,2} + \lambda_1 X_{1,4}, X_{1,3} + \lambda_2 X_{1,4}, X_{2,3} + \lambda_3 X_{1,4}, X_{2,4} + \lambda_4 X_{1,4}, X_{3,4} + \lambda_5 X_{1,4}\rangle$ $(\lambda_i \in \mathbb{C})$.
6. $\langle X_{1,2} + \lambda_1 X_{2,3}, X_{1,3} + \lambda_2 X_{2,3}, X_{1,4} + \lambda_3 X_{2,3}, X_{2,4} + \lambda_4 X_{2,3}, X_{3,4} + \lambda_5 X_{2,3}\rangle$ $(\lambda_i \in \mathbb{C})$.

The nonzero brackets are:

$$\begin{aligned}
&[X_{1,4}+\lambda_3X_{2,3}, X_{1,2}+\lambda_1X_{2,3}] = -\lambda_3X_{1,3},\\
&[X_{1,4}+\lambda_3X_{2,3}, X_{3,4}+\lambda_5X_{2,3}] = \lambda_3X_{2,4},\\
&[X_{1,3}+\lambda_2X_{2,3}, X_{1,2}+\lambda_1X_{2,3}] = -\lambda_2X_{1,3},\\
&[X_{1,3}+\lambda_2X_{2,3}, X_{3,4}+\lambda_5X_{2,3}] = X_{1,4}+\lambda_2X_{2,4},\\
&[X_{1,2}+\lambda_1X_{2,3}, X_{2,4}+\lambda_4X_{2,3}] = X_{1,4}+\lambda_4X_{1,3},\\
&[X_{1,2}+\lambda_1X_{2,3}, X_{3,4}+\lambda_5X_{2,3}] = \lambda_5X_{1,3}+\lambda_1X_{2,4},\\
&[X_{2,4}+\lambda_4X_{2,3}, X_{3,4}+\lambda_5X_{2,3}] = \lambda_4X_{2,4}.
\end{aligned}$$

Then, we distinguish:

6.1 $\lambda_4 \neq 0$. Then, $X_{2,4}$ belongs to the subalgebra and, hence, $\lambda_4 = 0$. $\rightarrow\leftarrow$

6.2 $\lambda_4 = 0$. Then, $X_{1,4}$ belongs to the subalgebra and, hence, $\lambda_3 = 0$. We distinguish again:

6.2.1 $\lambda_2 \neq 0$. Then, $X_{1,3}$ belongs to the subalgebra and, hence, $\lambda_2 = 0$. $\rightarrow\leftarrow$

6.2.2 $\lambda_2 = 0$. In this subcase, the unique nonzero brackets are:

$$\begin{aligned}
&[X_{1,3}+\lambda_2X_{2,3}, X_{3,4}+\lambda_5X_{2,3}] = X_{1,4},\\
&[X_{1,2}+\lambda_1X_{2,3}, X_{2,4}+\lambda_4X_{2,3}] = X_{1,4},\\
&[X_{1,2}+\lambda_1X_{2,3}, X_{3,4}+\lambda_5X_{2,3}] = \lambda_5X_{1,3}+\lambda_1X_{2,4}.
\end{aligned}$$

Then, so that the dimension of the derived algebra could be 1, it would be necessary that $\lambda_1 = \lambda_5 = 0$. Then, the algebra is $< X_{1,2}, X_{1,3}, X_{1,4}, X_{2,4}, X_{3,4} >$, which coincides with the algebra $\mathfrak{n}_5^6$, and it is not the algebra $\mathfrak{n}_5^7$, as we asked. $\rightarrow\leftarrow$

The last simply connected Lie group which could be a subgroup of G_4 would be the associated with the abelian Lie algebra $\mathfrak{n}_5^9$. It is studied in the following:

Proposition 3.14. *The simply connected Lie group associated with the abelian Lie algebra $\mathfrak{n}_5^9$ does not admit a representation as a subgroup of the Lie group G_4.*

Proof. According to Lemma 3.12, it is sufficient to consider the following six possibilities:

1. Let us consider subalgebras having a basis:
$$Y_1 = \lambda_1 X_{1,2} + \mu_1 X_{3,4}, \quad Y_2 = \lambda_2 X_{1,3} + \mu_2 X_{3,4}, \quad Y_3 = \lambda_3 X_{1,4} + \mu_3 X_{3,4},$$
$$Y_4 = \lambda_4 X_{2,3} + \mu_4 X_{3,4}, \quad Y_5 = \lambda_5 X_{2,4} + \mu_5 X_{3,4}.$$
By vanishing the ten possible brackets between basic elements, we obtain an equation system which contains, among others, the equations:
$$\lambda_4\mu_3 = 0, \quad \lambda_4\mu_2 = 0, \quad \lambda_2\mu_4 = 0, \quad \lambda_1\lambda_4 = 0, \quad \lambda_4\mu_1 = 0.$$
It is easy to see then that the system is incompatible, due to $\{Y_i\}_{i=1}^5$ is linearly independent.
2. Let us consider subalgebras having a basis:
$$Y_1 = \lambda_1 X_{1,2} + \mu_1 X_{2,4}, \quad Y_2 = \lambda_2 X_{1,3} + \mu_2 X_{2,4}, \quad Y_3 = \lambda_3 X_{1,4} + \mu_3 X_{2,4},$$
$$Y_4 = \lambda_4 X_{2,3} + \mu_4 X_{2,4}, \quad Y_5 = \lambda_5 X_{3,4} + \mu_5 X_{2,4}.$$
By proceeding as in the first case, we obtain the incompatible equations $\lambda_1\lambda_4 = 0$ and $\lambda_2\lambda_5 = 0$.
3. Let us consider subalgebras having a basis:
$$Y_1 = \lambda_1 X_{1,3} + \mu_1 X_{1,2}, \quad Y_2 = \lambda_2 X_{1,4} + \mu_2 X_{1,2}, \quad Y_3 = \lambda_3 X_{2,3} + \mu_3 X_{1,2},$$
$$Y_4 = \lambda_4 X_{2,4} + \mu_4 X_{1,2}, \quad Y_5 = \lambda_5 X_{3,4} + \mu_5 X_{1,2}.$$
By proceeding as in the first case, we obtain the incompatible equations system $\{\lambda_3\mu_2 = 0,\ \lambda_3\mu_1 = 0,\ \lambda_3\mu_4 = 0,\ \mu_3\lambda_4 = 0,\ \lambda_3\lambda_5 = 0,\ \lambda_3\mu_5 = 0\}$.
4. Let us consider subalgebras having a basis:
$$Y_1 = \lambda_1 X_{1,2} + \mu_1 X_{1,3}, \quad Y_2 = \lambda_2 X_{1,4} + \mu_2 X_{1,3}, \quad Y_3 = \lambda_3 X_{2,3} + \mu_3 X_{1,3},$$
$$Y_4 = \lambda_4 X_{2,4} + \mu_4 X_{1,3}, \quad Y_5 = \lambda_5 X_{3,4} + \mu_5 X_{1,3}.$$
By proceeding as in the first case, we obtain the following incompatible equations $\lambda_3\lambda_5 = 0$ and $\lambda_1\lambda_4 = 0$.
5. Let us consider subalgebras having a basis:
$$Y_1 = \lambda_1 X_{1,2} + \mu_1 X_{1,4}, \quad Y_2 = \lambda_2 X_{1,3} + \mu_2 X_{1,4}, \quad Y_3 = \lambda_3 X_{2,3} + \mu_3 X_{1,4},$$
$$Y_4 = \lambda_4 X_{2,4} + \mu_4 X_{1,4}, \quad Y_5 = \lambda_5 X_{3,4} + \mu_5 X_{1,4}.$$
By proceeding as in the first case, we obtain the following incompatible equations $\lambda_1\lambda_3 = 0$ and $\lambda_2\lambda_5 = 0$.

6. Let us consider subalgebras having a basis:

$$Y_1 = \lambda_1 X_{1,2} + \mu_1 X_{2,3}, \quad Y_2 = \lambda_2 X_{1,3} + \mu_2 X_{2,3}, \quad Y_3 = \lambda_3 X_{1,4} + \mu_3 X_{2,3},$$
$$Y_4 = \lambda_4 X_{2,4} + \mu_4 X_{2,3}, \quad Y_5 = \lambda_5 X_{3,4} + \mu_5 X_{2,3}.$$

By proceeding as in the first case, we obtain the following incompatible equations $\lambda_2\lambda_5 = 0$ and $\lambda_1\lambda_4 = 0$.

Then, as these six cases cover all the possible Lie subalgebras, it can be settled that the abelian Lie algebra $\mathfrak{n}_5^9$ is not a Lie subalgebra of the Lie algebra $\mathfrak{g}_4$. Consequently, the simply connected Lie group associated with $\mathfrak{n}_5^9$ does not admit a representation as a Lie subgroup of the Lie group G_4.

In this way, we have obtained all nilpotent Lie algebras which are Lie subalgebras of $\mathfrak{g}_4$. Let us remark that the subalgebras of $\mathfrak{g}_4$ have to be nilpotent because $\mathfrak{g}_4$ is nilpotent.

Moreover, according to Proposition 1.1, all the simply connected Lie subgroups of G_4 are those which are associated with the subalgebras obtained. Hence, all the simply connected Lie subgroups of G_4 have been so obtained.

We conclude the paper by systematizing previous results in the following:

Theorem 3.15.(Main Theorem) *Up to isomorphism, the simply connected Lie subgroups of the Lie group* G_4 (*each of them with their associated Lie algebra* (*which is a subalgebra of* $\mathfrak{g}_4$)) *are the following:*

Dimension 1, 2 and 3:

$$\mathfrak{n}_1^1 \quad \longrightarrow \quad \begin{pmatrix} 1 & x_{1,2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathfrak{n}_2^1 \quad \longrightarrow \quad \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathfrak{n}_3^1 \quad \longrightarrow \quad \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & 0 \\ 0 & 1 & x_{2,3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathfrak{n}_3^2 \quad \longrightarrow \quad \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Dimension 4:

$$\mathfrak{n}_4^1 \quad \longrightarrow \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & \frac{1}{2}x_{2,3}^2 \\ 0 & 0 & 1 & x_{2,3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathfrak{n}_4^2 \quad \longrightarrow \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathfrak{n}_4^3 \quad \longrightarrow \quad \begin{pmatrix} 1 & 0 & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Dimension 5:

$$\mathfrak{n}_5^4 \quad \longrightarrow \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{1,2} & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathfrak{n}_5^5 \quad \longrightarrow \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathfrak{n}_5^6 \quad \longrightarrow \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & x_{2,4} \\ 0 & 0 & 1 & x_{3,4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathfrak{n}_5^8 \quad \longrightarrow \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & x_{1,2} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark 3.16. The preceding result does not exclude that non-conjugated (and conjugated) subgroups have the same underlying Lie algebra.

Indeed, let us consider the following two subgroups of G_4, defined, respectively, by the matrices:

$$G_4^1 \equiv \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad G_4^2 \equiv \begin{pmatrix} 1 & 0 & x_{1,3} & 0 \\ 0 & 1 & x_{2,3} & x_{2,4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Their respective Lie algebras are $\mathfrak{g}_4^1$ and $\mathfrak{g}_4^2$, determined by:

$$\mathfrak{g}_4^1 = \langle X_{1,2}, X_{1,3}, X_{1,4} \rangle \qquad \text{y} \qquad \mathfrak{g}_4^2 = \langle X_{1,3}, X_{2,3}, X_{2,4} \rangle.$$

Note then that both subalgebras $\mathfrak{g}_4^1$ and $\mathfrak{g}_4^2$ of $\mathfrak{g}_4$ are isomorphic to the 3-dimensional abelian Lie algebra. Consequently, the simply connected Lie subgroups G_4^1 and G_4^2 associated with them are isomorphic, in virtue of Lie's Third Theorem, and thus they are representatives of the same isomorphism class of subgroups in the corresponding classification.

However, these two subgroups are non-conjugated. It is due to that $\mathfrak{g}_4^1$ is an ideal of $\mathfrak{g}_4$ and $\mathfrak{g}_4^2$ is a subalgebra but it is not an ideal. Hence, G_4^1 is a normal Lie subgroup of G_4 and G_4^2 is not. So the classification problem with respect to conjugacy classes need to determine all the non-conjugated classes for every subalgebra of $\mathfrak{g}_4$. Therefore, the classification of conjugacy classes would constitute a second step, after having identified which are the Lie algebras involved.

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