EXISTENCE OF INSENSITIZING CONTROLS FOR A SEMILINEAR HEAT EQUATION WITH A SUPERLINEAR NONLINEARITY

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Abstract

In this paper we consider a semilinear heat equation (in a bounded domain Ω of $\mathbb{R}^N$) with a nonlinearity that has a superlinear growth at infinity. We prove the existence of a control, with support in an open set $\omega \subset \Omega$, that insensitizes the L^2-norm of the observation of the solution in another open subset $\mathcal{O} \subset \Omega$ when $\omega \cap \mathcal{O} \neq \emptyset$, under suitable assumptions on the nonlinear term $f(y)$ and the right hand side term ξ of the equation. The proof, involving global Carleman estimates and regularizing properties of the heat equation, relies on the sharp study of a similar linearized problem and an appropriate fixed-point argument. For certain superlinear nonlinearities, we also prove an insensitivity result of a negative nature. The crucial point in this paper is the technique of construction of L^r–controls (r large enough) starting from insensitizing controls in L^2.

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1 Introduction and main results

Problem formulation

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded connected open set with boundary $\partial\Omega \in C^2$. Let f be a C^1 function defined on $\mathbb{R}$. Let ω and $\mathcal{O}$ be two open subsets of Ω (thought to be small, in practice). For $T > 0$, we denote $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$.

We consider a semilinear heat equation with partially known initial condition

$$\begin{cases} \partial_t y - \Delta y + f(y) = \xi + v 1_\omega \ \text{ in } Q, \\ y = 0 \ \text{ on } \Sigma, \quad y(x,0) = y_0(x) + \tau\, \hat{y}_0(x) \ \text{ in } \Omega, \end{cases} \tag{1}$$

where $\xi \in L^2(Q)$ is a given heat source, $y_0 \in L^2(\Omega)$ is a given initial data (although, by the reasons which will be seen later, we will address in this paper the case $y_0 = 0$), $\hat{y}_0 \in L^2(\Omega)$ is unknown with $\|\hat{y}_0\|_{L^2(\Omega)} = 1$, τ is an unknown small real number and $v \in L^2(Q)$ is a control function to be determined. Here 1_ω is the characteristic function of the control set ω.

Let us define

$$\phi(y) = \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |y(x,t;\tau,v)|^2 \, dx\, dt, \tag{2}$$

$y = y(\cdot,\cdot;\tau,v)$ being a solution to (1) associated to τ and v. A control function v is said to insensitize ϕ if

$$\left.\frac{\partial \phi(y(\cdot,\cdot;\tau,v))}{\partial \tau}\right|_{\tau=0} = 0, \quad \forall \hat{y}_0 \in L^2(\Omega) \text{ with } \|\hat{y}_0\|_{L^2(\Omega)} = 1. \tag{3}$$

This insensitivity condition means that we seek a control function v, acting on $\omega \times (0,T)$, such that ϕ is locally insensitive to small perturbations in the initial condition.

In [3] and [15], the existence of a control v satisfying (3) is proved to be equivalent to the existence of a control v solving the following problem:

$$\begin{cases} \partial_t y - \Delta y + f(y) = \xi + v 1_\omega \ \text{ in } Q, \\ y = 0 \ \text{ on } \Sigma, \quad y(x,0) = y_0(x) \ \text{ in } \Omega, \end{cases} \tag{4}$$

$$\begin{cases} -\partial_t q - \Delta q + f'(y) q = y 1_{\mathcal{O}} \ \text{ in } Q, \\ q = 0 \ \text{ on } \Sigma, \quad q(x,T) = 0 \ \text{ in } \Omega, \end{cases} \tag{5}$$

$$q(x,0) = 0 \ \text{ in } \Omega. \tag{6}$$

Thus the problem of seeking a control that insensitizes ϕ boils down to a non-classical null controllability problem. First, it is a null controllability problem of backward-forward nature for a cascade system of heat equations, the first one of semilinear type. In addition, the control enters on the second equation only indirectly through the first one, while q is the function we want to lead to zero after a time interval of length T.

Preliminaries and existing results

This problem, addressed by J.-L. Lions in [15], has been studied for globally Lipschitz-continuous nonlinearities and $\omega \cap \mathcal{O} \neq \emptyset$ (this last hypothesis is absolutely essential and to our knowledge nothing is known when the intersection is empty). First, in [3] the authors relaxed the notion of insensitizing controls, introducing the so–called ε–insensitizing controls: Given $\varepsilon > 0$, a control v is said to ε–insensitize ϕ if

$$\left| \left. \frac{\partial \phi(y(\cdot,\cdot;\tau,v))}{\partial \tau} \right|_{\tau=0} \right| \leq \varepsilon, \quad \forall \hat{y}_0 \in L^2(\Omega) \text{ with } \|\hat{y}_0\|_{L^2(\Omega)} = 1.$$

In the above-mentioned paper, the existence of ε–insensitizing controls for partially known data, both in the initial and boundary conditions, was proved. This problem is equivalent to an approximate controllability problem for a system of coupled heat equations and it was solved by using the techniques in [9].

The first results on the existence and non-existence of insensitizing controls were proved in [16]. To be precise, the author showed that when $f \equiv 0$ and $\Omega \backslash \overline{\omega} \neq \emptyset$, there exists $y_0 \in L^2(\Omega)$ such that, for every $v \in L^2(Q)$, the corresponding solution (y, q) to (4)–(5) satisfies $q(0) \neq 0$, that is to say, the functional ϕ cannot be insensitized (see Theorem 2 in [16]). On the other hand, when $\omega \cap \mathcal{O} \neq \emptyset$, $y_0 = 0$ and f is a C^1 globally Lipschitz-continuous function such that $f(0) = 0$, in [16] it is proved: *If* $\xi \in L^2(Q; \exp(\mathcal{M}/2t))$ *with* $\mathcal{M} > 0$ *large enough, there exists* $v \in L^2(Q)$ *such that the solution* (y, q) *to* (4)–(5) *satisfies* (6) (see Theorem 1 in [16]). Here $L^2(Q; \exp(\mathcal{M}/2t))$ stands for the weighted Hilbert space

$$L^2(Q; \exp(\mathcal{M}/2t)) = \{\xi \in L^2(Q) : \| \exp(\mathcal{M}/2t)\xi\|_{L^2(Q)} < \infty\}.$$

The proof combines a similar null controllability result for linear coupled parabolic systems and an appropriate fixed-point argument. More precisely, the author first linearizes the system and shows its null controllability with controls uniformly bounded in $L^2(Q)$ when the potentials of the linearized system lie in a bounded set of $L^\infty(Q)$. Since f is a globally Lipschitz-continuous function, this fact suffices for proving that the fixed-point mapping maps $L^2(Q)$ into a convex compact set of $L^2(Q)$.

The main goal of the present paper is to analyze the existence of controls that insensitize ϕ, that is to say, to study the null controllability properties of the coupled system (4)–(5) when f has a superlinear growth at infinity. In accordance with Theorems 1 and 2 in [16], we will assume from now on that $y_0 = 0$ and $\omega \cap \mathcal{O} \neq \emptyset$.

In the study of the null controllability of semilinear parabolic systems with superlinear nonlinearities, additional technical difficulties arise. Let us recall what happens in the simpler case of one semilinear heat equation. During the last years, the controllability properties of

$$\begin{cases} \partial_t y - \Delta y + f(y) = v 1_\omega \ \text{ in } Q, \\ y = 0 \ \text{ on } \Sigma, \quad y(x,0) = y_0(x) \ \text{ in } \Omega, \end{cases} \tag{7}$$

with a superlinear nonlinearity $f(y)$ has been thoroughly studied by several authors. As in the sublinear case, the technique to deal with this problem combines a fixed-point reformulation together with the study of the null controllability of linear problems of the form

$$\begin{cases} \partial_t y - \Delta y + ay = v 1_\omega \ \text{ in } Q, \\ y = 0 \ \text{ on } \Sigma, \quad y(x,0) = y_0(x) \ \text{ in } \Omega, \end{cases}$$

where $a \in L^\infty(Q)$. Due to the superlinear growth of the nonlinearity it is necessary, to perform a fixed point argument, that the solution of the controlled linear problem belongs to $L^\infty(Q)$. To this aim controls in $L^r(Q)$, with $r > (N+2)/2$, must be built. Moreover, in the linear case it is necessary to analyze how the L^r–norm of the controls depends on $\|a\|_\infty$. Let us mention some papers on this issue:

1. In [2], V. Barbu obtained null $L^{r(N)}$–controls v such that

$$\|v\|_{L^{r(N)}(Q)} \leq C(\|a\|_\infty)\|y_0\|_{L^2(\Omega)},$$

   with $r(N) \in \left[2, \dfrac{2(N+2)}{N-2}\right]$ if $N \geq 3$, $r(N) \in [2, \infty)$ if $N = 2$, and $r(N) \in [2, \infty]$ if $N = 1$. The proof of this estimate is based on a global Carleman inequality for the adjoint problem

$$\begin{cases} -\partial_t \varphi - \Delta \varphi + a\varphi = 0 \ \text{ in } Q, \\ \varphi = 0 \ \text{ on } \Sigma, \quad \varphi(x,T) = \varphi^0(x) \ \text{ in } \Omega, \end{cases} \tag{8}$$

due to A.V. Fursikov and O.Yu. Imanuvilov (see [11]). In fact, the author proved a sharp estimate of the constant appearing on the right-hand side of the Carleman inequality with respect to $\|a\|_\infty$. Nevertheless, this technique can only be applied to the study of the null controllability of the superlinear heat equation (7) when $N < 6$. For other controllability results proved in a similar way, see [1].

2. A second approach was developed by E. Fernández-Cara and E. Zuazua in [10]. They proved the "refined" observability inequality

$$\|\varphi(0)\|^2_{L^2(\Omega)} \leq C(T, \|a\|_\infty) \left(\iint_{\omega\times(0,T)} |\varphi| \, dx \, dt \right)^2, \tag{9}$$

for the solutions to (8), which implies the existence of a null control $v \in L^\infty(Q)$ satisfying

$$\|v\|_{L^\infty(Q)} \leq C(T, \|a\|_\infty)\|y_0\|_{L^2(\Omega)}.$$

In this case, the observability inequality (9) was deduced combining a global Carleman estimate for the adjoint problem and the regularizing effect of the heat equation. The authors give the explicit dependence on T and $\|a\|_\infty$ of the constant $C(T, \|a\|_\infty)$ in (9), which is essential to prove their nonlinear null controllability result. This technique was later used in [8] for a nonlinear heat equation with a superlinear term $f(y, \nabla y)$.

The study of the null controllability properties of the superlinear coupled system (4)–(5) is more intricate. In this case, as in [1], [2] and [10], null L^r–controls (with $r > (N+2)/2$) for the corresponding coupled linear system must also be built. Again, L^r–estimates of the controls are needed. The technique introduced by V. Barbu could be applied in this case if the condition $N < 6$ is imposed (which does not seem to be a natural restriction on N). On the other hand, the approach proposed by E. Fernández-Cara and E. Zuazua cannot be applied since the regularizing effect of the associated linear adjoint problem involves two functions φ and ψ (see (21)–(22)) while the corresponding "refined" observability inequality should only involve ψ (recall that the control v only appears in (4)).

In [4], the authors introduced a new technique of construction of null L^r–controls for coupled linear parabolic systems. This strategy made it possible to generalize Theorem 1 in [16] to more general nonlinearities. The proof of this insensitivity result as well as the above-mentioned technique, sketched in [4], are developed in the present paper. To our knowledge, this is the first insensitivity result in the literature for semilinear heat equations with a superlinear nonlinearity.

Main results

The first relevant result in this paper is the following one:

Theorem 1.1 *Assume that* $\omega \cap \mathcal{O} \neq \emptyset$ *and* $y_0 = 0$*. Let* f *be a* C^1 *function defined on* $\mathbb{R}$ *verifying* $f'' \in L^\infty_{\mathrm{loc}}(\mathbb{R})$*,* $f(0) = 0$ *and*

$$\lim_{|s|\to\infty} \frac{f'(s)}{\log(1+|s|)} = 0. \tag{10}$$

Let $r \in \left(\frac{N}{2}+1, \infty\right)$ *be given. Then, for any* $\xi \in L^r(Q)$ *such that*

$$\iint_Q \exp\left(\frac{1}{t^3}\right)|\xi|^2\, dx\, dt < \infty, \tag{11}$$

there exists a control function $v \in L^r(Q)$ *insensitizing the functional* ϕ *given by* (2).
□

Condition (11) means that the given source term ξ is asked to decay rapidly to zero close to the initial time $t = 0$. As seen before, a similar assumption, if a weaker one, is required in the case when a globally Lipschitz-continuous nonlinearity is considered. Observe that hypothesis $f(0) = 0$ is in accordance with assumptions on ξ (see [16] for both considerations).

As usual in the study of controllability problems for nonlinear equations, a controllability result for a linearized version of (4)–(6) will be first proved. We will then apply a fixed-point argument to deal with the general case. The structure of the proof is quite general (for other controllability results proved in a similar way, see [10], [17], ...).

Remark **1** Hypothesis (10) is fulfilled by certain superlinear nonlinearities f such as

$$|f(s)| = |p_1(s)| \log^\alpha(1+|p_2(s)|) \quad \text{for all } |s| \geq s_0 > 0,$$

with $\alpha \in [0,1)$, where p_1 and p_2 are real affine functions.

For nonlinearities $f \in C^1(\mathbb{R})$ satisfying hypothesis (10), system (1) admits a global solution when the data ξ, v, y_0 and $\hat{y}_0$ are regular enough. For instance, by linearization and the later application of a fixed-point argument, one can prove that for given ξ and v in $L^r(Q)$ and $y_0, \hat{y}_0 \in W^{2-2/r,r}(\Omega) \cap W^{1,r}_0(\Omega)$, with $r > N/2+1$, system (1) possesses a unique solution in $L^r(0,T;W^{2,r}(\Omega))$. Let us remark that throughout the paper, this regularity will be assumed on the data. □

Our second main result is of a negative nature.

Theorem 1.2 *There exist C^1 functions f verifying $f'' \in L^\infty_{\rm loc}(\mathbb{R})$, $f(0)=0$ and*

$$|f(s)| \sim |s| \log^\alpha(1+|s|) \qquad \text{as } |s| \to \infty, \tag{12}$$

with $\alpha > 2$, and there exist source terms $\xi \in L^r(Q)$ satisfying (11), for which it is not possible to find control functions insensitizing the functional ϕ given by (2). □

For the proof of this result, we choose

$$f(s) = \int_0^{|s|} \log^\alpha (1+|\sigma|)\, d\sigma \quad \text{for all } s \in \mathbb{R}$$

and we prove a localized estimate in $\Omega \setminus \overline{\omega}$ of the corresponding solution y of (4) that shows that for certain source terms ξ, the control v cannot compensate the blow up phenomena occurring in $\Omega \setminus \overline{\omega}$.

In view of Theorems 1.1 and 1.2, it would be interesting to analyze what happens when f satisfies (12), with $1 \le \alpha \le 2$ (see Subsection 6.4 for further comments).

The rest of this paper will be organized as follows. In the following Section, we present some technical results which will be proved in an appendix. Section 3 provides an exhaustive study of the linear case. In Section 4, we analyze the non-linear case and prove Theorem 1.1. The fifth Section is devoted to the proof of Theorem 1.2. In Section 6 we give other insensitivity results and discuss some open problems.

2 Some technical results

In this Section we state some technical results which will be used later. They are known results on the local regularity for the solutions to the linear heat equation. Nevertheless, we include the proof of these results in an appendix so as to obtain the explicit dependence of the constants on the potentials, which will be essential in our analysis.

First, let us present the following notation, which is used all along this paper. For $r \in [1,\infty]$ and a given Banach space X, $\|\cdot\|_{L^r(X)}$ will denote the norm in $L^r(0,T;X)$. For simplicity, the norm in $L^r(Q)$ will be represented by $\|\cdot\|_{L^r}$, for $r \in [1,\infty)$, and $\|\cdot\|_\infty$ will denote the norm in $L^\infty(Q)$. For $r \in [1,\infty)$ and any open set $\mathcal{V} \subset \mathbb{R}^N$, we will consider the Banach space

$$X^r(0,T;\mathcal{V}) = \left\{ u \in L^r(0,T;W^{2,r}(\mathcal{V})) : \, \partial_t u \in L^r(0,T;L^r(\mathcal{V})) \right\},$$

and its norm, defined by

$$\|u\|_{X^r(0,T;\mathcal{V})} = \|u\|_{L^r(W^{2,r}(\mathcal{V}))} + \|\partial_t u\|_{L^r(L^r(\mathcal{V}))}.$$

In particular, we will consider the space $X^r = X^r(0,T;\Omega)$ and its norm, denoted by $\|\cdot\|_{X^r}$. The norm in the space $L^2(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega))$ will be denoted by $\|\cdot\|_{L^2(H^2)\cap C(H^1)}$.

On the other hand, for $\alpha \in (0,1)$ and $u \in C^0(\overline{Q})$, we define the quantity

$$[u]_{\alpha,\frac{\alpha}{2}} = \sup_{\overline{Q}} \frac{|u(x,t)-u(x',t)|}{|x-x'|^\alpha} + \sup_{\overline{Q}} \frac{|u(x,t)-u(x,t')|}{|t-t'|^{\frac{\alpha}{2}}}.$$

We will consider the space $C^{\alpha,\frac{\alpha}{2}}(\overline{Q}) = \left\{u \in C^0(\overline{Q}) : [u]_{\alpha,\frac{\alpha}{2}} < \infty\right\}$, which is a Banach space with its natural norm $|u|_{\alpha,\frac{\alpha}{2};\overline{Q}} = \|u\|_\infty + [u]_{\alpha,\frac{\alpha}{2}}$. We will also consider the Banach space defined by

$$C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q}) = \left\{ u \in C^0(\overline{Q}) : \frac{\partial u}{\partial x_i} \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q})\ \forall i,\ \sup_{\overline{Q}} \frac{|u(x,t)-u(x,t')|}{|t-t'|^{\frac{1+\alpha}{2}}} < \infty \right\}.$$

The following holds:

Proposition 2.1 *Let $a \in L^\infty(Q)$ and $F \in L^2(Q)$ be given. Let us consider a solution $y \in L^2(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega))$ to*

$$\begin{cases} \partial_t y - \Delta y + ay = F \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \quad y(x,0) = 0 \quad \text{in } \Omega. \end{cases} \tag{13}$$

a) Let $\mathcal{V} \subset \Omega$ (resp. $\mathcal{B} \subset\subset \Omega$) be an open set. Let us suppose that $F \in L^r(0,T;L^r(\mathcal{V}))$ (resp. $F \in L^r(0,T;L^r(\Omega \setminus \overline{\mathcal{B}}))$, with $r \in (2,\infty)$. Then, for any open set $\mathcal{V}' \subset\subset \mathcal{V}$ (resp. $\mathcal{B} \subset\subset \mathcal{B}' \subset\subset \Omega$) one has

$$y \in X^r(0,T;\mathcal{V}') \quad (\text{resp. } y \in X^r(0,T;\Omega \setminus \overline{\mathcal{B}'})).$$

Moreover, there exist positive constants $C = C(\Omega,T,N,r,\mathcal{V},\mathcal{V}')$ (resp. $C = C(\Omega,T,N,r,\mathcal{B},\mathcal{B}')$) and $\mathcal{K} = \mathcal{K}(N)$ such that

$$\|y\|_{X^r(0,T;\mathcal{V}')} \leq C\left(1+\|a\|_\infty\right)^{\mathcal{K}} \left[\|F\|_{L^r(L^r(\mathcal{V}))} + \|y\|_{L^2(H^2)\cap C(H^1)}\right]. \tag{14}$$

$$(\text{resp. } \|y\|_{X^r(0,T;\Omega\setminus\overline{\mathcal{B}'})} \leq C\left(1+\|a\|_\infty\right)^{\mathcal{K}} \left[\|F\|_{L^r(L^r(\Omega\setminus\overline{\mathcal{B}}))} + \|y\|_{L^2(H^2)\cap C(H^1)}\right]).$$

b) Assume, in addition, that $F \in L^r(0,T;W^{1,r}(\mathcal{V}))$, *with* r *as above, and* $\nabla a \in L^\gamma(Q)^N$, *with*

$$\gamma = \begin{cases} \max\left(r, \dfrac{N}{2}+1\right) & \text{if } \ r \neq \dfrac{N}{2}+1, \\ \dfrac{N}{2}+1+\varepsilon & \text{if } \ r = \dfrac{N}{2}+1, \end{cases} \tag{15}$$

and ε *being an arbitrarily small positive number. Then, for any open set* $\mathcal{V}' \subset\subset \mathcal{V}$, *one has*

$$y \in L^r(0,T;W^{3,r}(\mathcal{V}')),\ \partial_t y \in L^r(0,T;W^{1,r}(\mathcal{V}'))$$

and, for a new positive constant $C = C(\Omega, T, N, r, \mathcal{V}, \mathcal{V}')$, *the following estimate holds*

$$\|y\|_{L^r(W^{3,r}(\mathcal{V}'))} + \|\partial_t y\|_{L^r(W^{1,r}(\mathcal{V}'))} \leq C\,\mathcal{H}\left[\|F\|_{L^r(W^{1,r}(\mathcal{V}))} + \|y\|_{L^2(H^2)\cap C(H^1)}\right],$$

where

$$\mathcal{H} = \mathcal{H}(N, \|a\|_\infty, \|\nabla a\|_{L^\gamma}) = (1+\|a\|_\infty)^{\mathcal{K}+1}(1+\|\nabla a\|_{L^\gamma}),$$

$\mathcal{K} = \mathcal{K}(N)$ *being as in* (14). □

We will also use the following result, which is immediately obtained rewriting Lemma 3.3, p. 80, in [14] with our notation.

Lemma 2.2 *Let* $\Omega \subset \mathbb{R}^N$, $N \geq 1$, *be an open set with* $\partial\Omega \in C^2$. *The following continuous embeddings hold:*

i. If $r < \dfrac{N}{2}+1$, *then* $X^r \hookrightarrow L^p(Q)$, *where* $\dfrac{1}{p} = \dfrac{1}{r} - \dfrac{2}{N+2}$.

ii. If $r = \dfrac{N}{2}+1$, *then* $X^r \hookrightarrow L^q(Q)$ *for all* $q < \infty$.

iii. If $\dfrac{N}{2}+1 < r < N+2$, *then* $X^r \hookrightarrow C^{\beta,\frac{\beta}{2}}(\overline{Q})$, *with* $\beta = 2 - \dfrac{N+2}{r}$.

iv. If $r = N+2$, *then* $X^r \hookrightarrow C^{l,\frac{l}{2}}(\overline{Q})$ *for all* $l \in (0,1)$.

v. If $r > N+2$, *then* $X^r \hookrightarrow C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{Q})$, *where* $\alpha = 1 - \dfrac{N+2}{r}$. □

3 The linear case

Let us consider the linear systems:

$$\begin{cases} \partial_t y - \Delta y + ay = \xi + v1_\omega \ \text{ in } Q, \\ y = 0 \ \text{ on } \Sigma, \quad y(x,0) = 0 \ \text{ in } \Omega, \end{cases} \tag{16}$$

$$\begin{cases} -\partial_t q - \Delta q + bq = y1_{\mathcal{O}} \ \text{ in } Q, \\ q = 0 \ \text{ on } \Sigma, \quad q(x,T) = 0 \ \text{ in } \Omega, \end{cases} \tag{17}$$

where $a, b \in L^\infty(Q)$, $\xi \in L^2(Q)$ and the open sets ω, $\mathcal{O}$ are given.

Recall that $\omega \cap \mathcal{O} \neq \emptyset$. The aim of this Section is to build a control $v \in L^r(Q)$, with r as in the statement of Theorem 1.1, such that the solution to (16)–(17) verifies

$$q(x,0) = 0 \ \text{ in } \Omega. \tag{18}$$

The estimate of $\|v\|_{L^r}$ with respect to the potentials is also given. To this end, additional assumptions on a, b and ξ will be required. The regularity of v, thus of y and q, will enable us to deal with the nonlinear case.

We will proceed in two steps. First, using an appropriate observability inequality, we will construct controls in $L^2(Q)$. This will be done in Subsection 3.1. Then, in view of the results in the precedent Section, we will exhibit more regular controls (see Subsection 3.2).

3.1 Insensitizing controls in $L^2(Q)$

We actually seek a regular control supported in $\omega \cap \mathcal{O} \neq \emptyset$. In the sequel, B_0 will be a fixed open set such that $B_0 \subset\subset \omega \cap \mathcal{O}$ and we will omit the dependence of the constants on B_0.

Under suitable assumptions on the given heat source ξ, the following result provides a control function $\hat{v}$ in $L^2(Q)$ with support in $\overline{B}_0 \times [0,T]$. The L^2- norm of $\hat{v}$ is also estimated with respect to ξ.

Proposition 3.1 *Let a, $b \in L^\infty(Q)$ be given and let $\xi \in L^2(Q)$ verify*

$$\iint_Q \exp\left(\frac{CM}{t}\right)|\xi|^2 \, dx \, dt < \infty, \tag{19}$$

with $C = C(\Omega, \omega, \mathcal{O})$ and $M = M(T, \|a\|_\infty, \|b\|_\infty)$ as in Proposition 3.2. Then, there exists a control function $\hat{v} \in L^2(Q)$ such that $\operatorname{supp} \hat{v} \subset \overline{B}_0 \times [0,T]$ *and the* *corresponding solution* $(\hat{y},\hat{q})$ *to* (16)–(17) *satisfies* (18). *Moreover,* $\hat{v}$ *can be chosen in such a way that*

$$\|\hat{v}\|_{L^2} \leq \exp\left(\frac{C}{2}H\right)\left(\iint_Q \exp\left(\frac{CM}{t}\right)|\xi|^2\,dx\,dt\right)^{1/2}, \tag{20}$$

$H = H(T,\|a\|_\infty,\|b\|_\infty)$ *being as in Proposition 3.2.*

The proof of this result can be found in [16] and it will be omitted here. The key point in this proof is to obtain an appropriate observability inequality for the corresponding adjoint systems:

$$\begin{cases} \partial_t\varphi - \Delta\varphi + b\varphi = 0 \ \text{ in } Q, \\ \varphi = 0 \ \text{ on } \Sigma, \quad \varphi(x,0) = \varphi^0(x) \ \text{ in } \Omega, \end{cases} \tag{21}$$

and

$$\begin{cases} -\partial_t\psi - \Delta\psi + a\psi = \varphi 1_{\mathcal{O}} \ \text{ in } Q, \\ \psi = 0 \ \text{ on } \Sigma, \quad \psi(x,T) = 0 \ \text{ in } \Omega, \end{cases} \tag{22}$$

with $\varphi^0 \in L^2(\Omega)$ and a, $b \in L^\infty(Q)$. The following result establishes the abovementioned observability inequality:

Proposition 3.2 *Let us assume that* $\omega\cap\mathcal{O} \neq \emptyset$. *Let* B_0 *be an arbitrary open subset of* $\omega\cap\mathcal{O}$. *Then, there exist positive constants* C, M *and* H *such that, for every* $\varphi^0 \in L^2(\Omega)$, *the corresponding solution* (φ,ψ) *to* (21)–(22) *satisfies*

$$\iint_Q \exp\left(-\frac{CM}{t}\right)|\psi|^2\,dx\,dt \leq \exp(CH)\iint_{B_0\times(0,T)}|\psi|^2\,dx\,dt,$$

with

$$M = M(T,\|a\|_\infty,\|b\|_\infty) = 1 + T\left(1 + \|a\|_\infty^{2/3} + \|b\|_\infty^{2/3} + \|a-b\|_\infty^{1/2}\right),$$

$$H = H(T,\|a\|_\infty,\|b\|_\infty) = 1 + \frac{1}{T} + T\left(1 + \|a\|_\infty + \|b\|_\infty\right) + \|a\|_\infty^{2/3} + \|b\|_\infty^{2/3} + \|a-b\|_\infty^{1/2}$$

and $C = C(\Omega,\omega,\mathcal{O},B_0)$. □

This inequality was first proved in [16] as a consequence of a global Carleman inequality for the heat equation (cf. [11]). In [5], the result is extended to the heat equation involving first order terms and the explicit dependence of the constants M and H on T and the size of the potentials is given. This is crucial to prove the existence of insensitizing controls in the nonlinear case when f is not a globally Lipschitz-continuous function.

3.2 Insensitizing controls in $L^r(Q)$

In this Subsection we prove that, under additional assumptions on ξ and the potentials a and b, starting from an insensitizing control $\hat{v} \in L^2(Q)$ supported in $\overline{B}_0 \times [0,T]$, we can build a more regular insensitizing control with a slightly larger support. Furthermore, we estimate the norm of the new control with respect to $\|\hat{v}\|_{L^2}$.

We proceed as follows. Let B, B_1 and B_2 be three open sets such that

$$B_0 \subset\subset B_1 \subset\subset B_2 \subset\subset B \subset \omega \cap \mathcal{O},$$

B_0 being the open set considered in the previous Subsection. For $a, b \in L^\infty(Q)$ and $\xi \in L^2(Q)$ satisfying (19), let $\hat{v}$ be a control (associated to B_0) provided by Proposition 3.1 and let $(\hat{y}, \hat{q})$ be the corresponding solution to (16)–(18).

Our goal is to construct a regular control supported in $\overline{B} \times [0,T]$. To this end, let us consider a function $\theta \in \mathcal{D}(B)$ such that $\theta \equiv 1$ in B_2. We set

$$q = (1-\theta)\, \hat{q} \tag{23}$$

and

$$y = (1-\theta)\, \hat{y} + 2\nabla\theta \cdot \nabla\hat{q} + (\Delta\theta)\hat{q}. \tag{24}$$

We will see that, under appropriate regularity assumptions on the data ξ, a and b, (y, q) solves (16)–(18) with the control function supported in $\overline{B} \times [0,T]$ given by

$$v = -\theta\xi + 2\nabla\theta \cdot \nabla\hat{y} + (\Delta\theta)\hat{y} + (\partial_t - \Delta + a)\left[2\nabla\theta \cdot \nabla\hat{q} + (\Delta\theta)\hat{q}\right]. \tag{25}$$

For $r \in (2, \infty)$, let us set

$$Z_r = \begin{cases} L^r(0,T; W^{1,r}(\Omega)) & \text{if} \quad r \in \left(2, \dfrac{N}{2}+1\right], \\ C^0(\overline{Q}) \cap L^r(0,T; W^{1,r}(\Omega)) & \text{if} \quad r > \dfrac{N}{2}+1. \end{cases} \tag{26}$$

In the sequel, unless otherwise specified, C will stand for a generic positive constant depending on Ω, ω, $\mathcal{O}$ and T (the dependence on N and r will be omitted for simplicity) whose value may change from line to line. One has the following result:

Proposition 3.3 *Let $\xi \in L^r(Q)$ verify (19), with $r \in (2,\infty)$, and let $\hat{v} \in L^2(Q)$ be a control provided by Proposition 3.1 (associated to B_0). The following holds:*

a) For $a, b \in L^\infty(Q)$, it holds that y, q given by (24) and (23) lie in Z_r and

$$\|y\|_{Z_r} + \|q\|_{Z_r} \leq C_1(\Omega, \omega, \mathcal{O}, T, \|a\|_\infty, \|b\|_\infty) \left(\|\xi\|_{L^r} + \|\hat{v}\|_{L^2}\right), \tag{27}$$

where

$$C_1(\Omega, \omega, \mathcal{O}, T, \|a\|_\infty, \|b\|_\infty) = \exp\left[C(1 + \|a\|_\infty + \|b\|_\infty)\right]. \tag{28}$$

b) Suppose, in addition, that $\nabla b \in L^\gamma(Q)^N$, with γ given by (15). Then, v defined by (25) satisfies $v \in L^r(Q)$, $\operatorname{supp} v \subset \overline{B} \times [0, T]$ and

$$\|v\|_{L^r} \leq C_2(\Omega, \omega, \mathcal{O}, T, \|a\|_\infty, \|b\|_\infty, \|\nabla b\|_{L^\gamma}) \left(\|\xi\|_{L^r} + \|\hat{v}\|_{L^2}\right), \tag{29}$$

where $C_2(\Omega, \omega, \mathcal{O}, T, \|a\|_\infty, \|b\|_\infty, \|\nabla b\|_{L^\gamma})$ is given by

$$C_2 = \exp\left[C(1 + \|a\|_\infty + \|b\|_\infty)\right] \left(1 + \|\nabla b\|_{L^\gamma}\right). \tag{30}$$

Proof:

a) First, let us see that, for $a, b \in L^\infty(Q)$, y and q lie in $L^r(0, T; W^{1,r}(\Omega))$. Let us write

$$y = y_1 + y_2 + y_3,$$

with

$$y_1 = (1 - \theta)\hat{y}, \qquad y_2 = 2\nabla\theta \cdot \nabla\hat{q} \quad \text{and} \quad y_3 = (\Delta\theta)\hat{q}.$$

Recalling that $(\hat{y}, \hat{q})$ solves (16)–(17) with a control $\hat{v}$ in $L^2(Q)$ provided by Proposition 3.1, classical energy estimates give

$$\hat{y}, \hat{q} \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap C([0, T]; H_0^1(\Omega)),$$

$$\|\hat{y}\|_{L^2(H^2 \cap H_0^1)} + \|\hat{y}\|_{C(H_0^1)} \leq \exp\left[C(1 + \|a\|_\infty)\right] \left(\|\xi\|_{L^2} + \|\hat{v}\|_{L^2}\right) \tag{31}$$

and

$$\|\hat{q}\|_{L^2(H^2 \cap H_0^1)} + \|\hat{q}\|_{C(H_0^1)} \leq \exp\left[C(1 + \|a\|_\infty + \|b\|_\infty)\right] \left(\|\xi\|_{L^2} + \|\hat{v}\|_{L^2}\right). \tag{32}$$

We can apply Proposition 2.1 to $\hat{y}$ and the open sets B_0 and B_1 together with (31) and conclude

$$\hat{y} \in X^r(0, T; \Omega \setminus \overline{B}_1)$$

and

$$\|\hat{y}\|_{X^r(0, T; \Omega \setminus \overline{B}_1)} \leq \exp\left[C(1 + \|a\|_\infty)\right] \left(\|\xi\|_{L^r} + \|\hat{v}\|_{L^2}\right). \tag{33}$$

Then, since $\operatorname{supp}(1-\theta) \subset \overline{\Omega} \setminus \overline{B}_1$, one has

$$y_1 \in X^r \quad \text{and} \quad \|y_1\|_{X^r} \leq \exp\left[C(1+\|a\|_\infty)\right] \left(\|\xi\|_{L^r} + \|\hat{v}\|_{L^2}\right). \tag{34}$$

On the other hand, as in particular $\hat{y}1_{\mathcal{O}} \in L^r((\Omega \setminus \overline{B}_1) \times (0,T))$, Proposition 2.1 (applied this time to $\hat{q}$ and the open sets B_1 and B_2), together with (33) and (32) yield

$$\hat{q} \in X^r(0,T;\Omega \setminus \overline{B}_2)$$

and

$$\|\hat{q}\|_{X^r(0,T;\Omega\setminus\overline{B}_2)} \leq C_1(\Omega,\omega,\mathcal{O},T,\|a\|_\infty,\|b\|_\infty)\left(\|\xi\|_{L^r} + \|\hat{v}\|_{L^2}\right),$$

with $C_1(\Omega,\omega,\mathcal{O},T,\|a\|_\infty,\|b\|_\infty)$ given by (28). Then, arguing as above, one deduces that

$$y_2 \in L^r(0,T;W^{1,r}(\Omega)),\ y_3, q \in X^r, \tag{35}$$

and

$$\|y_2\|_{L^r(W^{1,r}(\Omega))} + \|y_3\|_{X^r} + \|q\|_{X^r} \leq C_1\left(\|\xi\|_{L^r} + \|\hat{v}\|_{L^2}\right). \tag{36}$$

In particular, for $r \in \left(2, \dfrac{N}{2}+1\right]$ one has

$$y, q \in Z_r = L^r(0,T;W^{1,r}(\Omega)) \ \ (\text{indeed, } q \in X^r)$$

and inequality (27) holds.

Let us now suppose that $r > N/2+1$. In this case, in view of Lemma 2.2, the space X^r embeds continuously (and also compactly) in $C^0(\overline{Q})$. Hence, from (34), (35) and (36) one has

$$y_1, y_3, q \in Z_r = C^0(\overline{Q}) \cap L^r(0,T;W^{1,r}(\Omega))$$

and

$$\|y_1\|_{Z_r} + \|y_3\|_{Z_r} + \|q\|_{Z_r} \leq C_1\left(\|\xi\|_{L^r} + \|\hat{v}\|_{L^2}\right).$$

In order to complete the proof of point a), it suffices to see that $y_2 \in C^0(\overline{Q})$. To this end, let us observe that, for r greater than $N/2+1$, the function $\hat{y}1_{\mathcal{O}}$ lies in $L^\infty((\mathcal{O} \setminus \overline{B}_1) \times (0,T))$. Then, from Proposition 2.1 one deduces that

$$\hat{q} \in X^p(0,T;B \setminus \overline{B}_2) \quad \text{for all } p < \infty,$$

with

$$\|\hat{q}\|_{X^p(0,T;B\setminus\overline{B}_2)} \leq C_1\left(\|\xi\|_{L^r} + \|\hat{v}\|_{L^2}\right).$$

Thus, for fixed $p > N+2$, once again in view of Lemma 2.2, it holds that

$$\hat{q} \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{B \setminus B_2} \times [0,T]), \qquad \text{with } \alpha = 1 - \frac{N+2}{p}.$$

This gives $y_2 \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q})$. This space being continuously embedded in $C^0(\overline{Q})$, one has

$$y_2 \in C^0(\overline{Q}) \ \text{ and } \|y_2\|_{C^0(\overline{Q})} \leq C_1 \left(\|\xi\|_{L^r} + \|\hat{v}\|_{L^2}\right).$$

Is is then infered that $y \in Z_r = C^0(\overline{Q}) \cap L^r(0,T;W^{1,r}(\Omega))$ and (27) holds.

b) Let us suppose, in addition, that $\nabla b \in L^\gamma(Q)^N$, with γ given by (15). We first observe that v defined by (25) is supported in $\overline{B} \times [0,T]$. Moreover, $\hat{y}$ lies in $X^r(0,T;\Omega \setminus \overline{B}_1)$ and (33) holds. We can apply point b) of Proposition 2.1 to $\hat{q}$ for the open set $\mathcal{O} \setminus \overline{B}_1$ and deduce

$$\hat{q} \in L^r(0,T;W^{3,r}(B \setminus \overline{B}_2)), \ \partial_t \hat{q}|_{B \setminus \overline{B}_2} \in L^r(0,T;W^{1,r}(B \setminus \overline{B}_2))$$

and

$$\|\hat{q}\|_{L^r(W^{3,r}(B \setminus \overline{B}_2))} + \|\partial_t \hat{q}\|_{L^r(W^{1,r}(B \setminus \overline{B}_2))} \leq C_2 \left(\|\xi\|_{L^r} + \|\hat{v}\|_{L^2}\right),$$

with $C_2 = C_2(\Omega, \omega, \mathcal{O}, T, \|a\|_\infty, \|b\|_\infty, \|\nabla b\|_{L^\gamma})$ given by (30).

Now, in view of (25) and taking into account the previous considerations on $\hat{y}$ and $\hat{q}$, and the choice of θ, one concludes that v lies in $L^r(Q)$ and satisfies estimate (29). This ends the proof. □

Finally, from the regularity of y, q and v defined by (23)–(25), we deduce that (y,q) solves (16)–(18) with control function v. One then has the following insensitivity result for the linear case:

Corollary 3.4 *Let $\xi \in L^r(Q)$ satisfy (19), with $r \in (2,\infty)$. Let us assume that $a \in L^\infty(Q)$ and $b \in L^\infty(Q) \cap L^\gamma(0,T;W^{1,\gamma}(\Omega))$, with γ given by (15). Then, y, q defined by (24) and (23) lie in the space Z_r introduced in (26) and solve (16)–(18) for the control function $v \in L^r(Q)$ given by (25). Furthermore, there exists a positive constant C depending on Ω, ω, $\mathcal{O}$ and T such that the following estimates hold*

$$\|y\|_{Z_r} + \|q\|_{Z_r} \leq C_1 \left(\|\xi\|_{L^r} + \exp\left(\frac{C}{2}H\right) \left\| \exp\left(\frac{CM}{2t}\right) \xi \right\|_{L^2} \right)$$

and

$$\|v\|_{L^r} \leq C_2 \left(\|\xi\|_{L^r} + \exp\left(\frac{C}{2}H\right) \left\| \exp\left(\frac{CM}{2t}\right) \xi \right\|_{L^2} \right),$$

with C_1 and C_2 of the form (28) and (30), respectively, and H, M being as in Proposition 3.2. □

Let us remark that in the context of the heat equation, the previous technique provides a new method of construction of regular controls starting from controls in $L^2(Q)$. This will make it possible to give a new proof of known null controllability results for nonlinear heat equations. A local result on the null controllability for the classical heat equation and a local result on insensitizing controls for a semilinear heat equation, both with nonlinear Fourier boundary conditions, can also be obtained by using a similar construction (see [7] and [6]).

4 The nonlinear case: proof of Theorem 1.1

In this Section, we will apply an appropriate fixed-point argument to treat the nonlinear case. In a first step, f will be assumed to be a C^2 function. The general case will be studied in Subsection 4.2.

4.1 The case when f is a C^2 function

Let $f \in C^2(\mathbb{R})$ be a function verifying $f(0) = 0$ and (10). Let $\xi \in L^r(Q)$ satisfy hypothesis (11), with $r > N/2 + 1$.

Let us define

$$g(s) = \begin{cases} \dfrac{f(s)}{s} & \text{if } s \neq 0, \\ f'(0) & \text{if } s = 0. \end{cases}$$

Then g, $f' \in C^0(\mathbb{R})$ and $f(s) = g(s)\, s$ for all $s \in \mathbb{R}$. Since $f(0) = 0$, hypothesis (10) on f' implies a similar one on g, that is,

$$\lim_{|s| \to \infty} \frac{g(s)}{\log(1 + |s|)} = 0.$$

Thus, for each $\varepsilon > 0$, there exists a positive constant C_ε (which only depends on ε and on the function f) such that

$$|g(s)| + |f'(s)| \leq C_\varepsilon + \varepsilon \log(1 + |s|) \quad \text{for all } s \in \mathbb{R}. \tag{37}$$

Let us recall that, for $r > \dfrac{N}{2} + 1$, we defined

$$Z_r = C^0(\overline{Q}) \cap L^r(0, T; W^{1,r}(\Omega)).$$

For any $z \in \overline{B}(0;R) \subset Z_r$, $R > 0$ to be determined later, we consider the linear controllability problem

$$\begin{cases} \partial_t y - \Delta y + g(z)y = \xi + v1_\omega & \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \quad y(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{38}$$

$$\begin{cases} -\partial_t q - \Delta q + f'(z)q = y1_{\mathcal{O}} & \text{in } Q, \\ q = 0 \quad \text{on } \Sigma, \quad q(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{39}$$

$$q(x,0) = 0 \text{ in } \Omega. \tag{40}$$

Let us observe that (38)–(39) are of the form (16)–(17) with potentials

$$\begin{cases} a = a_z = g(z) \in L^\infty(Q), \\ b = b_z = f'(z) \in L^\infty(Q) \cap L^r(0,T;W^{1,r}(\Omega)) \ (\gamma = r \text{ in this case}). \end{cases}$$

In view of Corollary 3.4, for any $z \in \overline{B}(0;R) \subset Z_r$ there exists a control $v_z \in L^r(Q)$ such that the corresponding solution (y_z, q_z) to (38)–(39) lies in $Z_r \times Z_r$ and satisfies (40). Moreover, estimates

$$\|y_z\|_{Z_r} \le C_1(\Omega,\omega,\mathcal{O},T,z)\left(\|\xi\|_{L^r} + \exp\left(\frac{C}{2}H_z\right)\left\|\exp\left(\frac{CM_z}{2t}\right)\xi\right\|_{L^2}\right) \tag{41}$$

and

$$\|v_z\|_{L^r} \le C_2(\Omega,\omega,\mathcal{O},T,z)\left(\|\xi\|_{L^r} + \exp\left(\frac{C}{2}H_z\right)\left\|\exp\left(\frac{CM_z}{2t}\right)\xi\right\|_{L^2}\right) \tag{42}$$

hold, where

$$C_1(\Omega,\omega,\mathcal{O},T,z) = \exp\left[C(1 + \|g(z)\|_\infty + \|f'(z)\|_\infty)\right],$$
$$C_2(\Omega,\omega,\mathcal{O},T,z) = \exp\left[C(1 + \|g(z)\|_\infty + \|f'(z)\|_\infty)\right](1 + \|f''(z)\nabla z\|_{L^r}),$$
$$M_z = 1 + \|g(z)\|_\infty^{2/3} + \|f'(z)\|_\infty^{2/3} + \|g(z) - f'(z)\|_\infty^{1/2},$$
$$H_z = 1 + \|g(z)\|_\infty + \|f'(z)\|_\infty + \|g(z)\|_\infty^{2/3} + \|f'(z)\|_\infty^{2/3} + \|g(z) - f'(z)\|_\infty^{1/2},$$

and $C = C(\Omega,\omega,\mathcal{O},T) > 0$.

Due to hypothesis (11) on ξ, one has

$$\begin{aligned} \iint_Q \exp\left(\frac{CM_z}{t}\right)|\xi|^2\,dx\,dt &= \iint_Q \exp\left(\frac{CM_z}{t} - \frac{1}{t^3}\right)\exp\left(\frac{1}{t^3}\right)|\xi|^2\,dx\,dt \\ &\le \exp\left(C\,M_z^{3/2}\right)\iint_Q \exp\left(\frac{1}{t^3}\right)|\xi|^2\,dx\,dt, \end{aligned} \tag{43}$$

$C = C(\Omega, \omega, \mathcal{O}, T)$ being a new positive constant.

Then, from inequalities (41), (42) and (43), and using the convexity of the real function $s \mapsto s^{3/2}$, it can be estimated

$$||y_z||_{Z_r} \leq C_1(\Omega, \omega, \mathcal{O}, T, z) \left(\|\xi\|_{L^r} + \left\| \exp\left(\frac{1}{2t^3}\right) \xi \right\|_{L^2} \right), \tag{44}$$

and

$$\begin{aligned} ||v_z||_{L^r} &\leq C_2(\Omega, \omega, \mathcal{O}, T, z) \left(\|\xi\|_{L^r} + \left\| \exp\left(\frac{1}{2t^3}\right) \xi \right\|_{L^2} \right) \\ &\leq \tilde{C}(\Omega, \omega, \mathcal{O}, T, R) \left(\|\xi\|_{L^r} + \left\| \exp\left(\frac{1}{2t^3}\right) \xi \right\|_{L^2} \right), \end{aligned} \tag{45}$$

where $\tilde{C}(\Omega, \omega, \mathcal{O}, T, R)$ is a positive constant independent of z.

Let us define

$$\mathcal{A} : z \in \overline{B}(0; R) \subset Z_r \longmapsto \mathcal{A}(z) \subset L^r(Q),$$

with

$$\mathcal{A}(z) = \{ v \in L^r(Q) : (y, q) \text{ satisfies (38)–(40)}, \ v \text{ verifying (45)} \},$$

and let Λ be the set-valued mapping defined on Z_r as follows:

$$\Lambda : z \in \overline{B}(0; R) \subset Z_r \longmapsto \Lambda(z) \subset Z_r,$$

with

$$\Lambda(z) = \{ y \in Z_r : (y, q) \text{ solves (38)–(39) with } v \in \mathcal{A}(z), \, y \text{ satisfying (44)} \}.$$

Let us prove that Λ fulfills the assumptions of Kakutani's fixed-point Theorem. In the first place, one can check that $\Lambda(z)$ is a non-empty closed convex subset of Z_r for fixed $z \in Z_r$, due to the linearity of systems (38) and (39).

In fact, in view of Theorem 9.1 in [14] and (45), $\Lambda(z)$ is uniformly bounded in X^r, the space introduced in Section 2. Recall that for $r > N/2 + 1$, X^r is continuously embedded in the Hölder space $C^{\beta, \frac{\beta}{2}}(\overline{Q})$, with $\beta = 2 - (N+2)/r$ (see Lemma 2.2). Then, there exists a compact set $K \subset Z_r$, K only depending on R, such that

$$\Lambda(z) \subset K \quad \forall z \in \overline{B}(0; R). \tag{46}$$

Let us now prove that Λ is an upper hemicontinuous multivalued mapping, that is to say, for any bounded linear form $\mu \in Z_r'$, the real-valued function

$$z \in \overline{B}(0; R) \subset Z_r \longmapsto \sup_{y \in \Lambda(z)} \langle \mu, y \rangle$$

is upper semicontinuous. Correspondingly, let us see that

$$B_{\lambda,\mu} = \left\{ z \in \overline{B}(0;R) : \sup_{y\in\Lambda(z)} \langle \mu, y\rangle \ \geq \lambda \right\}$$

is a closed subset of Z_r for any $\lambda \in \mathbb{R}$, $\mu \in Z_r'$ (see [8] for a similar proof). To this end, we consider a sequence $\{z_n\}_{n\geq 1} \subset B_{\lambda,\mu}$ such that

$$z_n \to z \text{ in } Z_r.$$

Our aim is to prove that $z \in B_{\lambda,\mu}$. Since all the $\Lambda(z_n)$ are compact sets, by the definition of $B_{\lambda,\mu}$, for any $n \geq 1$ there exists $y_n \in \Lambda(z_n)$ such that

$$\langle \mu, y_n\rangle = \sup_{y\in\Lambda(z_n)} \langle \mu, y\rangle \ \geq \lambda. \tag{47}$$

Recalling now the definition of $\mathcal{A}(z_n)$ and $\Lambda(z_n)$, let $v_n \in \mathcal{A}(z_n)$, $q_n \in Z_r$ be such that (y_n, q_n) solves (38)–(40) with control v_n and potentials $g(z_n)$, $f'(z_n)$. From (44) and (45), y_n and v_n satisfy

$$||y_n||_{Z_r} \leq C_1(\Omega, \omega, \mathcal{O}, T, z_n) \left(\|\xi\|_{L^r} + \left\| \exp\left(\frac{1}{2t^3}\right)\xi \right\|_{L^2} \right),$$

and

$$||v_n||_{L^r} \leq C_2(\Omega, \omega, \mathcal{O}, T, z_n) \left(\|\xi\|_{L^r} + \left\| \exp\left(\frac{1}{2t^3}\right)\xi \right\|_{L^2} \right).$$

Thus, $\{y_n\}$ (resp. $\{v_n\}$) is uniformly bounded in Z_r (resp. in $L^r(Q)$). In particular, (46) gives us $\{y_n\} \subset K$. Hence there exist subsequences, still denoted by $\{y_n\}$ and $\{v_n\}$, such that

$$y_n \to \overline{y} \ \text{ strongly in } Z_r,$$

and

$$v_n \to \overline{v} \ \text{ weakly in } L^r(Q).$$

Since g and f'' are continuous functions, one also has

$$g(z_n) \to g(z) \text{ in } C^0(\overline{Q}),$$

$$f'(z_n) \to f'(z) \text{ in } C^0(\overline{Q}),$$

and

$$f''(z_n)\nabla z_n \to f''(z)\nabla z \text{ in } L^r(Q)^N.$$

Passing to the limit, one deduces that $\overline{y}$ and the associated function $\overline{q}$ solve (38)–(40) with control function $\overline{v}$ (and potentials $g(z)$, $f'(z)$). Moreover, $\bar{y}$ and $\bar{v}$ satisfy (44) and (45), that is, $\overline{v} \in \mathcal{A}(z)$ and $\overline{y} \in \Lambda(z)$. Then, taking limits in (47), it holds that

$$\sup_{y \in \Lambda(z)} \langle \mu, y \rangle \geq \langle \mu, \overline{y} \rangle \geq \lambda,$$

whence it is deduced that $z \in B_{\lambda,\mu}$ and hence, Λ is upper hemicontinuous.

Finally, let us see that there exists $R > 0$ such that

$$\Lambda\left(\overline{B}(0;R)\right) \subset \overline{B}(0;R). \tag{48}$$

Let $R > 0$ be, to be determined. For any $z \in \overline{B}(0;R) \subset Z_r$, from (44) and (37) it is observed that each $y \in \Lambda(z)$ satisfies

$$\begin{aligned} ||y||_{Z_r} &\leq \exp\left[C(1 + C_\varepsilon + \varepsilon \log(1 + ||z||_\infty))\right] \left(\|\xi\|_{L^r} + \left\| \exp\left(\frac{1}{2t^3}\right) \xi \right\|_{L^2} \right) \\ &\leq \exp\left[C(1 + C_\varepsilon)\right] (1 + R)^{C\varepsilon} \left(\|\xi\|_{L^r} + \left\| \exp\left(\frac{1}{2t^3}\right) \xi \right\|_{L^2} \right), \end{aligned}$$

with $C = C(\Omega, \omega, \mathcal{O}, T) > 0$. Thus, choosing $\varepsilon = 1/2C$, we get

$$||y||_{Z_r} \leq C(1 + R)^{1/2} \left(\|\xi\|_{L^r} + \left\| \exp\left(\frac{1}{2t^3}\right) \xi \right\|_{L^2} \right),$$

from which we infer the existence of $R > 0$ large enough such that (48) is satisfied.

The Kakutani Fixed-point Theorem thus applies, which ends the proof when f is a C^2 function.

***Remark* 2** Let us notice that two different nonlinearities f for which the positive constant C_ε in (37) coincide, lead to the same $R > 0$. This fact will be used in the following Subsection when studying the general case. $\square$

4.2 The general case

It is now assumed that f is a C^1 function satisfying $f'' \in L^\infty_{\text{loc}}(\mathbb{R})$, $f(0) = 0$ and (10) and let ξ be as above.

We consider a function $\rho \in \mathcal{D}(\mathbb{R})$ such that

$$\rho \geq 0 \text{ in } \mathbb{R}, \quad \operatorname{supp} \rho \subset [-1, 1] \quad \text{and} \quad \int_{\mathbf{R}} \rho(s)\, ds = 1.$$

For any $n \geq 1$, let us set

$$\rho_n(s) = n\rho(ns) \text{ for all } s \in \mathbb{R},$$

$$F_n = \rho_n * f, \quad f_n(\cdot) = F_n(\cdot) - F_n(0)$$

and

$$g_n(s) = \begin{cases} \dfrac{f_n(s)}{s} & \text{if } s \neq 0, \\ f_n'(0) & \text{if } s = 0. \end{cases}$$

Due to the properties of ρ_n and the convolution, as well as the hypothesis on f, one can prove that f_n and g_n have the following properties:

(i) g_n and f_n'' are continuous functions and $f_n(0) = 0$ for all $n \geq 1$.

(ii) $f_n \to f$ in $C^1(K)$ for all compact set $K \subset \mathbb{R}$.

(iii) $g_n \to g$ uniformly on compact sets of $\mathbb{R}$.

(iv) For any given $M > 0$ there exists a positive constant $C(M)$ such that

$$\sup_{|s|\leq M} (|g_n(s)| + |f_n'(s)| + |f_n''(s)|) \leq C(M)$$

for all $n \geq 1$.

(v) It also holds that:

$$\lim_{|s|\to\infty} \frac{|f_n'(s)| + |g_n(s)|}{\log(1+|s|)} = 0 \text{ uniformly in } n,$$

that is, for any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$|g_n(s)| + |f_n'(s)| \leq \varepsilon \log(1+|s|) \text{ for all } |s| \geq M_\varepsilon \text{ and } n \geq 1.$$

In particular, the last two properties imply that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, only depending on ε and not on the function f_n, such that

$$|g_n(s)| + |f_n'(s)| \leq C_\varepsilon + \varepsilon \log(1+|s|) \text{ for all } s \in \mathbb{R} \text{ and } n \geq 1. \tag{49}$$

As it was proved in the previous Subsection, for any $n \geq 1$ there exists a control function $v_n \in L^r(Q)$, with $\operatorname{supp} v_n \subset \omega \times [0,T]$, such that the following cascade of systems

$$\begin{cases} \partial_t y_n - \Delta y_n + f_n(y_n) = \xi + v_n 1_\omega \text{ in } Q, \\ y_n = 0 \text{ on } \Sigma, \quad y_n(x,0) = 0 \text{ in } \Omega, \end{cases} \tag{50}$$

$$\begin{cases} -\partial_t q_n - \Delta q_n + f_n'(y_n) q_n = y_n 1_{\mathcal{O}} \ \text{ in } Q, \\ q_n = 0 \ \text{ on } \Sigma, \quad q_n(x,T) = 0 \ \text{ in } \Omega, \end{cases} \tag{51}$$

admits a solution $(y_n, q_n) \in Z_r \times Z_r$ satisfying

$$q_n(x,0) = 0 \text{ in } \Omega. \tag{52}$$

Moreover, estimates

$$||y_n||_{Z_r} \leq \mathcal{C}_1(\Omega, \omega, \mathcal{O}, T, y_n) \left(\|\xi\|_{L^r} + \left\| \exp\left(\frac{1}{2t^3}\right) \xi \right\|_{L^2} \right), \tag{53}$$

and

$$||v_n||_{L^r} \leq \mathcal{C}_2(\Omega, \omega, \mathcal{O}, T, y_n) \left(\|\xi\|_{L^r} + \left\| \exp\left(\frac{1}{2t^3}\right) \xi \right\|_{L^2} \right), \tag{54}$$

hold, where

$$\mathcal{C}_1(\Omega, \omega, \mathcal{O}, T, y_n) = \exp\left[C(1 + \|g_n(y_n)\|_\infty + \|f_n'(y_n)\|_\infty)\right],$$
$$\mathcal{C}_2(\Omega, \omega, \mathcal{O}, T, y_n) = \exp\left[C(1 + \|g_n(y_n)\|_\infty + \|f_n'(y_n)\|_\infty)\right] (1 + \|f_n''(y_n) \nabla y_n\|_{L^r}),$$

and $C = C(\Omega, \omega, \mathcal{O}, T) > 0$.

Let us recall that y_n is, for any $n \geq 1$, a fixed point of a set-valued mapping Λ_n defined from Z_r onto itself. In view of estimates (53) and (54), and taking into account (49) and Remark 2, one deduces, arguing as in the previous Subsection, that there exists $R > 0$ large enough, and independent of n, such that

$$\Lambda_n\left(\overline{B}(0;R)\right) \subset \overline{B}(0;R).$$

In addition, $\{y_n\}$ (resp. $\{v_n\}$) is uniformly bounded in Z_r (resp. in $L^r(Q)$). Indeed, reasoning as in Subsection 4.1, $\{y_n\}$ is uniformly bounded in the space X^r and hence, r being grater than $N/2+1$, there exists a compact set K in Z_r such that $\{y_n\} \subset K$.

Thus, up to a subsequence, one has

$$y_n \to y \ \text{ strongly in } Z_r$$

and

$$v_n \to v \ \text{ weakly in } L^r(Q),$$

with $v \in L^r(Q)$ and $y \in K \subset Z_r$. Taking now into account properties (*ii*) and (*iii*), one also has

$$g_n(y_n) \to g(y) \quad \text{and} \quad f_n'(y_n) \to f'(y) \quad \text{in } C^0(\overline{Q}).$$

Hence, passing to the limit in (50)–(52), one infers that y and the corresponding q solve (38)–(40), that is, we have found a control v in $L^r(Q)$ insensitizing ϕ. This ends the proof of Theorem 1.1. $\square$

5 Proof of Theorem 1.2

This Section is devoted to proving the insensitivity result of a negative nature stated in Theorem 1.2. To do so, we will show that for certain functions f as in the statement and certain source terms $\xi \in L^r(Q)$ vanishing for $t \in (0, t_0)$, with $t_0 \in (0, T)$, whatever the control v is, the corresponding solution y to (4) blows up before the time $t = T$ and hence, the functional ϕ cannot be insensitized. We will follow the proof of Theorem 1.1 in [10], where the lack of null controllability of a semilinear heat equation is proved.

Let us consider the following function

$$f(s) = \int_0^{|s|} log^{\alpha}(1+\sigma)\, d\sigma \quad \text{for all } s \in \mathbb{R},$$

with $\alpha > 2$. It is easy to check that f is a convex function, $f(s)s < 0$ for all $s < 0$ and

$$|f(s)| \sim |s| \log^{\alpha}(1 + |s|) \quad \text{as} \quad |s| \to \infty.$$

Let $\rho \in \mathcal{D}(\Omega)$ be such that

$$\rho \geq 0 \ \text{ in } \Omega, \quad \rho \equiv 0 \ \text{ in } \omega, \quad \text{and} \quad \int_{\Omega} \rho(x)\, dx = 1.$$

For a fixed $t_0 \in (0, T)$, we set

$$\xi(x,t) = \begin{cases} 0 & \text{if} \quad t \in [0, t_0], \\ -(M + k) & \text{if} \quad t \in (t_0, T], \end{cases}$$

where M is a positive constant which will be chosen later and k is given by

$$k = \frac{1}{2} \int_{\Omega} \rho f^* \left(2 \frac{|\Delta \rho|}{\rho} \right) dx.$$

Here f^* is the *convex conjugate* of the convex function f (the function ρ can be taken in such a way that $\rho f^*(2|\Delta\rho|/\rho) \in L^1(\Omega)$, see [10]).

Let y be a solution to (4), associated to a control v and ξ, defined in the maximal interval $[0,T^*)$. Multiplying the equation in (4) by ρ and integrating in Ω, we get

$$\frac{d}{dt}\int_\Omega \rho y\,dx = \int_\Omega \rho\Delta y\,dx - \int_\Omega \rho f(y)\,dx + \int_\Omega \rho\xi\,dx.$$

We also set

$$z(t) = -\int_\Omega \rho(x)y(x,t)\,dx, \quad \forall t\in[t_0,T^*).$$

Using the properties of f (see [10] for the details), we obtain

$$\begin{cases} z'(t) \geq M + \dfrac{1}{2}f(z(t)), \quad t\in[t_0,T^*), \\ z(t_0) = z_0 = -\displaystyle\int_\Omega \rho(x)y(x,t_0)\,dx. \end{cases}$$

Defining

$$G(z_0;s) = \int_{z_0}^{s} \frac{2}{f(\sigma)+2M}\,d\sigma, \quad \forall s\geq z_0,$$

we can prove that

$$T^* \leq t_0 + \sup_{t\in[t_0,T^*)} G(z_0;z(t)) \leq t_0 + \int_{z_0}^{\infty}\frac{2}{f(\sigma)+2M}\,d\sigma.$$

Thus, for $M>0$ large enough, the solution y blows up in $L^1(\Omega)$ before T. This ends the proof.

6 Further comments, results and open problems

6.1 On the construction of regular controls for parabolic null controllability problems

The technique of construction of regular controls (starting from L^2–controls) introduced in the present paper can be applied to the study of the null controllability of (7) not only when f just depends on the state y but also when $f = f(y,\nabla y)$. In fact, controls in $C^{\alpha,\frac{\alpha}{2}}(\overline{Q})$, with $\alpha\in(0,1]$, can be obtained for potentials regular enough. Let us describe this strategy in the linear case. We consider the null controllability problem

$$\begin{cases} \partial_t y - \Delta y + B\cdot\nabla y + ay = v1_\omega \text{ in } Q, \\ y=0 \text{ on } \Sigma, \quad y(x,0)=y_0 \text{ in } \Omega, \end{cases} \tag{55}$$

$$y(x,T) = 0 \ \text{ in } \Omega,$$

where $y_0 \in L^2(\Omega)$, $a \in L^\infty(Q)$ and $B \in L^\infty(Q)^N$. The previous null controllability problem is equivalent to

$$\begin{cases} \partial_t q - \Delta q + B \cdot \nabla q + aq = -\eta'(t)Y + v 1_\omega \ \text{ in } Q, \\ q = 0 \ \text{ on } \Sigma, \quad q(x,0) = 0 \ \text{ in } \Omega, \\ q(x,T) = 0 \ \text{ in } \Omega, \end{cases} \tag{56}$$

where $q = y - \eta(t)Y$, $\eta \in C^\infty([0,T])$ satisfies

$$\eta \equiv 1 \ \text{ in } \ [0, T/3], \quad \eta \equiv 0 \ \text{ in } \ [2T/3, T],$$

and Y solves (55) with $v = 0$. Suppose that there exists a control $\hat{v} \in L^2(Q)$ solving (56), with $\operatorname{supp} \hat{v} \subset \overline{B}_0 \times [0,T]$, and let $\hat{q}$ be the associated state (here $B_0 \subset\subset \omega$ is a non-empty open set). Then $q = (1 - \theta(x))\hat{q}$ together with

$$v = \theta(x)\eta'(t)Y + 2\nabla\theta \cdot \nabla\hat{q} + (\Delta\theta)\hat{q} - (B \cdot \nabla\theta)\hat{q}$$

solve the null controllability problem (56), where $\theta \in \mathcal{D}(\omega)$ verifies $\theta \equiv 1$ in B_0.

Following Subsection 3.2, one can prove that $v \in L^\infty(Q)$ and

$$\|v\|_\infty \leq C(T, \Omega, \omega, \|a\|_\infty, \|B\|_\infty)\|\hat{v}\|_{L^2}.$$

The explicit dependence of the constant C on $\|a\|_\infty$ and $\|B\|_\infty$ can also be obtained. Moreover, if $B \in C^{\alpha,\frac{\alpha}{2}}(\overline{Q})$, the control v is proved to lie in $C^{\alpha,\frac{\alpha}{2}}(\overline{Q})$ and v verifies a $C^{\alpha,\frac{\alpha}{2}}$–estimate similar to the previous one.

This strategy has been used in [6] to construct hölderian controls for a null controllability problem for the heat equation with nonlinear boundary Fourier conditions (see also [7]).

6.2 Other insensitivity results

The proof of Theorem 1.1 can be adapted to prove other insensitivity results for system (1).

1. Theorem 1.1 is still true under a slightly more general condition on f. To be precise, there exists $l_1(\Omega, \omega, \mathcal{O}, T) > 0$ such that, if hypothesis (10) is replaced by

   $$\limsup_{|s|\to\infty} \frac{|f'(s)|}{\log(1 + |s|)} \leq l_1,$$

   a control function insensitizing the functional given by (2) can be found for any given source term ξ as in Theorem 1.1.

2. The following local insensitivity result can also be proved under no restrictions on the increasing of f:

Theorem 6.1 *Suppose that $\omega \cap \mathcal{O} \neq \emptyset$ and $y_0 = 0$. Let $f \in C^1(\mathbb{R})$ be such that $f'' \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ and $f(0) = 0$. Let $r \in \left(\frac{N}{2}+1, \infty\right)$ be given. Then, there exist two positive constants $M = M(\Omega, \omega, \mathcal{O}, T, f)$ and $\eta = \eta(\Omega, \omega, \mathcal{O}, T, f)$ such that, for any $\xi \in L^r(Q)$ verifying*

$$\|\xi\|_{L^r} + \left\|\exp\left(\frac{M}{2t}\right)\xi\right\|_{L^2} \leq \eta,$$

one can find a control function $v \in L^r(Q)$ insensitizing the functional defined in (2). □

6.3 Simultaneous null and insensitizing controls

As an extension of Theorem 1.1, one can prove the following result on the existence of simultaneous null and insensitizing controls, in other words, controls such that the solution (y, q) to the cascade of systems (4)–(5) (with $y_0 = 0$) verifies (6) and

$$y(x, T) = 0 \quad \text{in } \Omega. \tag{57}$$

Theorem 6.2 *Let us assume that $\omega \cap \mathcal{O} \neq \emptyset$ and $y_0 = 0$. Let f be a function satisfying hypothesis in Theorem 1.1 and let $r \in \left(\frac{N}{2}+1, \infty\right)$ be given. Then, for any $\xi \in L^r(Q)$ verifying*

$$\iint_Q \exp\left(\frac{1}{t^3(T-t)^3}\right)|\xi|^2 \, dx \, dt < \infty, \tag{58}$$

there exists a control function $v \in L^r(Q)$ insensitizing the functional given by (2) and such that the solution $y(\cdot,\cdot;\tau,v)|_{\tau=0}$ to (1) (associated to $\tau = 0$ and v) satisfies

$$y(x, T; \tau, v)|_{\tau=0} = 0 \ \ in \ \ \Omega.$$

Proof: The scheme of the proof is similar to that of Theorem 1.1. Let us observe that, in this case, the source term ξ is required to decay rapidly to zero, not only in the neighbourhood of $t = 0$, but also close to the final time $t = T$, which is a natural assumption. We will assume that f is a C^2 function verifying $f(0) = 0$ and

(10). The general case is studied following Subsection 4.2. Let $\xi \in L^r(Q)$ satisfy (58), with $r > N/2+1$.

Let us see that there exists a control v regular enough such that the solution (y,q) to the cascade of systems (4)–(5) (with $y_0 = 0$) simultaneously verifies (6) and (57). As usual, in a first step, a similar insensitivity result in the linear case is shown. A fixed-point argument is then applied to solve the nonlinear problem.

Let us start with the linear case. We consider the systems (16) and (17), with a and b in $L^\infty(Q)$ and the corresponding adjoint systems

$$\begin{cases} \partial_t \varphi - \Delta\varphi + b\varphi = 0 \ \text{ in } Q, \\ \varphi = 0 \ \text{ on } \Sigma, \quad \varphi(x,0) = \varphi^0(x) \ \text{ in } \Omega, \end{cases} \tag{59}$$

and

$$\begin{cases} -\partial_t \psi - \Delta\psi + a\psi = \varphi 1_{\mathcal{O}} \ \text{ in } Q, \\ \psi = 0 \ \text{ on } \Sigma, \quad \psi(x,T) = \psi^0(x) \ \text{ in } \Omega, \end{cases} \tag{60}$$

with φ^0 and ψ^0 in $L^2(\Omega)$. We first construct controls in $L^2(Q)$. Let B_0 be a fixed non-empty open subset of $\omega \cap \mathcal{O}$. In this case, one has the following observability inequality for the solutions to the previous systems, whose proof is similar to that of Proposition 3.2 (see [5]):

Lemma 6.3 *There exist two positive constants C and M such that*

$$\iint_Q \exp\left(-\frac{CM}{t(T-t)}\right)|\psi|^2\,dx\,dt \le C\iint_{B_0\times(0,T)} |\psi|^2\,dx\,dt \tag{61}$$

for all $\varphi^0, \psi^0 \in L^2(\Omega)$. More precisely, C only depends on Ω, ω and $\mathcal{O}$ and M is given by

$$M = 1 + T + T^2\left(1 + \|a\|_\infty^{2/3} + \|b\|_\infty^{2/3} + \|a-b\|_\infty^{1/2}\right).$$

□

Observe that in (60), $\psi(T)$ is allowed to be different from zero so that, in this case, a different observability inequality is obtained.

For each $\varepsilon > 0$, we consider the continuous and convex functional $\tilde{J}_\varepsilon$ defined on $L^2(\Omega)\times L^2(\Omega)$ by

$$\tilde{J}_\varepsilon(\varphi^0,\psi^0) = \frac{1}{2}\iint_{B_0\times(0,T)} |\psi|^2\,dx\,dt + \varepsilon\|\varphi^0\|_{L^2(\Omega)} + \varepsilon\|\psi^0\|_{L^2(\Omega)} + \iint_Q \xi\psi\,dx\,dt,$$

where φ and ψ solve (59)–(60). The following unique continuation property for the solutions to (59)–(60) holds:

If $\varphi^0, \psi^0 \in L^2(\Omega)$, (φ, ψ) is the associated solution to (59)–(60) and $\psi = 0$ in $\omega \times (0,T)$, then $\varphi \equiv \psi \equiv 0$.

Since $\mathcal{B}_0 \subset \omega \cap \mathcal{O}$, Lemma 6.3 gives that $\psi \equiv 0$ in Q and hence $\varphi \equiv 0$ in $\mathcal{O} \times (0,T)$. Classical unique continuation properties for the heat equation implies $\varphi \equiv 0$ in Q.

As a consequence of the previous continuation property for the solutions of (59)–(60), the functional $\tilde{J}_\varepsilon$ is, indeed, strictly convex and satisfies

$$\liminf_{\|\varphi^0\|_{L^2(\Omega)} + \|\psi^0\|_{L^2(\Omega)} \to +\infty} \frac{\tilde{J}_\varepsilon(\varphi^0, \psi^0)}{\|\varphi^0\|_{L^2(\Omega)} + \|\psi^0\|_{L^2(\Omega)}} \geq \varepsilon.$$

Consequently, $\tilde{J}_\varepsilon$ admits a unique minimizer $(\varphi^0_\varepsilon, \psi^0_\varepsilon) \in L^2(\Omega) \times L^2(\Omega)$. Let $(\varphi_\varepsilon, \psi_\varepsilon)$ be the solution to (59)–(60) associated to $(\varphi^0_\varepsilon, \psi^0_\varepsilon)$. Then, the control

$$v_\varepsilon = \psi_\varepsilon 1_{B_0}$$

is such that the corresponding solution $(y_\varepsilon, q_\varepsilon)$ to (16)–(17) satisfies

$$\|q_\varepsilon(0)\|_{L^2(\Omega)} \leq \varepsilon \quad \text{and} \quad \|y_\varepsilon(T)\|_{L^2(\Omega)} \leq \varepsilon. \tag{62}$$

In view of (61), the controls $\{v_\varepsilon\}$ are proved to be uniformly bounded in $L^2(Q)$ (observe that if ξ verifies (58), then it satisfies

$$\iint_Q \exp\left(\frac{CM}{t(T-t)}\right) |\xi|^2 \, dx \, dt < \infty, \tag{63}$$

with C and M as in Lemma 6.3). Thus, reasoning as in the proof of Proposition 3.1 and using (62), one infers the existence of a control $\hat{v} \in L^2(Q)$, with $\operatorname{supp} \hat{v} \subset \overline{B}_0 \times [0,T]$, such that the associated solution $(\hat{y}, \hat{q})$ to (16)–(17) satisfies (57) and (18). Moreover, one can estimate

$$\|\hat{v}\|^2_{L^2} \leq C \iint_Q \exp\left(\frac{CM}{t(T-t)}\right) |\xi|^2 \, dx \, dt, \tag{64}$$

with C and M as above.

The construction of a regular control starting from $\hat{v} \in L^2(Q)$ is the same as in Subsection 3.2. More precisely, let us define y and q by (24) and (23), respectively. As in Proposition 3.3, for $a \in L^\infty(Q)$ and $b \in L^\infty(Q) \cap L^r(0,T; W^{1,r}(\Omega))$ the functions y and q lie in $Z_r = C^0(\overline{Q}) \cap L^r(0,T; W^{1,r}(\Omega))$, they solve (16)–(18) for the control $v \in L^r(Q)$ given by (25), and (57) is satisfied (we use here that

$\hat{y}(T) = \hat{q}(T) = 0$ in $L^2(\Omega)$ and $\hat{q} \in C([0,T]; H_0^1(\Omega))$). In addition, from (27), (29), (63) and (64), one obtains the following estimates, similar to those in Corollary 3.4:

$$\|y\|_{Z_r} + \|q\|_{Z_r} \leq C_1 \left(\|\xi\|_{L^r} + \left\| \exp\left(\frac{CM}{2t(T-t)}\right) \xi \right\|_{L^2} \right)$$

and

$$\|v\|_{L^r} \leq C_2 \left(\|\xi\|_{L^r} + \left\| \exp\left(\frac{CM}{2t(T-t)}\right) \xi \right\|_{L^2} \right),$$

with C_1 and C_2 as in (28) and (30), M provided by Lemma 6.3 and C being a new positive constant depending on Ω, ω, $\mathcal{O}$ and T.

Finally, by applying a fixed-point argument completely analogous to the one used in the previous Section, one can infer the existence of a control $v \in L^r(Q)$ solving the nonlinear case when f is a C^2 function. As stated above, it is enough to follow Subsection 4.2 to deal with the general case. This ends the proof of Theorem 6.2.

□

6.4 Open problems

1. Theorem 1.1 has been proved under the essential assumption $y_0 = 0$. It would be interesting to give sufficient conditions on the initial data y_0 in order to insensitize (2). When $f \equiv 0$ and $\mathcal{O} = \Omega$, a positive result on the existence of insensitizing controls is proved for a small class of initial data (see Lemma 2 in [16]). The proof of this result is strongly based on properties of the semigroup of the classical heat equation. The general case is an open problem for a linear heat equation with an L^∞ potential and even in the case of the classical heat equation when $\mathcal{O} \neq \Omega$.

2. Hypothesis $\omega \cap \mathcal{O} \neq \emptyset$ is required in order to prove the existence of both ε–insensitizing and insensitizing controls. In the first case, the hypothesis is used to prove that certain functional is coercive, using a unique continuation property. In the case of insensitizing controls, this hypothesis is used to prove an observability inequality. At present both problems are far from being solved for disjoint control set and observation set (see [16]).

3. In view of known null controllability results for a semilinear heat equation with homogeneous Dirichlet boundary conditions (see Theorem 1.2 in [10]), one could think of extending Theorem 1.1 to nonlinearities f with growth at infinity of order $|s| \log^\alpha(1 + |s|)$ with $1 \leq \alpha < 3/2$. Notice that, in this

case, in the absence of control, blow up phenomena occur under suitable sign conditions. The idea in [10] is to control the system to zero in a short time to prevent the solution from blowing up and to set $v \equiv 0$ for the rest of the time interval. Observe that this argument cannot be applied when a source term ξ appears on the right-hand side of (7). This strategy also fails in our problem since, in insensitivity problems, both initial and final times are fixed and, in addition, there is a right-hand side term ξ in the equation. The problem then remains open for such nonlinearities.

4. In [5], the authors prove a result on the existence of insensitizing controls for a semilinear heat equation with a nonlinear term $f(y, \nabla y)$, with $f : \mathbb{R}\times\mathbb{R}^N \mapsto \mathbb{R}$ a globally Lipschitz-continuous function. It would be interesting to generalize this result to functions f with a superlinear growth at infinity. To do this, controls in L^r, with $r > N + 2$, should be built for the null controllability problem

$$\begin{cases} \partial_t y - \Delta y + B \cdot \nabla y + ay = \xi + v1_\omega \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \quad y(x, 0) = 0 \text{ in } \Omega, \end{cases} \tag{65}$$

$$\begin{cases} -\partial_t q - \Delta q - \nabla \cdot (Dq) + cq = y1_{\mathcal{O}} \text{ in } Q, \\ q = 0 \text{ on } \Sigma, \quad q(x, T) = 0 \text{ in } \Omega, \end{cases} \tag{66}$$

$$q(x, 0) = 0 \text{ in } \Omega,$$

where $a, c \in L^\infty(Q)$ and $B, D \in L^\infty(Q)^N$. Unfortunately, the technique introduced in the present paper cannot be applied in this case because of the lack of regularity introduced by the term $-\nabla\cdot(Dq)$. To be precise, assume that a null control $\hat{v} \in L^2(Q)$ for (65)–(66), with $\operatorname{supp} \hat{v} \subset \overline{B}_0 \times [0, T]$, has already been obtained (B_0 as in Subsection 6.1). Starting from $\hat{v}$ and the associated $(\hat{y}, \hat{q})$, the expression of a new control obtained by means of this strategy would be:

$$\begin{aligned} v &= -\theta\xi + 2\nabla\theta \cdot \nabla\hat{y} + (\Delta\theta)\hat{y} - \nabla\theta \cdot (B\hat{y}) \\ &+ (\partial_t - \Delta + a + B \cdot \nabla)\left[2\nabla\theta \cdot \nabla\hat{q} + (\Delta\theta)\hat{q} + \nabla\theta \cdot (D\hat{q})\right]. \end{aligned}$$

Here, $\theta \in \mathcal{D}(\omega)$ verifies $\theta \equiv 1$ in B_0. Observe that, if $D \in L^\infty(Q)^N$, some terms in this formula are not regular enough to make the state y lie in a suitable space to apply a fixed-point argument. Thus the problem remains open for such nonlinearities.

A Proof of Proposition 2.1

Let $a \in L^\infty(Q)$, $F \in L^2(Q)$ and y be as in Proposition 2.1. Let $\mathcal{V}$ and $\mathcal{V}'$ be arbitrary open sets such that $\mathcal{V}' \subset\subset \mathcal{V} \subset \Omega$. Throughout the proof, C will be a positive constant whose value may change from one line to another. The dependence of the constant C on Ω, T, N, r, $\mathcal{V}$ and $\mathcal{V}'$, which is not used in any essential way in our analysis, will not be specified for the sake of simplicity.

We will restrict our attention to the case when $N > 2$, the discussion being similar but more direct when $N = 1$ or $N = 2$.

a) Suppose that $F \in L^r(0,T;L^r(\mathcal{V}))$, with $r \in (2,\infty)$. We will proceed in several steps. Let us consider a family of open sets $\{\mathcal{V}_i\}_{0\le i\le I}$ such that

$$\mathcal{V}' = \mathcal{V}_I \subset\subset \mathcal{V}_{I-1} \subset\subset \cdots \subset\subset \mathcal{V}_1 \subset\subset \mathcal{V}_0 \subset\subset \mathcal{V},$$

where the integer I will be determined later.

In the first place, let $\zeta_0 \in \mathcal{D}(\mathcal{V})$ be a function such that $\zeta_0 \equiv 1$ in $\mathcal{V}_0$. Setting $w_0 = \zeta_0 y$, it is easy to check that w_0 solves the following problem

$$\begin{cases} \partial_t w - \Delta w = G \quad \text{in } Q, \\ w = 0 \ \text{ on } \Sigma, \quad w(x,0) = 0 \ \text{ in } \Omega, \end{cases} \tag{67}$$

with $G = G_0$ given by

$$G_0 = \zeta_0 F - [\zeta_0 a y + 2\nabla\zeta_0 \cdot \nabla y + (\Delta\zeta_0) y].$$

¿From the regularity assumptions on F and y, one has $\zeta_0 F \in L^r(Q)$ and

$$\zeta_0 a y + 2\nabla\zeta_0 \cdot \nabla y + (\Delta\zeta_0) y \in L^2(0,T;L^{2^\star}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),$$

with $\dfrac{1}{2^\star} = \dfrac{1}{2} - \dfrac{1}{N}$. Using classical interpolation inequalities, one obtains

$$G_0 \in L^r(0,T;L^{p_0}(\Omega)), \text{ with } p_0 = \min\left(r, \frac{2Nr}{Nr-4}\right),$$

and

$$\|G_0\|_{L^r(L^{p_0}(\Omega))} \le C(1+\|a\|_\infty)\left(\|F\|_{L^r(L^r(\mathcal{V}))} + \|y\|_{L^2(H^2)\cap C(H^1)}\right). \tag{68}$$

Due to the regularizing effect and the properties of the semigroup generated by the heat equation with Dirichlet boundary conditions (see, for instance, [13] and [12]), one deduces that

$$w_0 \in L^r(0,T;W^{2,p_0}(\Omega)) \ \text{ and } \ \|w_0\|_{L^r(W^{2,p_0}(\Omega))} \le C\|G_0\|_{L^r(L^{p_0}(\Omega))}. \tag{69}$$

Then, recalling that $w_0|_{\mathcal{V}_0} = y|_{\mathcal{V}_0}$, one infers from (69) and (68) that

$$y \in L^r(0,T;W^{2,p_0}(\mathcal{V}_0))$$

and the following estimate is satisfied

$$\|y\|_{L^r(W^{2,p_0}(\mathcal{V}_0))} \leq C(1+\|a\|_\infty)\left(\|F\|_{L^r(L^r(\mathcal{V}))} + \|y\|_{L^2(H^2)\cap C(H^1)}\right).$$

If $r \leq 2 + \dfrac{4}{N}$, that is, if $p_0 = r$, the first point is already proved. Let us now suppose that $r > 2 + \dfrac{4}{N}$ (i.e. $2 < p_0 = \dfrac{2Nr}{Nr-4} < r$). We will now use the following Lemma, which will be proved at the end of this Appendix.

Lemma A.1 *Let $a \in L^\infty(Q)$ and $F \in L^2(Q) \cap L^r(0,T;L^r(\mathcal{V}))$ be, with $\mathcal{V} \subset \Omega$ an arbitrary open set and $r \in (2,\infty)$. Let $y \in L^2(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega))$ satisfy (13). Let ω_0 and ω_1 be two open subsets of Ω such that $\omega_1 \subset\subset \omega_0 \subset \mathcal{V}$. Let us assume that $y \in L^r(0,T;W^{2,r_0}(\omega_0))$, with $r_0 \in [2,r)$. Then,*

$$y \in L^r(0,T;W^{2,r_1}(\omega_1)) \ \text{ and } \partial_t y \in L^r(0,T;L^{r_1}(\omega_1)),$$

with

$$r_1 = \begin{cases} \min\left(r, \dfrac{Nr_0}{N-r_0}\right) & \text{if } \ r_0 < N, \\ r & \text{if } \ r_0 \geq N. \end{cases} \tag{70}$$

Furthermore, the following estimate holds:

$$\|y\|_{L^r(W^{2,r_1}(\omega_1))} + \|\partial_t y\|_{L^r(L^{r_1}(\omega_1))} \leq C(1+\|a\|_\infty)\left(\|F\|_{L^r(L^r(\mathcal{V}))} + \|y\|_{L^r(W^{2,r_0}(\omega_0))}\right),$$

where C is a positive constant depending on Ω, T, N, r, ω_0 and ω_1. □

We apply this Lemma for $i = 1,\ldots,I$, replacing ω_0, ω_1, r_0 and r_1, respectively, by $\mathcal{V}_{i-1}$, $\mathcal{V}_i$, p_{i-1} and p_i, with

$$\frac{1}{p_i} = \frac{1}{p_0} - \frac{i}{N} \ \text{ for } \ i = 1,\ldots,I-1 \ \text{ and } \ p_I = r.$$

This yields $y \in L^r(0,T;W^{2,p_i}(\mathcal{V}_i))$, $\partial_t y \in L^r(0,T;L^{p_i}(\mathcal{V}_i))$, $1 \leq i \leq I$, and the corresponding estimates. In order to determine I, observe that we may go on applying Lemma A.1 while $p_{i-1} < N$ and $p_i < r$, that is to say, while $i < \dfrac{N(r-2)-4}{2r}$.

Thus, in I steps, with

$$I = \left[\frac{N(r-2)-4}{2r}\right] + 1$$

($[\sigma]$ being the integer part of the real number σ), we have $y \in X^r(0,T;\mathcal{V}')$ together with estimate (14), with $\mathcal{K} = \dfrac{N}{2} + 2$, which is a uniform bound of $I+1$.

b) Suppose now that $F \in L^r(0,T;W^{1,r}(\mathcal{V}))$, r as above, and $\nabla a \in L^\gamma(Q)^N$, with γ given by (15). Let us consider a new open set $\tilde{\mathcal{V}}$ such that

$$\mathcal{V}' \subset\subset \tilde{\mathcal{V}} \subset\subset \mathcal{V}.$$

In view of point a), $y \in X^r(0,T;\tilde{\mathcal{V}})$ and

$$\|y\|_{X^r(0,T;\tilde{\mathcal{V}})} \le C\,(1+\|a\|_\infty)^{\mathcal{K}}\left[\|F\|_{L^r(L^r(\mathcal{V}))} + \|y\|_{L^2(H^2)\cap C(H^1)}\right]. \qquad (71)$$

To end the proof, let us see that, in addition,

$$\partial_i y \in X^r(0,T;\mathcal{V}'),\ 1 \le i \le N,$$

and that the following estimate is satisfied

$$\|\partial_i y\|_{X^r(0,T;\mathcal{V}')} \le C\,(1+\|a\|_\infty)^{\mathcal{K}+1}(1+\|\partial_i a\|_{L^\gamma})\left[\|F\|_{L^r(W^{1,r}(\mathcal{V}))} + \|y\|_{L^2(H^2)\cap C(H^1)}\right],$$

where $\partial_i y$ denotes the derivative of y with respect to x_i, $1 \le i \le N$. To do so, let us set

$$w_i = \zeta_1 \partial_i y \ \text{ for a fixed } i \in \{1,\dots,N\},$$

with $\zeta_1 \in \mathcal{D}(\tilde{\mathcal{V}})$ a function such that $\zeta_1 \equiv 1$ in $\mathcal{V}'$. Then, w_i solves (67) with $G = G_i$ given by

$$G_i = \zeta_1\partial_i F - \zeta_1 a \partial_i y - \zeta_1 y \partial_i a - 2\nabla\zeta_1 \cdot \nabla(\partial_i y) - (\Delta\zeta_1)\partial_i y. \qquad (72)$$

Let us see that $G_i \in L^r(Q)$. We will study in detail the term $\zeta_1 y \partial_i a$. Under assumptions on y, a and F, it is direct to see that the other terms in (72) lie in $L^r(Q)$.

Let us observe that $\zeta_1 y \in X^r$. In view of Lemma 2.2, since $\nabla a \in L^\gamma(Q)^N$, with γ given by (15), and recalling that the Hölder space $C^{l,\frac{l}{2}}(\overline{Q})$ is continuously embedded in $C^0(\overline{Q})$, for all $r \in (2,\infty)$ one may infer that the term into consideration, $\zeta_1 y \partial_i a$, lies in $L^r(Q)$ and one has

$$\|\zeta_1 y \partial_i a\|_{L^r} \le C\|\zeta_1 y\|_{X^r}\|\partial_i a\|_{L^\gamma}.$$

Hence, coming back to (72), $G_i \in L^r(Q)$ and one can estimate

$$\|G_i\|_{L^r} \leq C \left[\|\partial_i F\|_{L^r(L^r(\mathcal{V}))} + (\|a\|_\infty + \|\partial_i a\|_{L^\gamma} + 1) \|y\|_{X^r(0,T;\tilde{\mathcal{V}})} \right]. \quad (73)$$

The regularizing effect of the heat equation (see [12] and [13]) yields $w_i \in X^r$, with

$$\|w_i\|_{X^r} \leq C\|G_i\|_{L^r}. \quad (74)$$

Then, from (74), (73) and (71), one deduces

$$\|w_i\|_{X^r} \leq C(1 + \|a\|_\infty)^{\mathcal{K}+1} (1 + \|\partial_i a\|_{L^\gamma}) \left[\|F\|_{L^r(W^{1,r}(\mathcal{V}))} + \|y\|_{L^2(H^2)\cap C(H^1)}\right],$$

with $C = C(\Omega, T, N, r, \mathcal{V}, \mathcal{V}')$ and $\mathcal{K} = \mathcal{K}(N)$ the same as above.

Finally, just taking into account that $w_i \equiv \partial_i y$ in $\mathcal{V}'$, point b) is proved. □

***Remark* 3** Following the previous proof, one easily observes that the same result can be obtained when replacing hypothesis $y \in L^2(0,T;H^2(\Omega)) \cap C([0,T];H^1(\Omega))$ by

$$y \in L^2(0,T;H^1_{\text{loc}}(\Omega)) \cap L^\infty(0,T;L^2_{\text{loc}}(\Omega)).$$

This fact just affects the number I of steps required to prove point a). □

We end this Appendix by giving the

Proof of Lemma A.1: Let us consider a function $\zeta \in \mathcal{D}(\omega_0)$ such that $\zeta \equiv 1$ in ω_1 and let us set $u = \zeta y$. Then, u solves (67), with

$$G = \zeta F - [\zeta a y + 2\nabla\zeta \cdot \nabla y + (\Delta\zeta) y].$$

The regularity of y and usual Sobolev embeddings give $G \in L^r(0,T;L^{r_1}(\Omega))$, where r_1 is given by (70), and the estimate

$$\|G\|_{L^r(L^{r_1}(\Omega))} \leq C(1 + \|a\|)_\infty \left[\|F\|_{L^r(L^r(\mathcal{V}))} + \|y\|_{L^r(W^{2,r_0}(\omega_0))}\right] \quad (75)$$

(here C depends on the open sets ω_0 and ω_1).

Then, again due to the regularizing properties of the heat equation, one deduces that

$$u \in L^r(0,T;W^{2,r_1}(\Omega)),\ \partial_t u \in L^r(0,T;L^{r_1}(\Omega))$$

and

$$\|u\|_{L^r(W^{2,r_1}(\Omega))} + \|\partial_t u\|_{L^r(L^{r_1}(\Omega))} \leq C\|G\|_{L^r(L^{r_1}(\Omega))}.$$

Finally, taking into account that $u|_{\omega_1} = y|_{\omega_1}$ and inequality (75), the result follows. □

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