ON THE NAVIER BOUNDARY CONDITION FOR VISCOUS FLUIDS IN ROUGH DOMAINS

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Abstract

In this paper we review some recent results concerning the study of the asymptotic behavior of viscous fluids in rough domains assuming Navier boundary conditions on the rough boundary. Our main interest is to study the relation between both the adherence and the Navier boundary conditions in the case of a boundary with weak rugosities. We show that the roughness acts on the fluid as a friction term. In particular, if the roughness is sufficiently strong, Navier condition implies adherence condition. This generalizes previous results of other authors.

1 Introduction

A relevant problem in fluid mechanics is the appropriate choice of the boundary conditions. For a viscous fluid in an open set $\Omega \subset \mathbb{R}^3$, a well accepted hypothesis is that if the boundary is impermeable, then the fluid adheres completely to it. Denoting by u the velocity of the fluid in Ω, this adherence condition becomes

$$u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

However, some other boundary conditions are often used. In this sense, for a viscous fluid governed by the Stokes or Navier-Stokes system (with viscosity coefficient equals one), Navier proposed the slip-friction boundary condition, see [22]:

$$u \cdot \nu = 0, \quad T\left(\frac{\partial u}{\partial \nu} - p\nu + \gamma u\right) = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where p is the pressure, ν is the unitary outside normal vector to Ω on $\partial\Omega$, T is the orthogonal projection on the tangent space to $\partial\Omega$ and γ is a nonnegative constant. In (2) we are assuming that the boundary is impermeable (so the normal component

of the velocity, $u \cdot \nu$, vanishes on $\partial\Omega$), and that it exerts to the fluid a friction force opposite and proportional to the velocity. Writing the equilibrium forces equation on the boundary, but only in the tangential components, and denoting by γ the friction coefficient, this gives the second equation in (2).

Taking into account that $p\nu$ is orthogonal to the tangent space to $\partial\Omega$, the Navier boundary condition can be also written as

$$u \cdot \nu = 0, \quad T\left(\frac{\partial u}{\partial \nu} + \gamma u\right) = 0 \quad \text{on } \partial\Omega. \tag{3}$$

Due to the freedom of choice of boundary conditions, a natural question is if there is any relationship between conditions (1) and (3). In this sense, it was considered in [10] a three-dimensional domain Ω_ε with a rough boundary described by the equation (see Figure 1)

$$x_3 = -\varepsilon\Psi\left(\frac{x'}{\varepsilon}\right), \quad \forall x' \in \omega, \tag{4}$$

(along this paper a point $x \in \mathbb{R}^3$ is decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$) with ω a bounded open set of $\mathbb{R}^2$ and Ψ a smooth periodic function such that

$$\text{Span}\left(\{\nabla\Psi(z') : z' \in \mathbb{R}^2\}\right) = \mathbb{R}^2. \tag{5}$$

It was proved that if u_ε is bounded in energy and satisfies $u_\varepsilon \cdot \nu = 0$ on the boundary described by (4), then the weak limit u of u_ε vanishes on $\omega \times \{0\}$. So, in this case, the Navier and adherence conditions are asymptotically equivalent. This means that the adherence condition, which is experimentally observed, may be due to the existence of microrugosities.

Generalizations of this result have been obtained in [3] for a non-periodic boundary described by

$$x_3 = \Phi_\varepsilon(x'), \quad \forall x' \in \omega,$$

where Φ_ε converges weakly-$*$ to zero in $W^{1,\infty}(\omega)$ and it is such that the support of the Young's measure associated to $\nabla\Phi_\varepsilon$ contains two linearly independent vectors. Remark that this last condition implies that $\nabla\Phi_\varepsilon$ does not converge to zero in $L^1(\omega)^2$.

Our main goal in the present paper is to study the relation between the Navier and the adherence boundary conditions in the case of weak rugosities. The article, which is a review of the results which appear in the Ph.D. Thesis of F.J. Suárez-Grau (see [24]), is organized as follows

In Section 2, we study the asymptotic behavior of viscous fluids in the open set Ω_ε described by (see Figure 2)

$$\Omega_\varepsilon = \left\{x = (x', x_3) \in \omega \times \mathbb{R} : -\delta_\varepsilon\Psi\left(\frac{x'}{\varepsilon}\right) < x_3 < 1\right\}, \tag{6}$$

where $\omega \subset \mathbb{R}^2$ is a Lipschitz bounded open set, $\Psi \in W_{loc}^{2,\infty}(\mathbb{R}^2)$ is periodic of period $Z' = (0,1)^2$, and $\delta_\varepsilon > 0$ satisfies $\lim_{\varepsilon \to 0} \dfrac{\delta_\varepsilon}{\varepsilon} = 0$.

We impose the Navier condition on the oscillating boundary Γ_ε of period ε and amplitude δ_ε (with $\delta_\varepsilon \ll \varepsilon$) given by (see Figure 2)

$$\Gamma_\varepsilon = \left\{ x = (x', x_3) \in \omega \times \mathbb{R} : x_3 = -\delta_\varepsilon \Psi \left(\frac{x'}{\varepsilon} \right) \right\}, \tag{7}$$

and, to simplify, the adherence condition on the rest of the boundary $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$.

Remark that in our case $\Phi_\varepsilon = \delta_\varepsilon \Psi(\frac{x'}{\varepsilon})$ converges strongly to zero in $W^{1,\infty}(\omega)$ and therefore the results in [3] do not apply.

Denoting by

$$\lambda = \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}} \in [0, +\infty], \tag{8}$$

(the limit exists at least for a subsequence) we show

• If $\lambda = +\infty$ and (5) holds, then the Navier and adherence boundary conditions are asymptotically equivalent. This extends the result obtained in [10] for $\delta_\varepsilon = \varepsilon$ to the case when $\delta_\varepsilon/\varepsilon$ tends to zero and $\delta_\varepsilon/\varepsilon^{\frac{3}{2}}$ tends to infinity.
• If $\lambda = 0$, the roughness is so small that it has no effect on the limit problem.
• If $\lambda \in (0, +\infty)$, the roughness is not strong enough to obtain the adherence condition, but it is large enough to make appear a new friction term. Namely, we obtain the following Navier boundary condition in the limit,

$$u_3 = 0, \quad -\partial_3 u' + \gamma u' + \lambda^2 R u' = 0, \quad \text{on } \omega \times \{0\},$$

where $R \in \mathbb{R}^{2 \times 2}$ is a symmetric and nonnegative matrix. The new term $\lambda^2 R$ is similar to the *strange term* obtained by D. Cioranescu and F. Murat in [17] for the homogenization of Dirichlet problems in perforated domains. This case can be considered as the general case, because it provides the other two ones tending λ to infinity or zero.

Related to this result, it has been studied in [6] the asymptotic behavior of viscous fluids confined in general rough domains, not necessarily periodic. In the particular case of a domain with a rough bottom described by

$$x_3 = \Psi_\varepsilon(x'), \quad \forall x' \in \omega,$$

with Ψ_ε converging weakly-$*$ to zero in $W^{1,\infty}(\omega)$, the results in [6] imply that the limit boundary condition is

$$u_3 = 0, \quad -\partial_3 u' + \gamma u' + H u' \mu = 0, \quad \text{on } \omega \times \{0\},$$

where μ is a nonnegative Borel measure, which can be infinity in compact sets of ω, and H is a μ-measurable matrix evaluated function. Our results provide an example where the extra term $H u' \mu$ is not zero. Another example of different nature for a ribbed boundary described by $x_3 = \varepsilon \Psi(\frac{x_1}{\varepsilon})$ is given in [4] and [5].

In Section 3 we consider the case of a thin domain $\Omega_\varepsilon^{thin}$ of small height h_ε tending to zero described by

$$\Omega_\varepsilon^{thin} = \left\{ x = (x', x_3) \in \omega \times \mathbb{R} : -\delta_\varepsilon \Psi\left(\frac{x'}{\varepsilon}\right) < x_3 < h_\varepsilon \right\}, \qquad (9)$$

with ω and Ψ as above, and the parameters h_ε and δ_ε satisfying

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{h_\varepsilon} = 0, \quad \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon} = 0. \qquad (10)$$

We obtain a Reynolds system in the limit which shows that near the rough bottom Γ_ε the behavior of the fluid is similar to the one obtained in Section 2 for fluids confined in domains of height one but with λ replaced by

$$\lambda_{thin} = \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}} h_\varepsilon^{\frac{1}{2}}. \qquad (11)$$

Remark that $\lambda = \lambda_{thin}$ if $h_\varepsilon = 1$.

The results obtained in Sections 2 and 3 show that the Navier boundary condition, which can be also written as

$$u_\varepsilon(x) \in T_\varepsilon(x), \quad \frac{\partial u_\varepsilon}{\partial \nu}(x) + \gamma u_\varepsilon(x) \in T_\varepsilon(x)^\perp, \quad \text{on } \Gamma_\varepsilon, \qquad (12)$$

with $T_\varepsilon(x)$ the tangent space in the point $x \in \Gamma_\varepsilon$, provides a new term in the limit equation. In Section 4 we study this phenomena in a more general setting. Instead of the Stokes or Navier-Stokes system, we consider a sequence of linear elliptic systems of M equations posed in varying open sets $\Omega_\varepsilon \subset \mathbb{R}^N$, not necessarily periodic, with a boundary condition similar to (12), where $T_\varepsilon(x)$ is replaced by an arbitrary linear space $V_\varepsilon(x) \subset \mathbb{R}^M$. This abstract formulation contains a lot of classical boundary conditions. For instance it allows us to study the asymptotic behavior of linear elliptic systems in rough domains Ω_ε where we impose Dirichlet and Neumann boundary conditions on varying subsets of $\partial\Omega_\varepsilon$. This problem has been studied in [7] and [8] for $\Omega_\varepsilon = \Omega$ fixed.

The results of Section 4 could be extended to viscous fluids. For the particular choice $V_\varepsilon(x) = T_\varepsilon(x)$, it would recover the results in [6].

To finish this introduction, we refer some open problems in which we are starting working and we hope provide results in the near future:

• Extension to non-Newtonian viscous fluids, which are involved in Biology.

• Behavior of fluids in thin rough domains described by (9), assuming different behaviors for the parameters from the imposed in (10).

• Problems with free boundaries and applications to lubrication and Oceanography.

2 Asymptotic behavior of viscous fluids in rough domains with fixed height

Given a Lipschitz bounded connected open set $\omega \subset \mathbb{R}^2$ and a function $\Psi \in W_{loc}^{2,\infty}(\mathbb{R}^2)$, periodic of period $Z' = (0,1)^2$, we define the domain Ω_ε by (6) and the rough portion of the boundary Γ_ε by (7). We also define $\Omega = \omega \times (0,1)$ and $\Gamma = \omega \times \{0\}$.

In Ω_ε, we consider the solution $(u_\varepsilon, p_\varepsilon)$ of the following Stokes system satisfying the Navier condition on the rough boundary Γ_ε and the adherence condition on the rest of the boundary $\partial\Omega_\varepsilon \setminus \Gamma_\varepsilon$,

$$
\begin{cases}
-\Delta u_\varepsilon + \nabla p_\varepsilon = f \text{ in } \Omega_\varepsilon, \quad \text{div } u_\varepsilon = 0 \text{ in } \Omega_\varepsilon \\[2mm]
u_\varepsilon \cdot \nu = 0 \text{ on } \Gamma_\varepsilon, \quad T_\varepsilon\left(\dfrac{\partial u_\varepsilon}{\partial \nu} + \gamma u_\varepsilon\right) = 0 \text{ on } \Gamma_\varepsilon \\[2mm]
u_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon,
\end{cases}
\tag{13}
$$

where $\gamma \geq 0$ is a friction coefficient, ν denotes the unitary outside normal vector to Ω_ε on Γ_ε, T_ε is the orthogonal projection on the tangent space to Γ_ε, and the second member f is in $L^2(\omega \times \mathbb{R})^3$ (more general second members can be considered).

The system (13) has a unique solution $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L_0^2(\Omega_\varepsilon)$ ($L_0^2(\Omega_\varepsilon)$ denotes the space of functions in $L^2(\Omega_\varepsilon)$ whose integral in Ω_ε is zero). Moreover, we prove that there exists $C > 0$ such that

$$
\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)^3} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C, \quad \forall \varepsilon > 0.
$$

Our problem is to describe the asymptotic behavior of the sequences u_ε and p_ε. This is given by the following theorem which is the main result of this section.

Theorem 1 *The solution $(u_\varepsilon, p_\varepsilon)$ of (13) satisfies*

$$
u_\varepsilon \rightharpoonup u \text{ in } H^1(\Omega)^3, \qquad p_\varepsilon \rightharpoonup p \text{ in } L^2(\Omega),
$$

with (u, p) the unique solution of

$$
\begin{cases}
-\Delta u + \nabla p = f \text{ in } \Omega, \quad \text{div u} = 0 \text{ in } \Omega \\[2mm]
u_3 = 0 \text{ on } \Gamma, \quad u = 0 \text{ on } \partial\Omega \setminus \Gamma,
\end{cases}
\tag{14}
$$

plus a boundary condition for u' which depends on the parameter λ defined by (8). More precisely we have

i) If $\lambda = 0$, then

$$
-\partial_3 u' + \gamma u' = 0 \quad \text{on } \Gamma.
\tag{15}
$$

ii) If $\lambda \in (0, +\infty)$, then defining $(\widehat{\phi}^i, \widehat{q}^i)$, $i = 1, 2$, as the solution of

$$
\begin{cases}
-\Delta_z \widehat{\phi}^i + \nabla_z \widehat{q}^i = 0, \quad \text{div}_z \widehat{\phi}^i = 0 \text{ in } \mathbb{R}^2 \times (0, +\infty) \\[2mm]
\widehat{\phi}_3^i(z', 0) + \partial_{z_i} \Psi(z') = 0, \ \partial_{z_3}(\widehat{\phi}^i)'(z', 0) = 0, \quad a.e. \ z' \in \mathbb{R}^2 \\[2mm]
\widehat{\phi}^i(., z_3), \ \widehat{q}^i(., z_3) \text{ periodic of period } Z', \quad a.e. \ z_3 \in (0, +\infty) \\[2mm]
\widehat{\phi}^i \in H^1(Z' \times (0, +\infty))^3, \ \widehat{q}^i \in L^2(Z' \times (0, +\infty)),
\end{cases}
\tag{16}
$$

and $R \in \mathbb{R}^{2 \times 2}$ by

$$R_{ij} = \int_{Z' \times (0,+\infty)} D_z \widehat{\phi}^i : D_z \widehat{\phi}^j \, dz, \quad \forall i, j \in \{1, 2\}, \qquad (17)$$

we have

$$-\partial_3 u' + \gamma u' + \lambda^2 R u' = 0 \ \ on \ \Gamma. \qquad (18)$$

iii) If $\lambda = +\infty$, then defining

$$W = Span(\{\nabla \Psi(z') : \ z' \in Z'\}), \qquad (19)$$

we have

$$u' \in W^{\perp} \ \ on \ \Gamma, \quad -\partial_3 u' + \gamma u' \in W, \ \ on \ \Gamma \qquad (20)$$

Remark 1 *For $\lambda = 0$, Theorem 1 shows that the roughness of Γ_ε is very slight and so the solution $(u_\varepsilon, p_\varepsilon)$ of (13) behaves as if Γ_ε coincides with the plane boundary Γ. For $\lambda \in (0, +\infty)$ (critical size), the boundary condition satisfied by the limit u of u_ε on the tangent space to Γ contains the new term $\lambda^2 R u'$. In this case, the effect of the roughness of Γ_ε is not worthless and it makes to appear this new term in the limit. Finally, for $\lambda = +\infty$ the roughness of Γ_ε is so strong that the limit u of u_ε does not only satisfies the condition $u_3 = 0$ on Γ, but also its tangent velocity on Γ, u', is orthogonal to the vectors $\nabla \Psi(z')$, with $z' \in Z'$. In particular, if the space W defined by (19) has dimension 2, then u satisfies the adherence condition $u = 0$ on Γ. This extends to the case where*

$$\lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon} = 0, \quad \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}} = +\infty,$$

the results obtained in [10] for $\delta_\varepsilon = \varepsilon$.

Remark 2 *The case $\lambda \in (0, +\infty)$ can be considered as the general one. In fact, if λ tends to zero or infinity in (18) we get (15) or (20) respectively.*

Sketch of the proof of Theorem 1 Since $(u_\varepsilon, p_\varepsilon)$ is bounded in $H^1(\Omega_\varepsilon)^3 \times L^2(\Omega_\varepsilon)$ it is clear that (at least for a subsequence) it converges weakly in $H^1(\Omega)^3 \times L^2(\Omega)$ to some (u, p) which is a solution of the Stokes system (14). The difficulty is then to obtain the boundary condition in the tangent space to Γ. For this purpose we need to study more carefully the behavior of u_ε near Γ_ε. This is carried out using an original adaptation of the unfolding method, [2], [9], [16], which is very related to the two-scale convergence method, [1], [21], [23]. The idea is to introduce suitable changes of variables which transform every periodic cell into a simpler reference set by using a supplementary variable (microscopic variable). In our case, given $(u_\varepsilon, p_\varepsilon)$ solution of (13), and defining

$$\hat{\omega}_\varepsilon = \bigcup_{\substack{k' \in \mathbb{Z}^2 \\ \varepsilon k' + \varepsilon Z' \subset \omega}} (\varepsilon k' + \varepsilon Z'), \quad K_\varepsilon = \left\{ z \in Z' \times \mathbb{R} : \ -\frac{\delta_\varepsilon}{\varepsilon} \Psi(z') < z_3 < \frac{1}{\varepsilon} \right\},$$

we introduce $\hat{u}_\varepsilon : \hat{\omega}_\varepsilon \times K_\varepsilon \to \mathbb{R}^3$, $\hat{p}_\varepsilon : \hat{\omega}_\varepsilon \times K_\varepsilon \to \mathbb{R}$ by

$$\hat{u}_\varepsilon(x', z) = \sum_{\substack{k' \in \mathbb{Z}^2 \\ \varepsilon k' + \varepsilon Z' \subset \omega}} u_\varepsilon(\varepsilon k' + \varepsilon z', \varepsilon z_3) \chi_{\varepsilon k' + \varepsilon Z'}(x').$$

Observe that $\hat{u}_\varepsilon$ is obtained transforming every column $\Omega_\varepsilon \cap \left((\varepsilon k' + \varepsilon Z') \times \mathbb{R} \right)$ in the set K_ε by using the change of variables $z = ((x' - \varepsilon k')/\varepsilon, x_3/\varepsilon)$. Here, x' is the macroscopic variable and z the microscopic one. Moreover, the set K_ε converges to the set $Z' \times (0, +\infty)$, while $(\varepsilon k' + \varepsilon Z') \times \mathbb{R}$ converges to the empty set. The asymptotic behavior of u_ε near Γ_ε is obtained by studying the asymptotic behavior of $\hat{u}_\varepsilon$. Namely, from u_ε bounded in $H^1(\Omega_\varepsilon)^3$, we deduce

$$\frac{1}{\varepsilon} \int_{\hat{\omega}_\varepsilon \times K_\varepsilon} |D_z \hat{u}_\varepsilon|^2 dx' dz \leq C$$

and thus (for a subsequence)

$$\hat{w}_\varepsilon(x', z) = \frac{1}{\sqrt{\varepsilon}} \left(\hat{u}_\varepsilon(x', z) - \int_{Z'} \hat{u}_\varepsilon(x', \rho', 0) \, d\rho' \right) \rightharpoonup \hat{w} \qquad (21)$$

in $L^2(\omega; H^1(Z' \times (0, M))^3)$, for every $M > 0$, with $\hat{w} \in L^2(\omega; H^1(Z' \times (0, M))^3)$, $D_z \hat{w} \in L^2(\omega; L^2(Z' \times (0, +\infty))^{3 \times 3})$. Moreover, we can prove that $\hat{u}$ is periodic with respect to z'.

Observing that

$$\int_{Z'} \hat{u}_\varepsilon(x', \rho', 0) \, d\rho' = \frac{1}{\varepsilon^2} \int_{\varepsilon k' + \varepsilon Z'} u_\varepsilon(y', 0) \, dy', \quad x' \in \varepsilon k' + \varepsilon Z',$$

we also deduce from (21) that

$$\hat{u}_\varepsilon(x', z) \rightharpoonup u \ \text{ in } L^2(\omega; H^1(Z' \times (0, M))^3), \quad \forall M > 0. \qquad (22)$$

On the other hand, the condition $u_\varepsilon \cdot \nu = 0$ on Γ_ε allows us to show

$$\frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}} \nabla \Psi(z') \cdot \hat{u}'_\varepsilon(x', z', 0) + \hat{w}_{\varepsilon,3}(x', z', 0) \to 0 \ \text{ in } L^2(\omega \times Z'). \qquad (23)$$

In order to obtain the boundary condition for the limit system in the tangent space to Γ the reasoning depends on the limit λ of $\delta_\varepsilon / \varepsilon^{\frac{3}{2}}$.

• If $\lambda = +\infty$, then (23) and (22) prove that u' belongs to $W^\perp$ a.e. on Γ, which gives the first assertion in (20). The second one is obtained using test functions v in (13) which a.e. in Γ satisfy $v_3 = 0$, $v' \in W^\perp$.

• If $\lambda \in (0, +\infty)$, passing to the limit in (23) we deduce

$$\lambda \nabla \Psi(z') \cdot u'(x', 0) + \hat{w}_3(x', z', 0) = 0.$$

On the other hand, (21) and $\hat{u}$ periodic in z' suggest that $u_\varepsilon(x)$ behaves as $u(x) + \sqrt{\varepsilon} \hat{w}(x', \frac{x}{\varepsilon})$. The proof of (18) is obtained using test functions in (13) of the form $v_\varepsilon(x) = v(x) + \sqrt{\varepsilon} \hat{v}(x', x/\varepsilon)$ (slightly modified to have $v_\varepsilon \cdot \nu = 0$ on Γ_ε) with $v' = 0$

on $\partial\Omega \setminus \Gamma$, $v_3 = 0$ on $\partial\Omega$, $\hat{v}$ periodic in z', $\hat{v}(x', z) = 0$ if z_3 is large, and satisfying $\hat{v}_3(x', z', 0) = -\lambda v'(x', 0)$. In particular this proves the equality

$$\hat{w}(x', z) = u_1(x', 0)\widehat{\Phi}^1(z) + u_2(x', 0)\widehat{\Phi}^2(z), \quad \text{a.e. in } \omega \times Z' \times (0, +\infty).$$

• If $\lambda = 0$, we consider v as above, such that $v' = 0$ on $\partial\Omega \setminus \Gamma$, $v_3 = 0$ on $\partial\Omega$ and then, we use as test function

$$v_\varepsilon(x) = \left(v'(x), v_3(x) - \frac{\delta_\varepsilon}{\varepsilon}\zeta(\frac{x_3}{\varepsilon})\nabla\Psi(\frac{x'}{\varepsilon}) \cdot v'(x') \right),$$

with ζ a smooth function such that $\zeta(t) = 0$ in $(-\infty, 0)$, $\zeta(t) = 1$ in $(1, +\infty)$. $\square$

Theorem 1 gives an approximation of $(u_\varepsilon, p_\varepsilon)$ in the weak topology of $H^1(\Omega)^3 \times L^2(\Omega)$. Indeed, we have the following result relative to the strong convergence of $(Du_\varepsilon\chi_{\Omega_\varepsilon}, p_\varepsilon\chi_{\Omega_\varepsilon})$ in $H^1(\Omega)^3 \times L^2(\Omega)$ (corrector result).

Theorem 2 *Under the assumptions of Theorem 1 we have*

i) If $\lambda = 0$ or $+\infty$, then

$$\lim_{\varepsilon \to 0} \left(\|u_\varepsilon\|_{H^1(\Omega_\varepsilon \setminus \Omega)^3} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus \Omega)} + \|u_\varepsilon - u\|_{H^1(\Omega)^3} + \|p_\varepsilon - p\|_{L^2(\Omega)} \right) = 0.$$

ii) If $\lambda \in (0, +\infty)$, then, taking $(\widehat{\phi}^i, \widehat{q}^i)$, $i = 1, 2$, as the solution of (16), and defining u^b and p^b by

$$u^b(x, z) = u_1(x)\widehat{\phi}^1(z) + u_2(x)\widehat{\phi}^2(z),$$

$$p^b(x, z) = u_1(x)\widehat{q}^1(z) + u_2(x)\widehat{q}^2(z),$$

we have

$$\lim_{\varepsilon \to 0} \left(\|u_\varepsilon\|_{H^1(\Omega_\varepsilon \setminus \Omega)^3} + \|u_\varepsilon - u\|_{L^2(\Omega)} + \|Du_\varepsilon - Du - \frac{\lambda}{\sqrt{\varepsilon}}D_z u^b(x, \frac{x}{\varepsilon})\|_{L^2(\Omega)^{3\times3}} \right) = 0,$$

$$\lim_{\varepsilon \to 0} \left(\|p_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus \Omega)} + \|p_\varepsilon - p - \frac{\lambda}{\sqrt{\varepsilon}}p^b(x, \frac{x}{\varepsilon})\|_{L^2(\Omega)} \right) = 0.$$

This corrector result can be improved obtaining an estimate for the difference of $(u_\varepsilon, p_\varepsilon)$ and its corrector. We focus in the case $\lambda \in (0, +\infty)$ which, as we said in Remark 2, can be considered as the general one. Assuming $\delta_\varepsilon = \lambda\varepsilon^{\frac{3}{2}}$, with $\lambda \in (0, +\infty)$, we prove the following theorem

Theorem 3 *If the function u defined by (14) and (18) belongs to $H^s(\Omega)^3$, with $s > 3/2$, then we have*

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon \setminus \Omega)^3} + \|u_\varepsilon - u - \lambda\sqrt{\varepsilon}u^b(x, \frac{x}{\varepsilon})\|_{H^1(\Omega)^3} \le C\sqrt{\varepsilon},$$

$$\|p_\varepsilon\|_{L^2(\Omega_\varepsilon \setminus \Omega)} + \|p_\varepsilon - p - \frac{\lambda}{\sqrt{\varepsilon}}p^b(x, \frac{x}{\varepsilon})\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}.$$

Sketch of the proof. The proof consists in showing that the pair $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon) = (u + \lambda\sqrt{\varepsilon}u^b(x, \frac{x}{\varepsilon}), p + \frac{\lambda}{\sqrt{\varepsilon}}p^b(x, \frac{x}{\varepsilon}))$ satisfies a Stokes system with right-hand side and boundary conditions close to the ones satisfied by $(u_\varepsilon, p_\varepsilon)$. Then, usual estimates for the Stokes problem applied to the difference of the equations satisfied by $(u_\varepsilon, p_\varepsilon)$ and $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ give the result. $\square$

In order to apply Theorem 3, we need the solution u of (14) and (18) in $H^s(\Omega)^3$, $s \geq 3/2$. A result in this sense is given by the following proposition.

Proposition 4 *The solution (u,p) of (14)-(18) is in $H^2(O \times (0,1))^3 \times H^1(O \times (0,1))$, for every open set O completely contained in ω.*

Sketch of the proof. Since u' belongs to $H^{\frac{1}{2}}(\Gamma)^2$, there exists (see e.g. [20]) $z' : \omega \times (0,1) \to \mathbb{R}^2$, with $z' \in H^2(O \times (0,1))^2$, for every open set O completely contained in ω, such that $z' = 0$ on $\omega \times \{1\}$, $\partial_3 z' = \lambda^2 R u'$ on Γ. Then, defining $v = (u' - z', u_3)$, $g = (f' + \Delta z', f_3)$ in Ω and extending v, p, g, z' to $\omega \times (-1, 1)$ by taking

$$v'(x', x_3) = v'(x', -x_3), \quad v_3(x', x_3) = -v_3(x', -x_3), \quad p(x', x_3) = p(x', -x_3),$$
$$g'(x', x_3) = g'(x', -x_3), \quad g_3(x', x_3) = -g_3(x', -x_3), \quad z'(x', x_3) = z'(x', -x_3),$$

for $x' \in \omega$, $x_3 \in (-1, 0)$, we deduce that (v, p) satisfies

$$\begin{cases} -\Delta v + \nabla p = g \text{ in } \omega \times (-1,1), \quad \text{div } v = -\text{div}_{x'} z' \text{ in } \omega \times (-1,1) \\ v = 0 \text{ on } \omega \times \{-1, 1\}, \quad \int_{\omega \times (-1,1)} p\, dx = 0. \end{cases}$$

Classical estimates for the Stokes problem with Dirichlet conditions (see e.g. [19]) show then the result. $\square$

The complete proofs of Theorems 1 and 2 and their generalizations to the Navier-Stokes system appear in [11]. For the proofs of Theorem 3 and Proposition 4 we refer to [13].

3   Asymptotic behavior of viscous fluids in rough domains with small height

In this section we consider the case of a viscous fluid confined in the thin rough domain $\Omega_\varepsilon^{thin}$ defined by (9). Remark that $\Omega_\varepsilon^{thin}$ has a rough bottom which agrees with Γ_ε defined by (7). We still denote $\Omega = \omega \times (0,1)$ and $\Gamma = \omega \times \{0\}$.

Analogously to Section 2, we consider the Stokes system in $\Omega_\varepsilon^{thin}$ together with the Navier boundary condition on Γ_ε and the adherence condition on the rest of the boundary $\partial\Omega_\varepsilon^{thin} \setminus \Gamma_\varepsilon$, that is

$$\begin{cases} -\Delta u_\varepsilon + \nabla p_\varepsilon = f \text{ in } \Omega_\varepsilon^{thin}, \quad \text{div } u_\varepsilon = 0 \text{ in } \Omega_\varepsilon^{thin}, \\[2mm] u_\varepsilon \cdot \nu = 0 \text{ on } \Gamma_\varepsilon, \quad T_\varepsilon\left(\frac{\partial u_\varepsilon}{\partial \nu} + \frac{\gamma}{h_\varepsilon}u_\varepsilon\right) = 0 \text{ on } \Gamma_\varepsilon, \\[2mm] u_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon^{thin} \setminus \Gamma_\varepsilon, \end{cases} \quad (24)$$

where $\gamma \geq 0$ is a friction coefficient, ν denotes the unitary outside normal vector to $\Omega_\varepsilon^{thin}$ on Γ_ε, T_ε is the orthogonal projection on the tangent space to Γ_ε and the second member f belongs to $C(\mathbb{R}; L^2(\omega))^3$ (more general second members can be considered).

The system (24) has a unique solution $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon^{thin})^3 \times L_0^2(\Omega_\varepsilon^{thin})$. Moreover, the following estimates hold

$$\frac{1}{|\Omega_\varepsilon^{thin}|} \int_{\Omega_\varepsilon^{thin}} |u_\varepsilon|^2 dx \leq C\, h_\varepsilon^4, \quad \frac{1}{|\Omega_\varepsilon^{thin}|} \int_{\Omega_\varepsilon^{thin}} |Du_\varepsilon|^2 dx \leq C\, h_\varepsilon^2,$$

$$\frac{1}{|\Omega_\varepsilon^{thin}|} \int_{\Omega_\varepsilon^{thin}} |p_\varepsilon|^2 dx \leq C, \quad \forall\, \varepsilon > 0. \tag{25}$$

Remark 3 *The proof of estimates (25) for u_ε and p_ε easyly follows taking u_ε as test function in (24) and then using the inequalities*

$$\|v\|_{L^2(\Omega_\varepsilon^{thin})} \leq Ch_\varepsilon \|\nabla v\|_{L^2(\Omega_\varepsilon^{thin})^3}, \quad \forall v \in H^1(\Omega_\varepsilon), \; v = 0 \text{ on } x_3 = h_\varepsilon.$$

$$\|q\|_{L^2(\Omega_\varepsilon^{thin})} \leq \frac{C}{h_\varepsilon} \|\nabla q\|_{H^{-1}(\Omega_\varepsilon^{thin})^3}, \quad \forall q \in L^2(\Omega_\varepsilon) \text{ with } \int_{\Omega_\varepsilon} q\,dx = 0.$$

Our goal is to study the asymptotic behavior of u_ε and p_ε when ε tends to zero. For this purpose, as usual, we use a dilatation in the variable x_3 to have functions defined in an open set of fixed height. Namely, we define $\tilde{u}_\varepsilon \in H^1(\Omega)^3$, $\tilde{p}_\varepsilon \in L_0^2(\Omega)$ by

$$\tilde{u}_\varepsilon(y) = u_\varepsilon(y', h_\varepsilon y_3), \quad \tilde{p}_\varepsilon(y) = p_\varepsilon(y', h_\varepsilon y_3), \quad \text{a.e. } y \in \Omega. \tag{26}$$

Then, the problem becomes in studying the asymptotic behavior of the functions $\tilde{u}_\varepsilon$ and $\tilde{p}_\varepsilon$. This is given by

Theorem 5 *Let $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon^{thin})^3 \times L_0^2(\Omega_\varepsilon^{thin})$ be the solution of the Stokes system (24) and let $\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon$ be defined by (26). Then, there exist $v' \in H^1(0,1; L^2(\omega))^2$, $w \in H^2(0,1; H^{-1}(\omega))$ and $p \in H^1(\omega)$, with null integral, such that*

$$\frac{\tilde{u}_\varepsilon}{h_\varepsilon} \rightharpoonup 0 \text{ in } H^1(\Omega)^3, \; \frac{\tilde{u}_\varepsilon'}{h_\varepsilon^2} \rightharpoonup v' \text{ in } H^1(0,1; L^2(\omega))^2,$$

$$\frac{\tilde{u}_{\varepsilon,3}}{h_\varepsilon^3} \rightharpoonup w \text{ in } H^2(0,1; H^{-1}(\omega)), \tag{27}$$

$$\tilde{p}_\varepsilon \to p \text{ in } L^2(\Omega),$$

where the functions v', w and p satisfy the following simplified Stokes system

$$\begin{cases} -\partial^2_{y_3\, y_3} v' + \nabla_{y'} p = f'(y', 0) & \text{in } \Omega \\[2mm] \mathrm{div}_{y'} v' + \partial_{y_3} w = 0 & \text{in } \Omega \\[2mm] \displaystyle\int_0^1 v'(y', y_3)\, dy_3 \cdot \nu = 0 & \text{on } \partial\omega \\[2mm] w(y', 0) = w(y', 1) = 0 & \text{in } \omega, \quad v'(y', 1) = 0 \quad \text{in } \omega. \end{cases} \tag{28}$$

Moreover, according to the value of λ_{thin} defined by (11), v' satisfies the following boundary condition on Γ:

i) *If $\lambda_{thin} = 0$, then we have*

$$-\partial_3 v' + \gamma v' = 0, \quad on\ \Gamma. \tag{29}$$

ii) *If $\lambda_{thin} \in (0, +\infty)$, then we have*

$$-\partial_3 v' + \gamma v' + \lambda_{thin}^2 Rv' = 0, \quad on\ \Gamma, \tag{30}$$

where R is defined by (17).

iii) *If $\lambda_{thin} = +\infty$, then we have*

$$-\partial_3 v' + \gamma v' \in W, \quad v' \in W^\perp, \quad on\ \Gamma, \tag{31}$$

where W is defined by (19).

Sketch of the proof. Since $(u_\varepsilon, p_\varepsilon)$ satisfies (25), and $\operatorname{div} u_\varepsilon = 0$ in Ω_ε, we get the estimates

$$\int_\Omega \left|\frac{\tilde{u}_\varepsilon}{h_\varepsilon^2}\right|^2 dy \le C, \quad \int_\Omega \left(\left|D_{y'}\left(\frac{\tilde{u}_\varepsilon}{h_\varepsilon}\right)\right|^2 + \left|\partial_{y_3}\left(\frac{\tilde{u}_\varepsilon}{h_\varepsilon^2}\right)\right|^2\right) dy \le C,$$

$$\int_\Omega |\tilde{p}_\varepsilon|^2 dy \le C, \quad \operatorname{div}_{y'}\left(\frac{\tilde{u}'_\varepsilon}{h_\varepsilon^2}\right) + \partial_{y_3}\left(\frac{\tilde{u}_{\varepsilon,3}}{h_\varepsilon^3}\right) = 0 \text{ in } \Omega, \tag{32}$$

for every $\varepsilon > 0$. This implies the existence of $v' \in H^1(0, 1; L^2(\omega))^2$, $w \in H^2(0, 1; H^{-1}(\omega))$ and $p \in H^1(\omega)$, such that (27) holds. Then, taking into account that $\tilde{u}_\varepsilon = 0$ on $\partial\Omega \setminus \Gamma$, and using as test function in (24) a sequence of the form $z_\varepsilon = h_\varepsilon^2(v(x', x_3/\varepsilon), 0)$ with v' smooth and vanishing on $\partial\Omega$, we easily get (28). In order to finish the proof of Theorem 5 it only remains to obtain the boundary condition satisfied by $\tilde{v}'$ on Γ. This follows reasoning similarly to the proof of Theorem 1 by introducing the sequence $\hat{u}_\varepsilon : \hat{\omega}_\varepsilon \times K_\varepsilon \to \mathbb{R}^3$ as

$$\hat{u}_\varepsilon(x', z) = \sum_{\substack{k' \in \mathbb{Z}^2 \\ \varepsilon k' + \varepsilon Z' \subset \omega}} u_\varepsilon(\varepsilon k' + \varepsilon z', \varepsilon z_3)\chi_{\varepsilon k' + \varepsilon Z'}(x')$$

with

$$\hat{\omega}_\varepsilon = \bigcup_{\substack{k' \in \mathbb{Z}^2 \\ \varepsilon k' + \varepsilon Z' \subset \omega}} (\varepsilon k' + \varepsilon Z'), \quad K_\varepsilon = \left\{z \in Z' \times \mathbb{R} : -\frac{\delta_\varepsilon}{\varepsilon}\Psi(z') < z_3 < \frac{h_\varepsilon}{\varepsilon}\right\}.$$

□

We remark that p only depends on the horizontal variables, i.e. $p = p(x')$. From (28), (29), (30) and (31), as usual in the asymptotic study of fluids in thin domains, we can obtain a Reynolds problem for p. Indeed, we have the following result

Corollary 6 *Let* $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon^{thin})^3 \times L_0^2(\Omega_\varepsilon^{thin})$ *be the solution of the Stokes system (24). Then, depending on the value of* λ_{thin} *defined by (11), the functions* v', w *and* p *in Theorem 5 are given by*

(i) *If* $\lambda_{thin} = 0$, p *is the solution of the Reynolds problem*

$$
\begin{cases}
-\text{div}_{y'} \left(\left(\frac{1}{3} + (1+\gamma)^{-1} \right) (\nabla_{y'} p - f'(y', 0)) \right) = 0 \ \text{in } \omega \\[2mm]
\left(\left(\frac{1}{3} + (1+\gamma)^{-1} \right) (\nabla_{y'} p - f'(y', 0)) \right) \cdot \nu = 0 \ \text{on } \partial\omega,
\end{cases}
$$

the function v' *is given by*

$$
v'(y) = \frac{1}{2} \left(y_3^2 + (1+\gamma)^{-1} \right) (\nabla_{y'} p(y') - f'(y', 0)), \quad a.e. \ y \in \Omega,
$$

and the distribution w *is zero.*

(ii) *If* $\lambda_{thin} \in (0, +\infty)$, *then defining* R *by (17), we have that* p *satisfies the following Reynolds problem*

$$
\begin{cases}
-\text{div}_{y'} \left(\left(\frac{1}{3} I + \left((1+\gamma)I + \lambda_{thin}^2 R \right)^{-1} \right) (\nabla_{y'} p - f'(y', 0)) \right) = 0 \ \text{in } \omega \\[2mm]
\left(\left(\frac{1}{3} I + \left((1+\gamma)I + \lambda_{thin}^2 R \right)^{-1} \right) (\nabla_{y'} p - f'(y', 0)) \right) \cdot \nu = 0 \ \text{on } \partial\omega,
\end{cases}
$$

the function v' *is given by*

$$
v'(y) = \frac{(y_3 - 1)}{2\mu} \left(y_3 I + \left((1+\gamma)I + \lambda_{thin}^2 R \right)^{-1} \right) (\nabla_{y'} p(y') - f'(y', 0)),
$$

a.e. $y \in \Omega$, *and the distribution* w *is given by*

$$
w(y) = - \int_0^{y_3} \text{div}_{y'} v(y', s) \, ds \ \text{in } \Omega. \tag{33}
$$

(iii) *If* $\lambda_{thin} = +\infty$, *then denoting by* $P_{W^\perp}$ *the orthogonal projection from* $\mathbb{R}^2$ *to the orthogonal space of* W *defined by (19), we have that* p *is given as the solution of the Reynolds problem*

$$
\begin{cases}
-\text{div}_{y'} \left(\left(\frac{1}{3} I + (1+\gamma)^{-1} P_{W^\perp} \right) (\nabla_{y'} p - f'(y', 0)) \right) = 0 \ \text{in } \omega \\[2mm]
\left(\left(\frac{1}{3} I + (1+\gamma)^{-1} P_{W^\perp} \right) (\nabla_{y'} p - f'(y', 0)) \right) \cdot \nu = 0 \ \text{on } \partial\omega,
\end{cases}
$$

the function v' *is given by*

$$
v'(y) = \frac{(y_3 - 1)}{2} \left(y_3 I + (1+\gamma)^{-1} P_{W^\perp} \right) (\nabla_{y'} p(y') - f'(y', 0)),
$$

a.e. $y \in \Omega$, *and the distribution* w *by (33).*

Remark 4 *The parameter λ_{thin} in Theorem 5 plays a similar role to that of λ defined in Theorem 1, i.e. if $\lambda_{thin} = 0$ the roughness is too slight and it has no effect on the solution. If $\lambda_{thin} = +\infty$, the roughness is so strong that, in Γ, v' belongs to the orthogonal space of W defined by (19). The case $\lambda_{thin} \in (0, +\infty)$ is the critical case where the roughness is not strong enough to imply $v' \in W^\perp$ in the limit, but it is enough to produce a new friction term $\lambda_{thin}^2 R v'$, where R is the matrix obtained in Theorem 1.*

Remark 5 *We remark that taking $h_\varepsilon = 1$ in (11), the parameters λ and λ_{thin} agree. In the case of thin domains, the expression of λ_{thin} does not only depend on the parameters δ_ε, ε which define Γ_ε, but also on the height h_ε of $\Omega_\varepsilon^{thin}$. This is due to the fact that far of the rough boundary the behavior of the fluid is different from the corresponding one in Section 2.*

Finally, we give corrector results for the velocity and the pressure in the following theorem.

Theorem 7 *Assume $\omega \in C^2$ and $f(x', 0) \in H^1(\omega)^2$. Let $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon^{thin})^3 \times L_0^2(\Omega_\varepsilon^{thin})$ be the solution of the Stokes system (24). Then we have*

i) *If $\lambda_{thin} = 0$ or $+\infty$, defining $\breve{u}_\varepsilon$ by*

$$\breve{u}_\varepsilon(x) = \left(h_\varepsilon^2 v'(x', \frac{x_3}{h_\varepsilon}), 0 \right), \quad a.e. \ x \in \Omega_\varepsilon^{thin},$$

we have

$$\frac{1}{h_\varepsilon^5} \int_{\Omega_\varepsilon^{thin}} |u_\varepsilon - \breve{u}_\varepsilon|^2 dx \to 0, \quad \frac{1}{h_\varepsilon^3} \int_{\Omega_\varepsilon^{thin}} |D(u_\varepsilon - \breve{u}_\varepsilon)|^2 dx \to 0, \quad (34)$$

$$\frac{1}{h_\varepsilon} \int_{\Omega_\varepsilon^{thin}} |p_\varepsilon - p|^2 dx \to 0. \quad (35)$$

ii) *If $\lambda_{thin} \in (0, +\infty)$, the above assertions still hold replacing $\breve{u}_\varepsilon$ by*

$$\breve{u}_\varepsilon(x) = \left(h_\varepsilon^2 v'(x', \frac{x_3}{h_\varepsilon}), 0 \right) + \lambda_{thin} h_\varepsilon^{\frac{3}{2}} \sqrt{\varepsilon} \left(v_1'(x', 0)\widehat{\phi}^1(\frac{x}{\varepsilon}) + v_2'(x', 0)\widehat{\phi}^2(\frac{x}{\varepsilon}) \right),$$

with $\widehat{\phi}^i$, $i = 1, 2$, the solutions of (16).

The results given in this section were announce in [12]. The generalization of these results to Navier-Stokes system will appear in a forthcoming paper [14].

4   Asymptotic behavior of elliptic systems in general rough domains

In the previous sections we have shown that the Navier boundary condition for the Stokes system provides a new term in the limit problem. In this section we study this phenomena for linear elliptic systems in rough domains $\Omega_\varepsilon \subset \mathbb{R}^N$, where Ω_ε has not

necessarily a periodic structure.

We consider a sequence of Lipschitz open sets $\Omega_\varepsilon \subset \mathbb{R}^N$ which converges to a Lipschitz open set $\Omega \subset \mathbb{R}^N$ in the following sense: For every $\rho > 0$, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, we have (see Figure 3)

$$\Omega^{\rho^-} = \{x \in \Omega : d(x, \partial\Omega) > \rho\} \subset \Omega_\varepsilon \subset \{x \in \mathbb{R}^N : d(x, \overline{\Omega}) < \rho\} = \Omega^{\rho^+}. \quad (36)$$

We denote by $\widetilde{\Omega}$ an open set containing completely Ω.

In Ω_ε, we consider the following homogenization problem

$$\begin{cases} -\operatorname{div} ADu_\varepsilon = f & \text{in } \Omega_\varepsilon \\ u_\varepsilon(x) \in V_\varepsilon(x), \quad \forall x \in \partial\Omega_\varepsilon, \quad ADu_\varepsilon(x) \cdot \nu \in V_\varepsilon(x)^\perp \quad \forall x \in \partial\Omega_\varepsilon, \end{cases} \quad (37)$$

where A belongs to $L^\infty(\widetilde{\Omega}; \mathcal{T}_{M \times N})$ ($\mathcal{T}_{M \times N}$ is the space of linear functions from the space of matrices $\mathcal{M}_{M \times N}$ into itself), $V_\varepsilon(x)$ is an arbitrary sequence of functions from $\partial\Omega_\varepsilon$ into the set of linear subspaces of $\mathbb{R}^M$, and the second member f is a function in $L^2(\widetilde{\Omega})^M$.

We also assume the following ellipticity condition: there exists $\alpha > 0$ such that

$$\begin{cases} \alpha\|v\|^2_{H^1(\Omega_\varepsilon)^M} \leq \int_{\Omega_\varepsilon} ADv : Dv \, dx, \\ \forall v \in H^1(\Omega_\varepsilon)^M, \ v(x) \in V_\varepsilon(x), \text{ a.e. } x \in \partial\Omega_\varepsilon. \end{cases} \quad (38)$$

Observe that this ellipticity condition is written in an integral form instead of in a pointwise one. This is more convenient for systems, where the pointwise and integral ellipticity conditions are not equivalent. In particular it permits to deal with the linear elasticity system, where the tensor only depends on the symmetric part of the derivative.

Assuming that $V_\varepsilon(x) = T_\varepsilon(x)$, with $T_\varepsilon(x)$ the tangent space in the point $x \in \partial\Omega_\varepsilon$, the oscillating boundary condition in (37) is similar to the Navier boundary condition (see (12)) considered in Sections 2 and 3. Some other choices of V_ε are also interesting. For example, taking S_ε an arbitrary subset of $\partial\Omega_\varepsilon$, and defining V_ε as $V_\varepsilon(x) = \{0\}$ for $x \in S_\varepsilon$, and $V_\varepsilon(x) = \mathbb{R}^N$ for $x \in \partial\Omega_\varepsilon \setminus S_\varepsilon$, the homogenization problem reads

$$\begin{cases} -\operatorname{div} ADu_\varepsilon = f_\varepsilon & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } S_\varepsilon, \quad ADu_\varepsilon(x) \cdot \nu = 0 & \text{on } \partial\Omega_\varepsilon \setminus S_\varepsilon. \end{cases}$$

In this case, we are studying the homogenization of elliptic partial systems with Dirichlet and Neumann conditions on varying subsets of the boundary. This problem has been studied in [7] and [8] in the particular case $\Omega_\varepsilon = \Omega$.

Our main result in this section is the following theorem

Theorem 8 *There exist a subsequence of ε, still denoted by ε, a Borel measure μ in $\partial\Omega$ which vanishes on the sets of null capacity, a μ-measurable function $R : \partial\Omega \to \mathcal{M}_{M\times M}$, with*

$$R\xi \cdot \xi \geq 0, \; |R\xi \cdot \eta| \leq \beta(R\xi \cdot \xi)^{\frac{1}{2}}(R\eta \cdot \eta)^{\frac{1}{2}}, \quad \forall \xi, \eta \in \mathbb{R}^M, \; \mu\text{-a.e. in } \partial\Omega,$$

for some $\beta > 0$, and a function V from $\partial\Omega$ into the set of linear subspaces of $\mathbb{R}^M$, satisfying

$$\alpha\|v\|_{H^1(\Omega)^M}^2 \leq \int_\Omega ADv : Dv\, dx + \int_{\partial\Omega} Rv \cdot v\, d\mu$$

$$\forall\, v \in H^1(\Omega)^M, \quad v \in V \; q.e. \text{ on } \partial\Omega,$$

with the following property: For every $f \in L^2(\widetilde{\Omega})^M$, the unique solution of (37) converges weakly in $H^1(\Omega^{\rho^-})$, for every $\rho > 0$, to the unique solution $u \in H^1(\Omega)^M$ of problem

$$\begin{cases} -\mathrm{div}\, ADu = f \quad in\ \Omega \\[2mm] u \in V, \quad q.e. \text{ in } \partial\Omega, \quad \displaystyle\int_{\partial\Omega} Ru \cdot u d\mu < +\infty \\[2mm] ADu \cdot \nu + Ru\mu \in V^\perp, \quad q.e. \text{ in } \partial\Omega. \end{cases} \quad (39)$$

Remark 6 *The solution of (39) is understood as the solution of the following variational problem,*

$$\begin{cases} u \in H^1(\Omega)^M, \; u(x) \in V(x)\ q.e. \text{ in } \partial\Omega, \; \displaystyle\int_{\partial\Omega} Ru \cdot u\, d\mu < +\infty \\[2mm] \displaystyle\int_\Omega ADu : Dv\, dx + \int_{\partial\Omega} Ru \cdot v\, d\mu = \int_\Omega f \cdot v\, dx \\[2mm] \forall\, v \in H^1(\Omega)^M, \; v(x) \in V(x)\ q.e. \text{ in } \partial\Omega, \; \displaystyle\int_{\partial\Omega} Rv \cdot v\, d\mu < +\infty. \end{cases} \quad (40)$$

Remark 7 *Theorem 8 applies for instance to study the behavior of the elasticity system*

$$\begin{cases} u_\varepsilon \in H^1(\Omega_\varepsilon)^N, \quad u_\varepsilon \in V_\varepsilon(x)\ q.e. \text{ in } \partial\Omega_\varepsilon \\[2mm] \displaystyle\int_{\Omega_\varepsilon} Be(u_\varepsilon) : e(v)\, dx = \int_{\Omega_\varepsilon} f \cdot v\, dx \\[2mm] \forall\, v \in H^1(\Omega_\varepsilon)^N, \quad v \in V_\varepsilon(x)\ q.e. \text{ in } \partial\Omega_\varepsilon, \end{cases}$$

where the rough domain Ω_ε is described by

$$\Omega_\varepsilon = \left\{ x = (x', x_N) \in \mathbb{R}^N \; : \; x' \in \omega, \quad \Phi_\varepsilon(x') \leq x_N \leq 1 \right\},$$

with $\omega \subset \mathbb{R}^{N-1}$ is a Lipschitz open set and Φ_ε converging $*$-weakly to zero in $W^{1,\infty}(\omega)$.

Here, $\Omega = \omega \times (0, 1)$, $B \in L^\infty(\widetilde{\Omega}; \mathcal{T}_{N,s})$ ($\mathcal{T}_{N,s}$ is the space of linear applications from the space of symmetric matrices of order $N \times N$, $\mathcal{M}_{N,s}$, into itself) is such that there exists $\alpha > 0$ satisfying

$$B(x)\xi : \xi \geq \alpha|\xi|^2, \quad \forall \xi \in \mathcal{M}_{N,s},\ a.e.\ x \in \partial\Omega,$$

and V_ε is a sequence of applications from $\partial\Omega_\varepsilon$ into the set of linear subspaces of $\mathbb{R}^N$ such that $V_\varepsilon(x', 1) = \{0\}$ for every $x' \in \omega$.

Remark 8 *Theorem 8 can also be extended to the Stokes system. In this case, taking $V_\varepsilon(x) = T_\varepsilon(x)$, we would recover the results in [6].*

Sketch of the proof of Theorem 8. The proof of this result is given in [15], where we also prove that problem (40) is stable by homogenization. The idea is the following one:

Thanks to assumption (38), the norm of the solution u_ε of problem (37) in $H^1(\Omega_\varepsilon)^M$ is bounded. Then, by a diagonal procedure, we can extract a subsequence of ε, still denoted by ε, such that for every $f \in L^2(\tilde{\Omega})^M$, there exists $u \in H^1(\Omega)^M$, such that the solution u_ε of (37) converges weakly to u in $H^1(\Omega^{\rho^-})^M$, for every $\rho > 0$. We define $W \subset H^1(\Omega)^M$ as the space of functions $u \in H^1(\Omega)^M$ which are the limit in the previous sense of a sequence u_ε solution of (37), for some $f \in L^2(\tilde{\Omega})^M$. We consider $u^1, u^2 \in W$, and sequences $u_\varepsilon^1, u_\varepsilon^2$ solutions of (37) with f replaced by some $f^1, f^2 \in L^2(\tilde{\Omega})^M$, such that u_ε^i converges weakly to u^i in $H^1(\Omega^{\rho^-})^M$, for every $\rho > 0$, $i = 1, 2$. For $\varphi \in C^1(\tilde{\Omega})$, we take $u_\varepsilon^2\varphi$ as test function in the equation for u_ε^1. Then, passing to the limit, we easily get

$$\int_\Omega ADu^1 : D(u^2\varphi)dx + \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} AD(u_\varepsilon^1 - u^1) : D(u_\varepsilon^2 - u^2)\,\varphi\,dx = \int_\Omega f^1u^2\varphi\,dx.$$

The conclusion of Theorem 8 follows proving that the bilinear function ν from $W \times W$ into the space of Radon measures in $\overline{\Omega}$, defined as

$$\int_{\overline{\Omega}} \varphi\,d\nu(u^1, u^2) = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} AD(u_\varepsilon^1 - u^1) : D(u_\varepsilon^2 - u^2)\,\varphi\,dx$$

is in the conditions of Theorem 4.1 in [15] (which is a variant of an integral representation result given in [18]). This proves the existence of V, μ an R in the conditions of the statement of Theorem 8, such that W is dense in the space of functions $u \in H^1(\Omega)^M$ with $Ru \cdot u \in L^1_\mu(\partial\Omega)$ and such that

$$\int_{\overline{\Omega}} \varphi\,d\nu(u^1, u^2) = \int_{\partial\Omega} Ru^1 \cdot u^2\,\varphi\,d\mu, \quad \forall\,u^1, u^2 \in W, \ \forall\,\varphi \in C^1(\tilde{\Omega}).$$

This proves that u^1 satisfies

$$\int_\Omega ADu^1 : D(u^2\varphi)dx + \int_{\partial\Omega} Ru^1 \cdot u^2\,\varphi\,d\mu = \int_\Omega f^1u^2\varphi\,dx, \quad \forall\,u^2 \in W, \ \forall\,\varphi \in C^1(\tilde{\Omega})$$

and then that the limit u of the solution u_ε of (37) is the solution of (40). $\square$

Acknowledgements
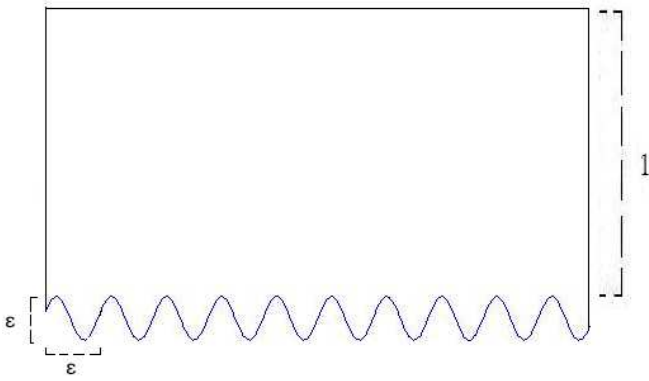
Figures

Figure 1: Rough domain Ω_ε with a rough bottom described by (4).
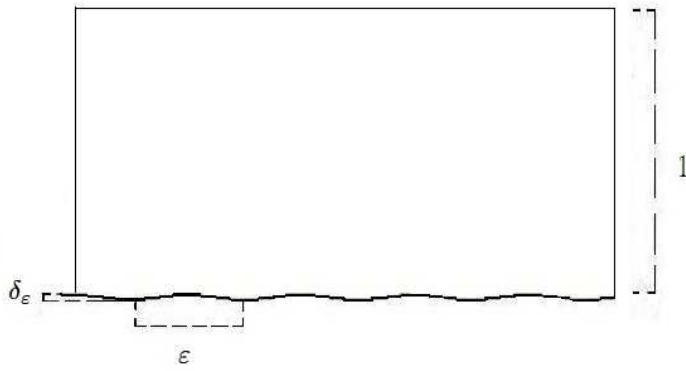
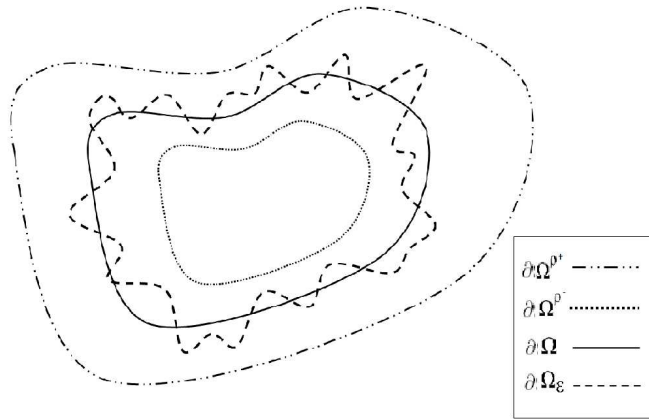Figure 2: Rough domain Ω_ε described by (6).

Figure 3: Convergence of the rough domain Ω_ε defined by (36).

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