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On the Approximate and Null Controllability of the Navier–Stokes Equations*

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Abstract. This paper presents some known results on the approximate and null controllability of the Navier–Stokes equations. All of them can be viewed as partial answers to a conjecture of J.-L. Lions.

Key words. approximate controllability, null controllability, Navier–Stokes equations

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1. Introduction. The Formulation and the Meaning.

Let $\Omega \subset \mathbf{R}^N$ be an open, bounded, and connected set with smooth boundary $\partial\Omega$ ($N = 2$ or $N = 3$); let $\mathcal{O} \subset \Omega$ be a (small) nonempty open subset; and assume $T > 0$. We will set $Q = \Omega \times (0, T)$, $\Sigma = \partial\Omega \times (0, T)$, and $\mathcal{U} = L^2(\mathcal{O} \times (0, T))^N$. For simplicity, the notation will be abridged such that we will give $L^2(\Omega)$ instead of $L^2(\Omega)^N$, etc. The usual norms in $L^2(Q)$ and $L^\infty(Q)$ will be denoted, respectively, by $\|\cdot\|_2$ and $\|\cdot\|_\infty$.

Let us consider the usual space of "test" functions

$$\mathcal{V} = \{\, \phi \in C_0^\infty(\Omega)^N;\ \nabla \cdot \phi = 0 \quad \text{in } \Omega \,\},$$

and let us denote by H (respectively V) the closure of $\mathcal{V}$ in $L^2(\Omega)$ (respectively $H_0^1(\Omega)$). H coincides with the space of all L^2-functions whose divergence vanishes in Ω and whose normal trace vanishes on $\partial\Omega$. The usual norm and scalar product in H will be denoted, respectively, by $|\cdot|$ and $(\cdot,\cdot)$. On the other hand, V coincides with the space of all H_0^1-functions whose divergence vanishes.

For each $v \in \mathcal{U}$ and each $y_0 \in H$, we consider the corresponding Navier–Stokes problem

$$\text{(1)} \qquad \begin{cases} \partial_t y + (y \cdot \nabla) y - \Delta y + \nabla p = 1_{\mathcal{O}} v, \quad \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(0) = y_0, \end{cases}$$

where (for convenience) the constant density and viscosity coefficients have been taken equal to 1. Here, $1_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$. In [8], Lions conjectured that

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(1) is *approximately controllable* in H for all $T > 0$. That is, he claimed that, for any given $y_d \in H$ and $\varepsilon > 0$, a control v can be found such that (1) possesses at least one solution $\{y, p\}$ satisfying

$$y \in C^0_w(0, T; H), \quad |y(T) - y_d| \leq \varepsilon$$

(and, also, that this has to be true for all $\mathcal{O}$, T, y_0, y_d, and ε). The conjecture is open, with the exception of some particular situations (for instance, when $N = 2$ and $\mathcal{O}$ is a neighborhood of $\partial\Omega$; see section 3).

The meaning of approximate controllability for (1) is the following. Provided it is true, once y_0 and y_d are fixed, it is always possible to drive the system described by (1) from y_0 to a final state $y(T)$ arbitrarily close to y_d. Let us emphasize that, here, the control region is $\mathcal{O} \times (0, T)$ and $\mathcal{O}$ is arbitrarily small. We can formulate (and interpret) in a similar way a conjecture concerning boundary controllability, with the control being exerted on a portion of the boundary.

It is also meaningful to ask whether or not (1) is *null controllable* for all T, that is, if, for each y_0, there exists $v \in \mathcal{U}$ such that (1) possesses at least one solution $\{y, p\}$ with

$$y \in C^0_w(0, T; H), \quad y(T) = 0.$$

Again, the answer to this question is unknown.

In recent years, several authors have given partial answers to these or related questions. Some of them are reviewed in the following sections.

2. A Fixed-Point Argument. In this context, the most significant contribution is the work of Fabre [3]. In principle, this seems to be a natural strategy. The goal is to reformulate the conjecture as a fixed-point equation. In order to make things meaningful, we have to be able to control linear systems of the Stokes kind, that is,

$$(2) \qquad \begin{cases} \partial_t y + \nabla \cdot (ay) - \Delta y + \nabla p = 1_{\mathcal{O}} v, \quad \nabla \cdot y = 0 \quad \text{in} \quad Q, \\ y = 0 \quad \text{on} \quad \Sigma, \\ y(0) = y_0, \end{cases}$$

where $a = a(x, t)$ is prescribed. In order to have compactness, we have to be able to do this when the coefficient a is not regular. Unfortunately, the argument does not provide approximate controllability for (1), but only for an approximation of it.

Let us be more precise. Assume $a \in L^\infty(Q)$ and consider system (2). It is not difficult to see that this system is approximately controllable in H if and only if the adjoint system

$$(3) \qquad \begin{cases} -\partial_t \phi - (a \cdot \nabla)\phi - \Delta\phi + \nabla q = 0, \quad \nabla \cdot \phi = 0 \quad \text{in} \quad Q, \\ \phi = 0 \quad \text{on} \quad \Sigma, \end{cases}$$

has the *unique continuation property* in $\mathcal{O} \times (0, T)$, that is, if and only if any couple $\{\phi, q\}$ verifying (3) and the equality

$$\phi = 0 \quad \text{in} \quad \mathcal{O} \times (0, T)$$

necessarily satisfies $\phi \equiv 0$. Recently, this property was established in [4]. Consequently, (2) is approximately controllable.

Once y_0, y_d, and ε are fixed, there is a completely natural method for determining the "best" control that drives (2) to a final state $y(T)$ satisfying $|y(T)-y_d| \le \varepsilon$. This is inspired by the usual convex duality techniques, and the conclusions can be sketched as follows (see [7] for a complete description). The control function $\hat{v}$ for which the norm in $L^2(\mathcal{O}\times(0,T))$ attains a minimum is the restriction to $\mathcal{O}\times(0,T)$ of the function $\hat{\phi}$, where

$$\text{(4)} \qquad \begin{cases} -\partial_t\hat{\phi} - (a\cdot\nabla)\hat{\phi} - \Delta\hat{\phi} + \nabla\hat{q} = 0, \quad \nabla\cdot\hat{\phi} = 0 \quad \text{in} \quad Q, \\ \hat{\phi} = 0 \quad \text{on} \quad \Sigma, \\ \hat{\phi}(T) = \hat{\phi}^0. \end{cases}$$

In (4), $\hat{\phi}^0$ is the unique function satisfying

$$\text{(5)} \qquad J(\hat{\phi}^0;a) \le J(\phi^0;a) \quad \forall\phi^0 \in H, \quad \hat{\phi}^0 \in H,$$

with

$$J(\phi^0;a) = \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |\phi|^2\,dx\,dt + \varepsilon|\phi^0| - (y'_d,\phi^0)\,;$$

finally, $y'_d = y_d - Y(T)$, with Y being, together with P, the unique solution to

$$\begin{cases} \partial_t Y + \nabla\cdot(aY) - \Delta Y + \nabla P = 0, \quad \nabla\cdot Y = 0 \quad \text{in} \quad Q, \\ Y = 0 \quad \text{on} \quad \Sigma, \\ Y(0) = y_0. \end{cases}$$

Notice that $a \mapsto \hat{v}$ is a well-defined continuous mapping. This is implied by the fact that $\phi^0 \mapsto J(\phi^0;a)$ is *strictly convex* and *continuous* and satisfies

$$\liminf_{|\phi^0|\to\infty} \frac{J(\phi^0;a)}{(\phi^0)} \ge \varepsilon$$

(a consequence, again, of the unique continuation property).

Now, let $\Lambda_0 : L^2(Q) \mapsto L^\infty(Q)$ be a continuous mapping. For each $z \in L^2(Q)$, let $\Lambda(z) = y$, with y being the solution to (2) with $a = \Lambda_0(z)$ and $v = \hat{v}$ (the corresponding minimal norm control). Then $\Lambda : L^2(Q) \mapsto L^2(Q)$ is *continuous* and *compact*. If we could affirm that Λ maps a ball into itself, we would be able to deduce, by virtue of Schauder's theorem, that it possesses a fixed point $\hat{y}$. Of course, we would have

$$\text{(6)} \qquad \begin{cases} \partial_t\hat{y} + \nabla\cdot(\Lambda_0(\hat{y})\hat{y}) - \Delta\hat{y} + \nabla\hat{p} = 1_{\mathcal{O}}\hat{v}, \quad \nabla\cdot\hat{y} = 0 \quad \text{in} \quad Q, \\ \hat{y} = 0 \quad \text{on} \quad \Sigma, \\ \hat{y}(0) = y_0 \end{cases}$$

and also $|\hat{y}(T) - y_d| \le \varepsilon$.

Consequently, our task is "reduced" to finding a mapping Λ_0 such that:

1. Λ maps a ball of $L^2(Q)$ into itself;
2. there exists a fixed point $\hat{y}$ of Λ such that $\Lambda_0(\hat{y}) = \hat{y}$.

At present, the way that Λ_0 has to be constructed is not known. In fact, the only thing we know is a triviality: the first condition above is verified whenever Λ_0 takes values in a ball of $L^\infty(Q)$. This leads easily to controllability results for systems like (6).

For instance, assume $M > 0$:

$$T_M(s) = \begin{cases} M & \text{if } s > M, \\ s & \text{if } |s| \le M, \\ -M & \text{if } s < -M. \end{cases}$$

Then set $\Lambda_M(\xi) = (T_M(\xi_1), \dots, T_M(\xi_N))$. Let us also denote by Λ_M the corresponding Nemitskii operator, which is well defined and continuous in $L^2(Q)$ and takes values in the ball $B(0; M) \subset L^\infty(Q)$. We then have the following system:

$$(6') \qquad \begin{cases} \partial_t \hat{y} + \nabla \cdot (\Lambda_M(\hat{y})\hat{y}) - \Delta \hat{y} + \nabla \hat{p} = 1_{\mathcal{O}} \hat{v}, \quad \nabla \cdot \hat{y} = 0 \quad \text{in} \quad Q, \\ \hat{y} = 0 \quad \text{on} \quad \Sigma, \\ \hat{y}(0) = y_0. \end{cases}$$

In accordance with the previous argument, once y_0, y_d, and ε are fixed, there exists a $v \in \mathcal{U}$ and a solution $\{y, p\}$ to (6′) satisfying $|y(T) - y_d| \le \varepsilon$.

3. The Analysis of a Galerkin Approximation.

In this section, we will present controllability results for a finite-dimensional (Galerkin) approximation to the Navier–Stokes equations. The main contributors to results of this kind have been Lions and Zuazua (see [9]).

Since we will be working in finite-dimensional spaces, approximate and exact controllability will be equivalent. Furthermore, the state equation will be reversible in time. This means that, for practical purposes, we only have to consider the case in which the initial state vanishes.

Thus, let E be a finite-dimensional subspace of V. For each $v \in \mathcal{U}$, we consider the following approximation to (1):

$$(7) \qquad \begin{cases} y_E : [0, T] \mapsto E, \\ (\partial_t y_E, e) + ((y_E \cdot \nabla) y_E, e) + (\nabla y_E, \nabla e) = (1_{\mathcal{O}} v, e), \quad t \text{ a.e. in } [0, T],\ e \in E, \\ y_E(0) = 0. \end{cases}$$

Let $E_{\mathcal{O}}$ be the linear space formed by the restriction to $\mathcal{O}$ of the functions of E. We will see that, under the assumption

$$(8) \qquad \dim E_{\mathcal{O}} = \dim E,$$

the finite-dimensional system (7) is exactly controllable in E at time T. That is, for each $z_E \in E$, there exists $v \in \mathcal{U}$ such that the corresponding solution to (7) satisfies $y_E(T) = z_E$. We will also provide an estimate of the associated cost.

As in the previous section, it is convenient to begin with a similar linear problem. Let us fix $a \in L^2(0, T; E)$ and consider the following system:

$$(9) \qquad \begin{cases} y_E : [0, T] \mapsto E, \\ (\partial_t y_E, e) + ((a \cdot \nabla) y_E, e) + (\nabla y_E, \nabla e) = (1_{\mathcal{O}} v, e), \quad t \text{ a.e. in } [0, T],\ e \in E, \\ y_E(0) = 0. \end{cases}$$

This is exactly controllable at time T if and only if the corresponding adjoint system

$$
\begin{cases} \phi_E : [0,T] \mapsto E, \\ -(\partial_t \phi_E, e) + (a\phi_E, \nabla e) + (\nabla \phi_E, \nabla e) = 0, \quad t \text{ a.e. in } [0,T],\ e \in E, \\ \phi_E(T) = \phi^0 \end{cases} \tag{10}
$$

has the following property:

$$
\text{if } \phi^0 \in E \text{ and } (y_E(T), \phi^0) = 0 \text{ for all } v \in \mathcal{U}, \text{ then } \phi_E \equiv 0. \tag{11}
$$

But (11) is implied by (8). Consequently, we have exact controllability for (9).

Let us fix $z_E \in E$. We can use arguments similar to those in the previous section to determine the minimal L^2-norm control $\hat{v}$ that drives (9) to z_E. Thus, let us set

$$
I(\phi^0; a) = \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |\phi_E|^2 \, dx \, dt - (z_E, \phi^0) \quad \forall \phi^0 \in E
$$

and let $\hat{\phi}^0$ be the unique function in E satisfying

$$
I(\hat{\phi}^0; a) \leq I(\phi^0; a) \quad \forall \phi^0 \in E. \tag{12}
$$

Then $\hat{v}$ is given by the restriction to $\mathcal{O}\times(0,T)$ of the function $\hat{\phi}_E$, which is determined by $\hat{\phi}^0$ through (10). The corresponding cost is

$$
C(z_E, a) = \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |\hat{\phi}_E|^2 \, dx \, dt.
$$

Note that this quantity can be bounded independently of a. Indeed, there exist constants $C_1(\mathcal{O}, E)$ and $C_2(E)$ such that

$$
\int_{\mathcal{O}} |e|^2 \geq C_1 |e|^2 \quad \text{and} \quad |e|^2 \geq C_2 \|e\|^2 \quad \forall e \in E.
$$

This is a consequence of (8) and the fact that E is finite-dimensional. Hence,

$$
\begin{aligned} C(z_E, a) &= -\inf_{\phi^0 \in E} I(\phi^0; a) \\ &\leq -\inf_{\phi^0 \in E} \left\{ \frac{C_1 C_2 T}{2(2T + C_2)} |\phi^0|^2 - |z_E||\phi^0| \right\} \\ &= \frac{2T + C_2}{2C_1 C_2 T} |z_E|^2. \end{aligned} \tag{13}
$$

Let us now go back to the nonlinear problem (9). Again, let us fix z_E in E. For each $a \in L^2(0,T;E)$, let us denote by $\hat{y}_E$ the solution to (9) corresponding to the control function $\hat{v}$. Then $a \mapsto \hat{y}_E$ is a well defined, continuous, and compact mapping from $L^2(0,T;E)$ into itself. Furthermore, (13) shows that it maps the hole space $L^2(0,T;E)$ into a ball. Consequently, there exists at least one fixed point for this mapping. In other words, there exists $\hat{v} \in \mathcal{U}$, with norm

$$
\|\hat{v}\|^2 \leq \frac{2T + C_2}{C_1 C_2 T} |z_E|^2, \tag{14}
$$

such that the unique solution to

$$\begin{cases} \hat{y}_E : [0,T] \mapsto E, \\ (\partial_t \hat{y}_E, e) + ((\hat{y}\cdot\nabla)\hat{y}_E, e) + (\nabla\hat{y}_E, \nabla e) = (1_{\mathcal{O}}\hat{v}, e), \quad t \text{ a.e. in } [0,T],\ e \in E, \\ \hat{y}_E(0) = 0 \end{cases}$$

satisfies $\hat{y}_E(T) = z_E$. Since z_E is arbitrary in E, this implies exact controllability.

Remark 1. Of course, the bound (14) depends on E. It may be interesting to modify the previous argument by changing exact controllability to ε-approximate controllability at the finite-dimensional level. At that point, it might be reasonable to search for a bound of the cost depending on ε but not on E. □

4. A Variant of the Return Method and Its Consequences. The methods in this section were introduced by Coron (see [1] and the references therein). The main idea is to construct specific solutions $\{y_\alpha, p_\alpha\}$ of the Navier–Stokes equations such that the linearized Euler equations at $\{y_\alpha, p_\alpha\}$ are, in a certain sense, "almost" exactly controllable. Once y_0 and y_d are fixed, this furnishes a first control v^0. In a second step, after a correction of v^0, we are able to drive the Navier–Stokes system to a final state that is close to y_d.

This method has several important limitations. First, we must have $N = 2$; on the other hand, the boundary conditions have to be of the Navier slip type. In practice, this is equivalent to prescribing the values on the boundary of the stream function and the vorticity function. Furthermore, we obtain approximate controllability only in $W^{-1,\infty}(\Omega)$, not in $L^2(\Omega)$ (however, the same arguments lead to approximate controllability in $W^{1,\infty}(K)$ for each compact set $K \subset \Omega$).

Let us assume that the state equation is

$$\begin{cases} \partial_t y + (y\cdot\nabla)y - \Delta y + \nabla p = 1_{\mathcal{O}} v, \quad \nabla\cdot y = 0 \quad \text{in} \quad Q, \\ y\cdot n = 0, \quad \nabla\times y = 0 \quad \text{on} \quad \Sigma, \\ y(0) = y_0 \quad \text{in} \quad \Omega \end{cases} \tag{15}$$

(where $\Omega \subset \mathbf{R}^2$). Let us fix y_d (for simplicity, regular enough). The argument is as follows:

1. For each $\alpha > 0$, it is possible to find v_α and a couple, $\{y_\alpha, p_\alpha\}$, such that y_α is a gradient outside $\mathcal{O}\times(0,T)$,

$$\begin{cases} \partial_t y_\alpha + (y_\alpha\cdot\nabla)y_\alpha - \Delta y_\alpha + \nabla p_\alpha = 1_{\mathcal{O}} v_\alpha, \quad \nabla\cdot y_\alpha = 0 \quad \text{in} \quad Q, \\ y_\alpha\cdot n = 0, \quad \nabla\times y_\alpha = 0 \quad \text{on} \quad \Sigma, \\ y_\alpha(0) = y_\alpha(T) = 0 \quad \text{in} \quad \Omega \end{cases} \tag{16}$$

and, furthermore, the linearized Euler system at y_α, that is,

$$\begin{cases} \partial_t z + (y_\alpha\cdot\nabla)z + (z\cdot\nabla)y_\alpha + \nabla\pi = 1_{\mathcal{O}} w, \quad \nabla\cdot z = 0 \quad \text{in} \quad Q, \\ z\cdot n = 0 \quad \text{on} \quad \Sigma, \end{cases} \tag{17}$$

is α-controllable in the following sense: for any given z_0 and z_d of class C^∞, there exists a control $w \in \mathcal{U}$ such that (17) possesses at least one solution satisfying

$$z(0) = z_0 \quad \text{in} \quad \Omega$$

and also

$$z(T) = z_d \quad \text{in} \quad \{\, x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) \geq \alpha \,\},$$

and is bounded in $C^3(\overline{Q})$ independently of α. (Note that we are saying that, for every α, this property is satisfied for all z_0 and z_d.)

2. Let us set $v(x,t) = 0$ in (15) for $t \in [0, (1-\delta)T]$ (δ will be determined below). This defines $\{y, p\}$ without ambiguity in $\Omega \times [0, (1-\delta)T]$, and, in particular, we can speak of $y((1-\delta)T)$. In $[(1-\delta)T, T]$, we do the following:

a. First, v_α and $\{y_\alpha, p_\alpha\}$ are rescaled. On the basis of the α-controllability of (17), we introduce a first control function v^0:

$$v^0(x,t) \equiv \frac{1}{\delta} v_\alpha \left(x, \frac{1}{\delta}(t - (1-\delta)T) \right) + w \left(x, \frac{1}{\delta}(t - (1-\delta)T) \right).$$

The associated state is

$$\{y^0, p^0\} \equiv \frac{1}{\delta}\{y_\alpha, p_\alpha\} \left(x, \frac{1}{\delta}(t - (1-\delta)T) \right) + \{z, \pi\} \left(x, \frac{1}{\delta}(t - (1-\delta)T) \right).$$

Here, w and $\{z, \pi\}$ are perturbations corresponding to α (which will also be fixed below), the initial state $y((1-\delta)T)$, and the desired state y_d. In order to drive (15) to a final state close to y_d, it is natural (at least formally) to look for a control close to v^0 for $t \in [(1-\delta)T, T]$.

b. We introduce a second control function by modifying v^0 as needed. Thus, we solve the following problem:

$$(18) \quad \begin{cases} \partial_t y + (y \cdot \nabla) y - \Delta y + \nabla p = 1_{\mathcal{O}} v^0 + (\nabla \times \overline{y})(y - y^0)^\perp, \quad \nabla \cdot y = 0 \quad \text{in} \quad Q, \\ \hfill y \cdot n = 0, \quad \nabla \times y = 0 \quad \text{on} \quad \Sigma, \\ \hfill y(0) = y_0 \quad \text{in} \quad \Omega. \end{cases}$$

In (18), we have introduced the following notation:

$$\overline{y}(x,t) = \frac{1}{\delta} y_\alpha \left(x, \frac{1}{\delta}(t - (1-\delta)T) \right), \quad (a_1, a_2)^\perp = (-a_2, a_1).$$

Since $\nabla \times \overline{y}$ vanishes outside $\mathcal{O} \times (0,T)$, it is clear that (18) can be written in the form (15), with

$$v = v^0 + (\nabla \times \overline{y})(y - y^0)^\perp.$$

Now, the task is reduced to showing that, for every $\varepsilon > 0$, there exist positive α and δ such that

$$(19) \qquad \|y(T) - y_d\|_{W^{-1,\infty}} \leq \varepsilon.$$

This can be achieved in the following way. Let us set $R = y - y^0$, $\omega = \nabla \times R$. Then

$$\partial_t \omega + (R + y^0) \cdot \nabla \omega - \Delta \omega = -(R + z) \cdot \nabla(\nabla \times z) + \Delta(\nabla \times z)$$

in $\Omega \times ((1-\delta)T, T)$. Furthermore, $R(x, (1-\delta)T) \equiv 0$ and z and all its derivatives of order ≤ 3 are uniformly bounded. This leads first to a pointwise estimate of ω

and then to an estimate of $R(\cdot, T)$ in $W^{-1,\infty}(\Omega)$ when α and δ are sufficiently small. Consequently, for any given $\varepsilon > 0$, there exist $\alpha^0 > 0$ and $\eta : (0, \alpha^0) \mapsto \mathbf{R}_+$ such that, whenever $0 < \alpha < \alpha^0$ and $0 < \delta < \eta(\alpha)$, one has (19) (for further details, see [1]).

It is only at this last step of the argument that the type of boundary condition becomes important. At present, it is not known how the method has to be modified in order to maintain its validity in the context of (1). Moreover, it is not clear whether it can be generalized to a similar three-dimensional situation (see, however, [2]).

5. Local Results Concerning Null Controllability. In this section, we will refer to the null controllability of (1). The intention is, in a second step, to prove exact controllability to any regular solution and, as a consequence, to prove approximate controllability. The most important contributions in this context are those of Fursikov and Imanuvilov (see [5], [6], and the references therein).

Again, there are important limitations. In particular, only local results can be proved: starting from an initial state that is close to zero, we can control the system in such a way that the final state is exactly zero at $t = T$. For simplicity, we will present the argument when Ω is a bounded simply connected domain of $\mathbf{R}^2$, the boundary conditions are of the Navier slip type (as in (15)), and the control is the trace of the vorticity function ω on a portion γ of the boundary.

Remark 2. When $N = 2$ and several other additional conditions are satisfied, a combination of the results presented in this and the previous section leads to global null controllability, that is, with no restriction on the size of the initial data (see [2]). □

Let $\gamma \subset \partial\Omega$ be a nonempty open set. With standard notation, the problem is to find a real-valued function $h = h(x, t)$, defined and regular enough on $\gamma \times (0, T)$, such that the solution $\{\omega, \psi, y\}$ to

$$(20) \qquad \begin{cases} \partial_t \omega + \nabla \cdot (\omega y) - \Delta\omega = 0, \quad -\Delta\psi = \omega, \quad y = \nabla \times \psi & \text{in} \quad Q, \\ \omega = h 1_\gamma, \quad \psi = 0 & \text{on} \quad \Sigma, \\ \omega(0) = \nabla \times y_0 & \text{in} \quad \Omega \end{cases}$$

verifies $\omega(T) = 0$. Observe that, for any given $y_0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$ and $h \in C^1(\overline{\gamma} \times [0, T])$, there exists exactly one solution $\{\omega, \psi, y\}$ to (20). Among other things, one has $\omega \in C^0(0, T; H^1(\Omega))$.

We are going to introduce a sequence $\{\omega^n, \psi^n, y^n, h^n\}$ that converges, when the initial state y_0 is "small," towards a solution $\{\omega, \psi, y, h\}$ to (20). By definition, each ω^n will solve a null controllability problem for a linear system. In particular, this will give $\omega(T) = 0$. The sequence $\{\omega^n, \psi^n, y^n, h^n\}$ is defined as follows:

a. First, we choose a constant $R > 0$ and we take $\omega^0 \equiv 0$, $h^0 \equiv 0$.

b. Then, once $n \geq 0$ and ω^n and h^n are given, we set

$$(21) \qquad \overline{\omega}^n = T_R(\omega^n), \quad \psi^n = (-\Delta)^{-1}\overline{\omega}^n, \quad y^n = \nabla \times \psi^n \,;$$

finally, ω^{n+1} is (together with h^{n+1}) the solution to the null controllability problem

$$(22) \qquad \begin{cases} \partial_t \omega^{n+1} + \nabla \cdot (\omega^{n+1} y^n) - \Delta\omega^{n+1} = 0 & \text{in} \quad Q, \\ \omega^{n+1} = h^{n+1} 1_\gamma & \text{on} \quad \Sigma, \\ \omega^{n+1}(0) = \nabla \times y_0, \quad \omega^{n+1}(T) = 0 & \text{in} \quad \Omega \end{cases}$$

furnished by the method of Fursikov and Imanuvilov (see [5]).

Let $M > 0$ be such that

$$\|\omega\|_{L^\infty} \leq R \;\Rightarrow\; \|\nabla \times ((-\Delta)^{-1}\omega)\|_{L^\infty} \leq M.$$

Then every y^n satisfies $\|y^n\|_{L^\infty} \leq M$. A first consequence is that the corresponding functions ω^n are uniformly bounded in $L^\infty(Q)$ and $L^2(0,T;H^2(\Omega))$. More precisely,

$$\|\omega^n\|_{L^2(0,T;H^2(\Omega))} + \|\omega^n\|_\infty \leq C(\|y_0\|_{H^2} + \|y_0\|_{W^{1,\infty}}),$$

where C only depends on Ω, γ, T, and M. From these estimates, it can be deduced in a second step that, under the assumption

$$\|y_0\|_{H^2} + \|y_0\|_{W^{1,\infty}} \leq \varepsilon, \tag{23}$$

one has

$$\|\omega^{n+1} - \omega^n\|_{L^2(0,T;H^2(\Omega))} \leq \eta(\varepsilon)\|\omega^n - \omega^{n-1}\|_{L^2(0,T;H^2(\Omega))} \quad \forall n \geq 1, \tag{24}$$

where $\lim_{\varepsilon\to 0^+} \eta(\varepsilon) = 0$. Hence, the complete sequences ω^n, ψ^n, y^n, and h^n converge in appropriate spaces provided (23) is satisfied for a sufficiently small ε.

Remark 3. There is another way to present the same argument. Let

$$\Phi_R : L^2(0,T;H^2(\Omega)) \mapsto L^2(0,T;H^2(\Omega))$$

be as follows: $\Phi_R(z) = \omega$ if and only if ω is (together with ψ, y, and h) the solution to

$$\begin{cases} \partial_t\omega + \nabla\cdot(\omega y) - \Delta\omega = 0, \quad -\Delta\psi = T_R(z), \quad y = \nabla\times\psi & \text{in} \quad Q, \\ \omega = h1_\gamma & \text{on} \quad \Sigma, \\ \omega(0) = \nabla\times y_0, \quad \omega(T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{25}$$

obtained as in [5]. If $\|y_0\|_{H^2} + \|y_0\|_{W^{1,\infty}}$ is sufficiently small, then Φ_R is a contraction. □

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