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# A review of basic theoretical results concerning the Navier-Stokes and other similar equations

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**Abstract**

These notes are devoted to provide an introductory approach to the Navier-Stokes and some other related equations. Most concepts and arguments recalled below are very general and we believe that this presentation can be of help for the theoretical analysis of many PDE's arising from Sciences and Engineering. First, we recall the Navier-Stokes equations, we explain the meaning of the variables and data and we state some technical results needed for our study. Then, we state and give the proofs of some basic existence, uniqueness and regularity results. In the proof of existence, we apply usual compactness arguments to a family of Galerkin approximations. We also discuss briefly some of the main open problems arising in the three-dimensional case. In a final section, we review briefly the state of the art for other similar equations and we indicate some related open questions.

**Key words:** *Navier-Stokes equations; nonlinear PDE's in fluid mechanics; compactness methods for nonlinear PDE's; Galerkin's approximations; existence, uniqueness and regularity results.*

**AMS subject classifications:** *35Q30, 35Q35, 35K55, 65M60*

## 1 Introduction. Formulation of the problem and main results

In these notes, $\Omega \subset \mathbb{R}^N$ is a bounded connected open set at least of class $C^{0,1}$ ($N = 2$ or $N = 3$) and we have $0 < T \leq +\infty$. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$ and we will denote by $C$ a generic positive constant, usually depending on $\Omega$, $T$ and maybe other data.

We will first be concerned with the nonlinear problem

$$\begin{cases} u_t + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = u^0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $\nu > 0$, $f = f(x,t)$ and $u^0 = u^0(x)$ are given.

This serves to model the behavior of a Newtonian viscous incompressible fluid whose particles fill the spatial domain $\Omega$ during the time interval $(0,T)$.

In (1), the unknowns are the $\mathbb{R}^N$-valued function $u = u(x,t)$ (the velocity field) and the real-valued function $p = p(x,t)$ (the pressure). The data are $f = f(x,t)$ (a density of external forces) and $u^0 = u^0(x)$ (an initial velocity field). It is assumed that the mass density of the fluid is equal to 1. The first equality is the conservation of momentum law, i.e. Newton's second law, written along the trajectories. The second one indicates that the fluid is incompressible, i.e. that the volume occupied by a set of particles is independent of time. We have complemented these equations with boundary conditions on $\Sigma$ that express that the particles adhere to the wall (and therefore do not slip) and initial conditions at time $t = 0$.

Our first aim in this paper is to recall the main known existence, uniqueness and regularity results that hold for (1), as well as the main ideas needed in their proofs; a complete analysis can be found for instance in [7, 13, 19, 20, 22, 31, 33]. In view of the generality of the concepts and arguments presented below, we believe that this can be of help for the theoretical analysis of the Navier-Stokes and many other PDE's arising from Sciences and Engineering.

From the viewpoint of mechanics, to try to solve (1) is full of sense: we assume that the mechanical state of the fluid at time $t = 0$ and the external forces acting on the fluid during $(0,T)$ are known and we try to determine the mechanical state for $t \in (0,T)$.

However, the equations in (1) are not always appropriate for the description of the flow of an incompressible fluid. Thus, there are many (realistic) modifications of (1) that lead to related mathematical problems. Let us mention some of them:

• The term $-\nu\Delta u$ in (1) is the contribution of viscosity to the motion of particles. In some *idealized* situations, it may be adequate to assume that $\nu = 0$ (this means that the role of viscosity is negligible). Then we find the so called Euler equations:

$$u_t + (u \cdot \nabla)u + \nabla p = f, \quad \nabla \cdot u = 0. \tag{2}$$

• Sometimes, it is more accurate to assume that the mass density of the fluid is not a constant. Then we must introduce an additional unknown in (1), the positive real-valued function $\rho = \rho(x,t)$ and we must complete (1) with the *mass conservation* law. The resulting equations are the following:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho\,(u_t + (u \cdot \nabla)u) - \mu\Delta u + \nabla p = \rho f, \\ \nabla \cdot u = 0. \end{cases} \tag{3}$$

These are the nonhomogeneous incompressible Navier-Stokes equations.

• For more complex flows, $\nu$ is not a constant but depends on the mechanical state of the fluid. It is then usual to assume that $\nu$ is a positive function of $|Du|$, where $Du = \frac{1}{2}(\nabla u + \nabla u^t)$ is the symmetric part of the gradient of $u$. This leads to the equations for the so called *quasi-Newtonian* fluids:

$$u_t + (u \cdot \nabla)u - 2\nabla \cdot (\nu(|Du|)Du) + \nabla p = f, \quad \nabla \cdot u = 0. \tag{4}$$

• More generally, it may happen that viscous effects depend on $u$ globally, for instance through the solution of a transport equation governed by $u$. There are many situations where this is the right way to model the fluid. For instance, this is the case for the so called *visco-elastic* fluids of the Oldroyd kind, which obey to the following system:

$$\begin{cases} u_t + (u \cdot \nabla)u - \nu\Delta u + \nabla p = \nabla \cdot \tau + f, \\ \nabla \cdot u = 0, \\ \tau_t + (u \cdot \nabla)\tau + c\tau + g_a(\nabla u, \tau) = bDu. \end{cases} \tag{5}$$

Here, $\tau = \tau(x,t)$ is a symmetric tensor known as the extra elastic stress tensor, $c$ and $b$ are positive constants and $g_a : \mathbb{R}^{N\times N} \times \mathbb{R}^{N\times N} \mapsto \mathbb{R}^{N\times N}$ is an appropriate bilinear tensor-valued function.

For more details on these and other equations arising in fluid mechanics, see for instance [6, 11, 15, 22, 27, 29]. See also section 5 for a brief review of known results.

There are mainly two reasons to consider problem (1). First, as mentioned above, it can be used for the description of an important family of flows. On the other hand, from a mathematical viewpoint, the analysis of (1) is of high interest, since it leads in practice to the main difficulties one usually encounters when dealing with nonlinear PDE's. To get an idea of the large number of relevant open questions raised by (1), see [14].

Of course, before presenting the main results, we have to give a sense to (1) and specify the kind of solution we are looking for.

We will need some function spaces and basic results. First, let us introduce

$$\mathcal{V} := \{ \varphi \in C_0^\infty(\Omega)^N : \nabla \cdot \varphi = 0 \ \text{ in } \ \Omega \},$$

where $C_0^\infty(\Omega)$ stands for the space formed by the functions $\varphi : \Omega \mapsto \mathbb{R}$ of class $C^\infty$ with compact support in $\Omega$. Then we have the following well known De Rham's lemma:

**Lemma 1** *Let $S \in \mathcal{D}'(\Omega)^N$ be given, with*

$$\langle S, \varphi \rangle = 0 \quad \forall \varphi \in \mathcal{V}.$$

*Then there exists $q \in \mathcal{D}'(\Omega)$ such that $S = \nabla q$.*

Another version of De Rham's lemma will be given below; see lemma 13 in section 3.

We will denote by $H$ (resp. $V$) the adherence of $\mathcal{V}$ in the Hilbert space $L^2(\Omega)^N$ (resp. in $H_0^1(\Omega)^N$). Of course, $H$ (resp. $V$) is a new Hilbert space for the norm of $L^2(\Omega)^N$ (resp. the norm of $H_0^1(\Omega)^N$), which will be denoted by $|\cdot|$ (resp. $\|\cdot\|$).

Moreover, we have

$$H = \{\, v \in L^2(\Omega)^N : \nabla \cdot v = 0 \;\text{ in }\; \Omega,\; v \cdot n = 0 \;\text{ on }\; \partial\Omega \,\}. \tag{6}$$

Here, $n(x)$ is by definition the outward normal vector to $\Omega$ at $x$ (a point of $\partial\Omega$). It is known that, for any $v \in L^2(\Omega)^N$ such that $\nabla \cdot v \in L^2(\Omega)$, we can give a sense to the normal trace $v \cdot n$ on $\partial\Omega$ in a space that contains $L^2(\partial\Omega)$. This justifies (6).

As a consequence of lemma 1, we also find that

$$V = \{\, v \in H_0^1(\Omega)^N : \nabla \cdot v = 0 \;\text{ in }\; \Omega \,\}. \tag{7}$$

Another property of the Hilbert spaces $H$ and $V$ is the following:

$$V \hookrightarrow H \equiv H' \hookrightarrow V', \tag{8}$$

with dense and compact embeddings; furthermore, $V'$ can be identified (isomorphically and isometrically) to the quotient space $H^{-1}(\Omega)^N / \nabla L^2(\Omega)$. In other words, the "points" of $V'$ can be viewed as the classes of $H^{-1}(\Omega)^N$ determined by the following equivalence relation

$$f \sim g \;\text{ if and only of }\; f - g = \nabla q \;\text{ for some }\; q \in L^2(\Omega).$$

We will also need to speak of the space of distributions $\mathcal{D}'(D; X)$, where $D \subset \mathbb{R}^m$ is an open set and $X$ is a Banach space; very often, we will have $D = (0, T)$. By definition, $\mathcal{D}'(D; X)$ is the space of linear sequentially continuous mappings $S : \mathcal{D}(D) \mapsto X$[1].

For given $S \in \mathcal{D}'(D; X)$ and $\varphi \in \mathcal{D}(D)$, we will denote by $\langle S, \varphi \rangle_{\mathcal{D}'(D;X),\mathcal{D}(D)}$ or more simply $\langle S, \varphi \rangle$ the point of $X$ assigned by $S$ to $\varphi$. We will say that the sequence $\{S_n\}$ converges to $S$ in $\mathcal{D}'(D; X)$ if

$$\langle S_n, \varphi \rangle \to \langle S, \varphi \rangle \;\text{ in }\; X \;\text{ for every }\; \varphi \in \mathcal{D}(D).$$

Every $f \in L^1_{\text{loc}}(D; X)$ determines uniquely a distribution $S_f \in \mathcal{D}'(D; X)$ through the formula

$$\langle S_f, \varphi \rangle = \int_D f(\xi)\varphi(\xi)\, d\xi \quad \forall \varphi \in \mathcal{D}(D).$$

Furthermore, the mapping $f \mapsto S_f$ is linear, sequentially continuous and one-to-one. Accordingly, $L^1_{\text{loc}}(D; X)$ can be identified to a subspace of $\mathcal{D}'(D; X)$

[1] Recall that $\varphi_n \to \varphi$ in $\mathcal{D}(D)$ if all the supports of the functions $\varphi_n$ are contained in the same compact set $K \subset \Omega$ and any derivative of any order of $\varphi_n$ converges uniformly in $K$ to the corresponding derivative of $\varphi$.

and $S_f$ can be denoted by $f$. This will be made in the sequel. In particular, for any $p \in [1, +\infty]$, $L^p(D; X)$ is also a subspace of $\mathcal{D}'(D; X)$.

Notice however that there are (many) distributions in $\mathcal{D}'(D; X)$ that do not belong to $L^1_{\rm loc}(D; X)$. Indeed, if $\xi$ is a point of $D$ and we set

$$\langle \delta_\xi, \varphi \rangle = \varphi(\xi) \quad \forall \varphi \in \mathcal{D}(D),$$

then $\delta_\xi \in \mathcal{D}'(D; X)$, but there is no function $f \in L^1_{\rm loc}(D; X)$ such that $\delta_\xi = S_f$.

If $S \in \mathcal{D}'(D; X)$ is given, we can give a sense to any derivative of $S$ of any order. Thus, if $\alpha = (\alpha_1, \dots \alpha_m)$ is a standard multi-index and we set $|\alpha| = \alpha_1 + \cdots + \alpha_m$, $\partial^\alpha S$ is by definition the $X$-valued distribution given as follows:

$$\langle \partial^\alpha S, \varphi \rangle = (-1)^{|\alpha|} \langle S, \partial^\alpha \varphi \rangle \quad \forall \varphi \in \mathcal{D}(D).$$

In particular, we can speak of any derivative of a function in $L^1_{\rm loc}(D; X)$. Notice that, for each $\alpha$, the linear operator $\partial^\alpha : \mathcal{D}'(D; X) \mapsto \mathcal{D}'(D; X)$ is sequentially continuous.

The following results are well known. Their proofs can be found for instance in [8] and [31].

**Lemma 2** *Let $X$ be a Banach space. Assume that $1 \le p_0, p_1 \le +\infty$, $f \in L^{p_0}(0, T; X)$ and $f_t \in L^{p_1}(0, T; X)$. Then $f \in C^0([0, T]; X)$ and we have the estimate*

$$\|f\|_{C^0([0,T];X)} \le C \left( \|f\|_{L^{p_0}(0,T;X)} + \|f_t\|_{L^{p_1}(0,T;X)} \right),$$

*where $C$ only depends on $p_0$ and $p_1$.*

**Lemma 3** *Let $V$ and $H$ be Hilbert spaces satisfying (8) with dense and continuous embeddings. Assume that $1 < p < +\infty$, $f \in L^p(0, T; V)$ and $f_t \in L^{p'}(0, T; V')$. Then $f \in C^0([0, T]; H)$ and we have the estimate*

$$\|f\|_{C^0([0,T];H)} \le C \left( \|f\|_{L^p(0,T;V)} + \|f_t\|_{L^{p'}(0,T;V')} \right),$$

*where $C$ only depends on $p$. Furthermore, the function $t \mapsto \|f(t)\|_H^2$ is absolutely continuous and one has*

$$\frac{d}{dt} \|f(t)\|_H^2 = 2 \langle f_t(s), f(s) \rangle_{V',V} \quad \text{a.e. in} \ \ (0, T). \tag{9}$$

*Consequently, the following identity holds for any $t_1, t_2 \in [0, T]$:*

$$\frac{1}{2} \|f(t_2)\|_H^2 - \frac{1}{2} \|f(t_1)\|_H^2 = \int_{t_1}^{t_2} \langle f_t(s), f(s) \rangle_{V',V} \, ds. \tag{10}$$

**Lemma 4** *Let $X$ and $Y$ be Banach spaces. Assume that $X$ is reflexive, $X \hookrightarrow Y$ with a continuous embedding and $v \in L^\infty(0, T; X) \cap C^0([0, T]; Y)$. Then $v \in C^0_w([0, T]; Y)$, i.e. for every $L \in Y'$ the real-valued function $t \mapsto \langle L, v(t) \rangle_{Y',Y}$ is continuous.*

Following [20], we can now present a first rigorous formulation of (1):

Problem I: Given $f \in L^2(Q)^N$ and $u^0 \in H$, find $u$ and $p$ such that

$$\begin{cases} u \in L^2(0,T;V) \cap L^\infty(0,T;H), \quad p \in \mathcal{D}'(Q), \\ u_t + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f \ \text{ in } \ \mathcal{D}'(Q), \\ u|_{t=0} = u^0. \end{cases} \tag{11}$$

It will be seen below that any function $u$ satisfying $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and $u_t + (u\cdot\nabla)u - \nu\Delta u + \nabla p = f$ in $\mathcal{D}'(Q)$ for some $p \in \mathcal{D}'(Q)$ also satisfies $u_t \in L^\sigma(0,T;V')$ for an appropriate $\sigma > 1$. Thus, in view of lemma 2, $u \in C^0([0,T];V')$ and it is meaningful to speak of $u|_{t=0}$ and to ask for the initial condition in (11) *at least as an equality in* $V'$.

Notice that, if $u$ and $p$ solve (11), we automatically have $u(\cdot,t) \in V$ for $t$ a.e. in $(0,T)$. Consequently, we have in some sense $\nabla \cdot u = 0$ in $Q$ and $u = 0$ on the lateral boundary $\Sigma$.

Let us now give a second formulation of (1):

Problem II: Given $f \in L^2(Q)^N$ and $u^0 \in H$, find $u$ such that

$$\begin{cases} u \in L^2(0,T;V) \cap L^\infty(0,T;H), \\ \langle u_t, v\rangle_V + b(u,u,v) + \nu a(u,v) = \langle \ell, v\rangle \quad \forall v \in V, \\ u|_{t=0} = u^0. \end{cases} \tag{12}$$

Here, $\langle\cdot\,,\cdot\rangle_V$ stands for the duality pairing associated to $V$ and $V'$, $\ell = \ell(t)$ with

$$\langle \ell(t), v\rangle_V = \int_\Omega f(x,t)v(x)\,dx \quad \forall v \in V, \quad t \in [0,T] \ \text{ a.e.}$$

and we have introduced the bilinear and trilinear forms $a(\cdot\,,\cdot)$ and $b(\cdot\,,\cdot\,,\cdot)$, given as follows:

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx \quad \forall u,v \in V, \tag{13}$$

$$b(u,v,w) = \int_\Omega (u\cdot\nabla)v \cdot w\,dx \quad \forall u,v,w \in V. \tag{14}$$

Since $f \in L^2(Q)^N$, we have $\ell \in L^2(0,T;V')$ and $\langle \ell, v\rangle_V \in L^2(0,T)$ for any $v \in V$. On the other hand, if $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$, it is not difficult to check that, for each $v \in V$, we have $a(u,v) \in L^2(0,T)$ and, at least,

$$b(u,u,v) \in L^1(0,T) \ \text{ and } \ \langle u_t, v\rangle \in \mathcal{D}'(0,T).$$

Consequently, the equalities in (12) can be understood in the sense of $\mathcal{D}'(0,T)$.

Let us introduce the linear operator $A : V \mapsto V'$, with

$$\langle Au, v\rangle_V = a(u,v) \quad \forall u,v \in V$$

and the bilinear operator $B : V \times V \mapsto V'$, with

$$\langle B(u,v), w\rangle_V = b(u,v,w) \quad \forall u,v,w \in V.$$

Then an equivalent formulation of (12) is the following:

$$\begin{cases} u \in L^2(0,T;V) \cap L^\infty(0,T;H), \\ u_t + B(u,u) + \nu Au = \ell \ \text{ in } \ \mathcal{D}'(0,T;V'), \\ u|_{t=0} = u^0. \end{cases} \tag{15}$$

It will be seen below that any function $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ that satisfies $u_t + B(u,u) + \nu Au = \ell$ belongs to $C^0([0,T];V')$. Thus, the initial conditions in (12) and (15) again make sense.

The main results concerning the existence and uniqueness of solution for problem II are the following:

**Theorem 5** *Assume that $f \in L^2(Q)^N$ and $u^0 \in H$ are given. Then there exists at least one solution of problem II.*

**Theorem 6** *Assume that $N = 2$ and $f \in L^2(Q)^2$ and $u^0 \in H$ are given. Then problem II possesses exactly one solution.*

We will see in section 3 that any solution of problem II is, together with some $p$, a solution of problem I. As a consequence, theorems 5 and 6 show that the original system (1) can be solved in an appropriate class.

## 2 Proof of uniqueness

In this section, we will prove that, when $N = 2$, problem II possesses at most one solution. We will need some previous results:

**Lemma 7** *Assume that $N = 2$. Then there exists a constant $C > 0$ such that*

$$\|v\|_{L^4} \le C|v|^{1/2}\|v\|^{1/2} \quad \forall v \in H_0^1(\Omega). \tag{16}$$

The proof can be found in [20]. A consequence of this lemma is the following:

**Lemma 8** *Assume that $N = 2$. Then for every $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ one has $B(v,v) \in L^2(0,T;V')$. Furthermore, there exists a constant $C > 0$ such that*

$$\|B(v,v)\|_{L^2(0,T;V')} \le C\|v\|_{L^\infty(0,T;H)}\|v\|_{L^2(0,T;V)} \tag{17}$$

*for all such $v$.*

**Proof:** Assume that $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$. Let us try to estimate $\langle B(v(t),v(t)), w\rangle_V$ for each $w \in V$.

We have:

$$\langle B(v(t),v(t)),w\rangle_V = \int_\Omega (v(t)\cdot\nabla)v(t)\cdot w\,dx = -\int_\Omega (v(t)\cdot\nabla)w\cdot v(t)\,dx$$
$$\leq C\,\|v(t)\|_{L^4}^2\|w\| \leq C\,|v(t)|\,\|v(t)\|\,\|w\|.$$

Hence,

$$\|B(v(t),v(t))\|_{V'} \leq C\,|v(t)|\,\|v(t)\|$$

for $t$ a.e. in $(0,T)$ and we obviously have $B(v,v)\in L^2(0,T;V')$ and (17). ∎

**Remark 1** When $N=3$, the estimates (16) and (17) do not hold. In this case, instead of (16) we only have

$$\|v\|_{L^4} \leq C|v|^{1/4}\|v\|^{3/4} \quad \forall v\in H_0^1(\Omega). \tag{18}$$

Accordingly, it can be proved that for every $v\in L^2(0,T;V)\cap L^\infty(0,T;H)$ one has $B(v,v)\in L^{4/3}(0,T;V')$ and the estimates

$$\|B(v(t),v(t))\|_{V'} \leq C\,|v(t)|^{1/2}\|v(t)\|^{3/2}$$

a.e. in $(0,T)$ and

$$\|B(v,v)\|_{L^{4/3}(0,T;V')} \leq C\|v\|_{L^\infty(0,T;H)}^{1/2}\|v\|_{L^2(0,T;V)}^{3/2}, \tag{19}$$

but nothing better than this. ∎

Now, assume that $N=2$ and $u$ and $u'$ are two solutions of (15), where the data $f\in L^2(Q)^2$ and $u^0\in H$ are given.

Observe that, in this case, $u_t$ and $u'_t$ belong to $L^2(0,T;V')$.

Indeed, we have

$$u_t = \ell - B(u,u) - \nu Au.$$

It is immediate that $\ell, Au\in L^2(0,T;V')$ and, on the other hand, in view of lemma 8, we also have $B(u,u)\in L^2(0,T;V')$. Consequently, $u_t\in L^2(0,T;V')$. A similar argument holds for $u'_t$.

Let us set $w=u-u'$. Then $w\in L^2(0,T;V)\cap L^\infty(0,T;H)$, $w_t\in L^2(0,T;V')$,

$$w_t + \nu Aw = -B(u,u) - B(u',u') \equiv -B(u,w) - B(w,u') \tag{20}$$

and $w|_{t=0}=0$.

In view of lemma 3, $w\in C^0([0,T];H)$ and we have

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 = \langle w_t(t),w(t)\rangle_V \quad \text{a.e. in } (0,T).$$

This, together with (20), yields the following:

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\|w(t)\|^2 = -b(w(t),u'(t),w(t))$$
$$\leq C\,|w(t)|\,\|w(t)\|\,\|u'(t)\|$$
$$\leq \frac{\nu}{2}\|w(t)\|^2 + C\,\|u'(t)\|^2|w(t)|^2.$$

After integration in time, we deduce at once that

$$|w(t)|^2 \leq C \int_0^t \|u'(s)\|^2 |w(s)|^2 \, ds \quad \forall t \in [0,T] \tag{21}$$

and, from Gronwall's lemma, we find that $w \equiv 0$ and $u$ and $u'$ must coincide. This ends the proof of theorem 6. ■

**Remark 2** With an argument a little more complicate, it can also be proved that the solution $u$ of problem II depends continuously of the data $f$ and $u^0$. For more details, see for instance [31]. ■

**Remark 3** In general, the uniqueness of solution of (12) with $N = 3$ and not necessarily small data $f$ and $u^0$ is unknown. Actually, this is a major open problem in Navier-Stokes theory[2]. ■

When $N = 3$, we have uniqueness of regular solution. For instance, if the solution of (12) satisfies

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H) \cap L^s(0,T;L^r(\Omega)^3)$$

with $2/s + 3/r \leq 1$ and $r > 3$, then $u$ is the unique solution in this class.

This is a result from [20] (see also [28]) that can be proved as follows:

• It is sufficient to consider the case in which $2/s + 3/r = 1$ and $r > 3$. Let $u$ and $u'$ be two solutions with the previous regularity and let us set $w = u - u'$. Then $u_t$ and $u'_t$ belong to $L^2(0,T;V')$. Indeed, we have for instance that

$$|\langle B(u(t), u(t), v\rangle_V| \leq |u(t)|^{2/s} \|u(t)\|^{3/r} \|u(t)\|_{L^r} \|v\|$$

for all $v \in V$ and the function $|u|^{2/s}\|u\|^{3/r}\|u\|_{L^r} \in L^2(0,T)$. Consequently, we also have $w_t \in L^2(0,T;V')$.

• Proceeding as in the proof of theorem 6, we have

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\|w(t)\|^2 = -b(w(t), u'(t), w(t))$$

a.e. in $(0,T)$. But now

$$|\langle B(u(t), u(t), v\rangle_V| \leq |u(t)|^{2/s} \|u(t)\|^{3/r} \|u(t)\|_{L^r} \|v\|,$$

whence we have again (21) and $w \equiv 0$.

**Remark 4** More recently, under the assumption

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H) \cap C^0(0,T;L^3(\Omega)^3),$$

the uniqueness of solution of (12) with $N = 3$ has been established; see [23]. ■

[2] When $N = 3$, regularity results for non necessarily small data are also open; see section 4.

## 3 Proof of existence. Galerkin approximations and the compactness method

This section is devoted to provide the proof of theorem 5. We will use a compactness method in combination with a Galerkin approximation scheme with special basis.

More precisely, we will first choose a very particular basis of $V$, we will construct a family of approximated solutions $u^m$ $(m \geq 1)$ and we will deduce appropriate estimates for $u^m$ and $u_t^m$ which lead to the existence of "weakly convergent" subsequences. Then, we will check that any of these subsequences converge strongly somewhere and consequently the corresponding limit is a solution to problem II. This last point is needed (and even crucial) in the proof, since the problem is nonlinear and therefore, roughly speaking, weak convergence does not suffice to pass to the limit in the equations.

Let us then begin with the proof. Recall that, in the sequel, $\langle \cdot\,, \cdot \rangle_V$ stands for the duality pairing connecting $V$ and $V'$; we will denote by $(\cdot\,, \cdot)$ the usual scalar products in $L^2(\Omega)$ and $H$.

We will use the following well known result, which is a consequence of Hilbert-Schmidt theorem and the compactness and density of the embeddings in (8):

**Lemma 9** *There exists a Hilbert basis $\{w^1, w^2, \ldots\}$ of $V$ with the $w^m$ satisfying*

$$\begin{cases} (\nabla w^m, \nabla v) = \lambda_m (w^m, v) \quad \forall v \in V, \quad w^m \in V, \\ |w^m| = 1, \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_m \to +\infty. \end{cases} \tag{22}$$

*The functions $w^m$ are orthogonal in $V$ and orthonormal (and also complete) in $H$.*

Let $V_m$ be the space spanned by the $m$ first eigenfunctions $w^1, \ldots, w^m$ and let $\widetilde{P}_m : V' \mapsto V_m$ the orthogonal projector defined by

$$\widetilde{P}_m g = \sum_{i=1}^{m} \langle g, w^i \rangle_V \, w^i \quad \forall g \in V'.$$

The important property satisfied by $\{w^1, w^2, \ldots\}$ is that, regarded as a linear bounded operator from $V'$ into itself, the norm of $\widetilde{P}_m$ is equal to 1:

$$\|\widetilde{P}_m\|_{\mathcal{L}(V';V')} = 1 \quad \forall m \geq 1. \tag{23}$$

Let $\{f^m\}$ be a sequence in $C^0(\overline{Q})^N$ satisfying $f^m \to f$ strongly in $L^2(Q)^N$ as $m \to \infty$. We will divide the proof in five steps:

Step 1: Construction of the approximated solutions.

For each $m \geq 1$, we consider the following finite dimensional problem: Find a $C^1$ function $u^m : [0,T] \mapsto V_m$ such that

$$\begin{cases} (u_t^m, v) + b(u^m, u^m, v) + \nu a(u^m, v) = (f^m, v) \quad \forall v \in V_m, \quad t \in (0,T), \\ u^m|_{t=0} = u^{0m}. \end{cases} \tag{24}$$

Here, $u^{0m} = P_m u^0$, where $P_m : H \mapsto V_m$ is the usual orthogonal projector, given as follows:

$$P_m h = \sum_{i=1}^{m} (h, w^i)\, w^i \quad \forall h \in H$$

(compare with the definition of $\widetilde{P}_m$).

If we put

$$u^m(t) = \sum_{i=1}^{m} \eta_{im}(t) w^i, \tag{25}$$

then (24) can be regarded as a Cauchy problem for a first order ordinary differential system where the unknowns are the functions $\eta_{im}$. More precisely, an equivalent formulation of (24) is the following:

$$\begin{cases} \eta'_{jm} + \sum_{i,k=1}^{m} b(w^i, w^k, w^j)\eta_{im}\eta_{km} + \nu\lambda_j \eta_{jm} = (f^m, w^j) \quad \forall j = 1, \ldots, m, \\ \eta_{jm}(0) = (u^0, w^j) \quad \forall j = 1, \ldots, m. \end{cases} \tag{26}$$

Therefore, the classical existence and uniqueness theory for ordinary differential systems can be applied and we can affirm that, for each $m \geq 1$, there exist $T_m > 0$ and a unique function $u^m : [0, T_m) \mapsto V_m$ that solves (24) at least in $[0, T_m)$. Furthermore, for every $m$ the following alternative holds:

$$\text{Either } T_m = T, \text{ or } \limsup_{t \to T_m} |u^m(t)| = +\infty. \tag{27}$$

The estimates we are going to deduce for the functions $u^m$ will show that only the first of these two assertions can be true.

Step 2: A priori estimates of $u^m$.

The aim in this and the following step is to get estimates of $u^m$ independent of $m$. We will then apply some general principles in Functional Analysis that state that, in many Banach spaces, bounded sets are relatively weakly sequentially compact and consequently possess weakly convergent subsequences (see propositions 10 and 11 below).

The first (and most important) estimates that we can find are the so called *energy estimates*. They are obtained as follows.

For each $t \in [0, T_m)$, let us take $v = u^m(t)$ in (24). We get the identity

$$\frac{1}{2}\frac{d}{dt}|u^m(t)|^2 + \nu\|u^m(t)\|^2 = (f^m(\cdot,t), u^m(t)). \tag{28}$$

Integrating in the time interval $[0,t]$, we also have

$$\frac{1}{2}|u^m(t)|^2 + \nu \int_0^t \|u^m(s)\|^2\,ds = \frac{1}{2}|u^{0m}|^2 + \int_0^t (f^m(\cdot,s), u^m(s))\,ds. \tag{29}$$

Now, using Young's inequality and the definitions of $u^{0m}$ and $\ell$, it is immediate that

$$\begin{aligned}\frac{1}{2}|u^m(t)|^2 + \nu \int_0^t \|u^m(s)\|^2\,ds &= \frac{1}{2}|u^{0m}|^2 + \int_0^t (f^m(\cdot,s), u^m(s))\,ds \\ &\le \frac{1}{2}|u^0|^2 + \frac{\nu}{2}\int_0^t \|u^m(s)\|^2\,ds + C\int_0^t |f(\cdot,s)|^2\,ds\end{aligned} \tag{30}$$

and

$$|u^m(t)|^2 + \nu \int_0^t \|u^m(s)\|^2\,ds \le |u^0|^2 + C\|f\|^2_{L^2(Q)}. \tag{31}$$

This holds for all $t \in [0,T_m)$. Consequently, in view of (27), we have $T_m = T$ and

$$\|u^m\|_{L^\infty(0,T;H)} + \|u^m\|_{L^2(0,T;V)} \le C, \tag{32}$$

where $C$ depends on $\Omega$, $T$, $\nu$, $|u^0|$ and $\|f\|_{L^2(Q)}$ but is independent of $m$.

The identity (29) (that hold for all $t \in [0,T]$) is known as the energy equality for $u^m$. It can be interpreted as follows:

> The sum of the kinetic energy of the fluid particles at time $t$ and the energy that has been dissipated (or lost) as a consequence of viscosity during the interval $(0,t)$ is equal to the sum of the initial kinetic energy and the mechanical work due to external forces.

For this reason, the Banach space $E = L^2(0,T;V) \cap L^\infty(0,T;H)$ (endowed with the sum of the norms in the left hand side of (32)) is called the *energy space* of the solutions of problem II. Accordingly, (31) and (32) are the so called energy estimates.

Step 3: A priori estimates of $u_t^m$.

It will be seen below that, due to the nonlinear terms, the estimates (32) do not suffice to pass to the limit in (24). We will also need some uniform estimates of $u_t^m$.

In order to obtain them, we argue as follows. First, we notice that

$$(u_t^m(t), v) = \langle \ell^m(t) - B(u^m(t), u^m(t)) - \nu A u^m(t), v\rangle_V$$

for all $v \in V_m$ a.e. in $(0,T)$, where $\ell^m \in L^2(0,T;V')$ is defined by the equalities

$$\langle \ell^m(t), v\rangle_V = \int_\Omega f^m(x,t)v(x)\,dx \quad \forall v \in V, \quad t \in [0,T] \ \text{ a.e.}$$

Consequently, since $u_t^m(t) \in V_m$, we have

$$u_t^m(t) = \widetilde{P}_m(\ell^m(t) - B(u^m(t), u^m(t)) - \nu A u^m(t))$$

a.e. in $(0, T)$. Recalling (23), we see that

$$\|u_t^m(t)\|_{V'} \leq \|\ell^m(t)\|_{V'} + \|B(u^m(t), u^m(t))\|_{V'} + \nu \|Au^m(t)\|_{V'} \quad \text{a.e.} \tag{33}$$

It is clear that $\|\ell^m(t)\|_{V'} \leq |f^m(\cdot, t)| \leq C|f(\cdot, t)|$. When $N = 2$, we can apply lemma 8 and (32) and deduce that $u_t^m$ is uniformly bounded in $L^2(0, T; V')$. When $N = 2$, we can instead apply remark 1; we see in this case that $u_t^m$ is uniformly bounded in $L^{4/3}(0, T; V')$. Summarizing, we have:

$$\|u_t^m\|_{L^\sigma(0,T;V')} \leq C \quad \text{with } \sigma = 2 \text{ if } N = 2 \text{ and } \sigma = 4/3 \text{ if } N = 3. \tag{34}$$

Step 4: The choice of convergent subsequences.

In this step, we will use (32) and (34) to extract several subsequences of $\{u^m\}$ with appropriate convergence properties. For simplicity, all them will be indexed again with $m$.

We will need below the following results, whose proofs can be found for instance in [3]:

**Proposition 10** *Let $X$ be a reflexive Banach space and let $B \subset X$ be a bounded set. Then $B$ is weakly relatively sequentially compact. In other words, any sequence in $B$ possesses weakly convergent subsequences. In particular, this holds in any Hilbert space.*

**Proposition 11** *Let $X$ be a separable Banach space, let us denote by $X'$ the associated dual space and let $B \subset X'$ be a bounded set. Then $B$ is weakly-$*$ relatively sequentially compact. In other words, any sequence in $B$ possesses weakly-$*$ convergent subsequences.*

Notice that $L^2(0, T; V)$ is a Hilbert space and $u^m$ is uniformly bounded in $L^2(0, T; V)$. Hence, in view of proposition 10, there exists a first subsequence satisfying

$$u^m \to u \;\text{ weakly in }\; L^2(0, T; V). \tag{35}$$

From (35), one also has $u^m \to u$ in $\mathcal{D}'(0, T; V')$, whence

$$u_t^m \to u_t \;\text{ in }\; \mathcal{D}'(0, T; V'). \tag{36}$$

Secondly, observe that $L^\infty(0, T; H)$ can be identified with the dual of the separable Banach space $L^1(0, T; H)$. Accordingly, using proposition 11 we find a new subsequence satisfying (35) and

$$u^m \to u \;\text{ weakly-}* \text{ in }\; L^\infty(0, T; H). \tag{37}$$

Let us now consider the sequence $\{u_t^m\}$ in $L^\sigma(0,T;V')$. Taking into account (34) and the fact that $L^\sigma(0,T;V')$ is a reflexive Banach space and applying proposition 11, we see that, at least for a new subsequence, we have (35), (37) and

$$u_t^m \to z \quad \text{weakly in} \quad L^\sigma(0,T;V'). \tag{38}$$

From (36), it is clear that $z = u_t$ and thus

$$u_t^m \to u_t \quad \text{weakly in} \quad L^\sigma(0,T;V'). \tag{39}$$

We will now use a *compactness result* in order to deduce from (35) and (39) a strong convergence property for $u^m$:

**Theorem 12** *Let $X_0$, $X$ and $X_1$ be three Banach spaces such that $X_0$ and $X_1$ are reflexive and*

$$X_0 \hookrightarrow X \hookrightarrow X_1, \tag{40}$$

*where the first embedding is compact and the second one is continuous. Assume that $1 < p_0, p_1 < +\infty$ and let us introduce the linear space*

$$W^{p_0,p_1}(0,T;X_0,X_1) = \{\, z \in L^{p_0}(0,T;X_0) : z_t \in L^{p_1}(0,T;X_1) \,\},$$

*which is endowed with the "natural" norm*

$$\|z\|_{W^{p_0,p_1}(0,T;X_0,X_1)} = \|z\|_{L^{p_0}(0,T;X_0)} + \|z_t\|_{L^{p_1}(0,T;X_1)}.$$

*Then $W^{p_0,p_1}(0,T;X_0,X_1)$ is a reflexive Banach space and the embedding*

$$W^{p_0,p_1}(0,T;X_0,X_1) \hookrightarrow L^{p_0}(0,T;X)$$

*is compact.*

This result is due to J.-L. Lions and J. Peetre; see [21]. It will be applied here with $X_0 = V$, $X = H$, $X_1 = V'$, $p_0 = 2$ and $p_1 = \sigma$. Thus, in view of (35) and (39), extracting if necessary a new subsequence, one has:

$$u^m \to u \quad \text{strongly in} \quad L^2(0,T;H). \tag{41}$$

Obviously, it is then not restrictive to assume that

$$u^m \to u \quad \text{strongly in} \quad L^2(Q)^N \quad \text{and a.e.} \tag{42}$$

A crucial point is that (35), (37) and (42) together imply

$$B(u^m,u^m) \to B(u,u) \quad \text{weakly in} \quad L^\sigma(0,T;V'). \tag{43}$$

Indeed, $B(u^m,u^m)$ is uniformly bounded in $L^\sigma(0,T;V')$. Consequently, at least for a subsequence one has $B(u^m,u^m) \to \widetilde{B}$ weakly in this space. For any $\varphi \in \mathcal{V}$ and any $\psi \in \mathcal{D}(0,T)$, we then have

$$\int_0^T \langle \widetilde{B}(t), \varphi\rangle_V \, \psi(t)\, dt = \lim_{m\to\infty} \int_0^T \langle B(u^m(t),u^m(t)), \varphi\rangle_V \, \psi(t)\, dt$$
$$= \lim_{m\to\infty} \iint_Q (u^m\cdot\nabla)u^m \cdot \varphi\,\psi\, dx\, dt = -\lim_{m\to\infty} \iint_Q (u^m\cdot\nabla)(\varphi\psi)\cdot u^m\, dx\, dt.$$

But this coincides with

$$-\iint_Q (u\cdot\nabla)(\varphi\psi)\cdot u\,dx\,dt = \iint_Q (u\cdot\nabla)u\cdot\varphi\,\psi\,dx\,dt,$$

because all the functions $u_i^m u_j^m$ converge strongly in $L^1(Q)$. We thus find that

$$\int_0^T \langle \widetilde{B}(t),\varphi\rangle_V\,\psi(t)\,dt = \int_0^T \langle B(u(t),u(t)),\varphi\rangle_V\,\psi(t)\,dt$$

for all $\varphi\in\mathcal{V}$ and $\psi\in\mathcal{D}(0,T)$. Obviously, this implies that $\widetilde{B}=B(u,u)$. Since this argument can be applied to any subsequence converging weakly in $L^\sigma(0,T;V')$, we get (43).

STEP 5: THE PASSAGE TO THE LIMIT.

We are now ready to pass to the limit in (24).

Thus, let us consider a subsequence $\{u^m\}$ satisfying (35), (37), (39), (42) and (43). Of course, $u\in L^2(0,T;V)\cap L^\infty(0,T;H)$ and $u_t\in L^\sigma(0,T;V')$, whence in particular $u\in C^0([0,T];V')$ and $u\in C^0_w([0,T];H)$.

First of all, notice that

$$u^m(0)\to u(0)\ \text{ weakly in }\ V'. \tag{44}$$

Indeed, from (35), (39) and lemma 3 we know that $u^m\to u$ weakly in $C^0([0,T];V')$. Since the linear mapping $v\mapsto v(0)$ is continuous from $C^0([0,T];V')$ into $V'$, we have (44).

From (44) and the facts that $u^m(0)=u^{0m}$ and $u^{0m}\to u^0$ strongly in $H$, the following holds:

$$u|_{t=0}=u^0.$$

This shows that the initial condition in problem II is satisfied by $u$.

Now, let us fix $v$ in the space $\cup_{j\ge1}V_j$. For any sufficiently large $m$, one has

$$(u_t^m(t),v)+\langle B(u^m(t),u^m(t)),v\rangle_V+\nu\langle Au^m(t),v\rangle_V=\langle \ell^m(t),v\rangle_V \tag{45}$$

a.e. in $(0,T)$. We will check that the three terms in the left hand side of (45) converge in $\mathcal{D}'(0,T)$ respectively to $\langle u_t,v\rangle_V$, $\langle B(u,u),v\rangle_V$ and $\nu\langle Au,v\rangle_V$.

Indeed, since $(u^m,v)\to(u,v)$ strongly in $L^2(0,T)$, we have $(u_t^m,v)=(u^m,v)_t\to(u,v)_t=\langle u_t,v\rangle_V$ in $\mathcal{D}'(0,T)$. It is also clear in view of (43) that $\langle B(u^m,u^m),v\rangle_V\to\langle B(u,u),v\rangle_V$ weakly in $L^\sigma(0,T)$. Finally, from (35), we also have $Au^m\to Au$ in $L^2(0,T;V')$ and $\langle Au^m,v\rangle_V\to\langle Au,v\rangle_V$ weakly in $L^2(0,T)$.

Taking into account that $\langle \ell^m,v\rangle_V\to\langle \ell,v\rangle_V$ strongly in $L^2(0,T)$, we conclude that the function $u$ satisfies

$$\langle u_t,v\rangle_V+\langle B(u,u),v\rangle_V+\nu\langle Au,v\rangle_V=\langle \ell,v\rangle_V\ \text{ in }\ \mathcal{D}'(0,T) \tag{46}$$

for any $v\in\cup_{j\ge1}V_j$. By density, it is obvious that (46) must also hold for any $v\in V$. Consequently,

$$u_t+B(u,u)+\nu Au=\ell\ \text{ in }\ \mathcal{D}'(0,T;V').$$

This shows that $u$ solves problem II.

Consequently, theorem 5 is proved. ■

**Remark 5** It is not difficult to prove that the solution we have found satisfies the energy inequalities

$$\begin{aligned} &\frac{1}{2}|u(\cdot,t')|^2 + \nu \int_t^{t'} \|u(\cdot,s)\|^2\,ds \\ &\qquad \le \frac{1}{2}|u(\cdot,t)|^2 + \int_t^{t'} (f(\cdot,s),u(s))\,ds \end{aligned} \tag{47}$$

for all $t,t' \in [0,T]$ with $t < t'$. However, in general it is unknown whether similar equalities hold. For more details about energy inequalities, see [22].

To end this section, let us prove that we have solved in fact problem I. More precisely, let us show that problems I and II are equivalent.

Thus, let $u$ and $p$ solve problem I. Then, for any $\varphi \in \mathcal{V}$ and any $\psi \in \mathcal{D}(0,T)$, we have:

$$\begin{aligned} 0 &= \langle u_t + (u\cdot\nabla)u - \nu\Delta u + \nabla p - f, \varphi\psi\rangle_{\mathcal{D}'(Q)^N,\mathcal{D}(Q)^N} \\ &= \iint_Q \left(-u\cdot(\psi\varphi)_t + (u\cdot\nabla)u\cdot(\psi\varphi) + \nu\nabla u\cdot\nabla(\psi\varphi)\right)\,dx\,dt \\ &- \iint_Q f\cdot(\psi\varphi)\,dx\,dt \\ &= -\int_0^T \left(\int_\Omega u\cdot\varphi\,dx\right)\psi_t\,dt + \int_0^T (b(u,u,\varphi) + \nu a(u,\varphi) - \langle \ell,\varphi\rangle_V)\,\psi\,dt. \end{aligned}$$

Hence,

$$\frac{d}{dt}\left(\int_\Omega u\cdot\varphi\,dx\right) + b(u,u,\varphi) + \nu a(u,\varphi) = \langle \ell,\varphi\rangle_V \quad \text{in} \quad \mathcal{D}'(0,T)$$

for all $\varphi \in \mathcal{V}$. This proves that $u$ is a solution of problem II.

In order to prove the reciprocal, we will use the following Banach-valued version of De Rham's lemma:

**Lemma 13** *Let $E$ be a Banach space and let $S \in \mathcal{D}'(\Omega;E)^N$ be given, with*

$$\langle S,\varphi\rangle = 0 \quad \forall \varphi \in \mathcal{V}.$$

*Then there exists $q \in \mathcal{D}'(\Omega;E)$ such that $S = \nabla q$. Furthermore, if $r \in (1,+\infty)$, $s \in \mathbb{Z}$ and $S \in W^{s,r}(\Omega;E)^N$, we can choose $q$ in $W^{s+1,r}(\Omega;E)$ and depending continuously of $S$, i.e. such that the mapping $S \in W^{s,r}(\Omega;E)^N \mapsto q \in W^{s+1,r}(\Omega;E)$ is continuous.*

Let $u$ be a solution to problem II and let us set

$$S = u_t + (u \cdot \nabla)u - \nu\Delta u - f.$$

It is then easy to check that $S \in W^{-1,\infty}(0,T;H) + L^\sigma(0,T;H^{-1}(\Omega)^N) \subset W^{-1,\infty}(0,T;H^{-1}(\Omega)^N)$. But this last Banach space is isomorphic and isometric to $H^{-1}(\Omega;W^{-1,\infty}(0,T))^N$. Therefore, we can apply lemma 13 with $E = W^{-1,\infty}(0,T)$ (recall that $\langle S,\varphi\rangle = 0$ for all $\varphi \in \mathcal{V}$). The conclusion is that there exists $p \in L^2(0,T;W^{-1,\infty}(0,T))$ such that $S = -\nabla p$. In other words, for some $p \in W^{-1,\infty}(0,T;L^2(\Omega))$, one has

$$u_t + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f \ \text{ in } \ \mathcal{D}'(Q)^N.$$

This proves that $u$ and $p$ solve problem II.

**Remark 6** There are other ways to prove the existence of solution of problem II. Some of them are nonconstructive and rely on appropriate fixed point theorems; see for instance [19, 33]. There are also other constructive proofs; see [20, 31, 22]. ■

**Remark 7** With similar arguments, the existence of solutions of problems I and II can be proved under slightly more general assumptions. Thus, $\Omega \subset \mathbb{R}^N$ can be an arbitrary connected open set (not necessarily bounded), the right hand side $f$ can belong to the space $L^1(0,T;H^{-1}(\Omega)^N)$, etc. ■

**Remark 8** Assume that the nonlinear term $b(u,u,v)$ is omitted in (12). Then we can argue as in steps 1 and 2 of the proof of theorem 5 and construct Galerkin approximations $u^m$ satisfying (32). This suffices to choose a subsequence satisfying (35) and (37). But now this is enough to pass to the limit in the approximated problems in all the terms. Consequently, the need of the estimates (34) for the approximated solutions of (12) comes from the fact that this system contains nonlinear terms. ■

**Remark 9** Lemma 12 is interesting by itself. It provides a criterion to ensure the relative compactness of a family $\mathcal{F} \in L^{p_0}(0,T;X)$. This subject has been investigated by J. Simon in [30]. There, the following assertion is proved:

*Assume that $X$ is a Banach space, $1 \le p_0 < +\infty$ and $\mathcal{F} \subset L^{p_0}(0,T;X)$ is given. Then $\mathcal{F}$ is relatively compact in $L^{p_0}(0,T;X)$ if and only if one has:*

- *The set*

$$\{ \int_{t_1}^{t_2} f(s)\,ds : f \in \mathcal{F} \}$$

  *is relatively compact in $X$ for any $t_1, t_2 \in [0,T]$ with $t_1 \le t_2$.*
- $\|\tau_h f - f\|_{L^{p_0}(0,T-h;X)} \to 0$ *uniformly in $f \in \mathcal{F}$ as $h \to 0$. Here, $\tau_h f(t) \equiv f(t+h)$ for any $t \in [0,T-h]$ and any $h \in (0,T)$.*

■

## 4 Some regularity results

Besides existence and uniqueness results, it is also interesting to investigate *regularity properties* of the solutions of problem II. Indeed, it is reasonable to expect that, when the open set $\Omega$ and the data $f$ and $u^0$ are more regular than in theorem 5, so are the associated solutions.

When $N = 2$, this can be established rigorously. For instance, we have the following:

**Theorem 14** *Assume that $N = 2$, $\Omega \subset \mathbb{R}^2$ is a bounded connected open set of class $C^{1,1}$, $f \in L^2(Q)^2$ and $u^0 \in V$. Then the unique solution of problem II satisfies*

$$u \in L^2(0,T;H^2(\Omega)^2) \cap C^0([0,T];V), \quad u_t \in L^2(0,T;H). \tag{48}$$

For the proof, it suffices to get uniform estimates of the Galerkin approximations $u^m$ in $L^2(0,T;H^2(\Omega)^2)$ and $L^\infty(0,T;V)$ and uniform estimates of the time derivatives $u_t^m$ in $L^2(0,T;H)$.

The estimates of $u^m$ can be obtained by taking in (24) $v = Au^m(t)$ for each $t \in (0,T)$. Indeed, with the notation (25), we have

$$Au^m(t) = \sum_{i=1}^{m} \lambda_i \eta_{im}(t) w^i$$

and this shows that this choice of $v$ is admissible. We easily deduce that

$$\begin{aligned}
\frac{1}{2}\|u^m(t)\|^2 + \nu \int_0^t |Au^m(s)|^2\,ds &= \frac{1}{2}\|u^{0m}\|^2 + \int_0^t (f^m(\cdot,s), Au^m(s))\,ds \\
&\quad - \int_0^t b(u^m(s), u^m(s), Au^m(s))\,ds \\
&\le \frac{1}{2}\|u^0\|^2 + \frac{\nu}{2}\int_0^t |Au^m(s)|^2\,ds + C\int_0^t |f(\cdot,s)|^2\,ds \\
&\quad + C\int_0^t |u^m(s)|^2 \|u^m(s)\|^4\,ds.
\end{aligned}$$

Here, we have used that $u^0 \in V$. These inequalities, together with (32), Gronwall's lemma and the regularity of $\Omega$, yield the desired bounds for $u^m$.

For the estimates of $u_t^m$, we take $v = u_t^m(t)$ in (24). Now, we find that

$$\begin{aligned}
|u_t^m(t)|^2 + \frac{\nu}{2}\frac{d}{dt}\|u^m(t)\|^2 &= (f^m(\cdot,t), u_t^m(t)) - b(u^m(t), u^m(t), u_t^m(t)) \\
&\le \frac{1}{2}|u_t^m(t)|^2 + |f(\cdot,t)|^2 + C\|u^m(t)\|_{H^2}^2 \|u^m(t)\|^2
\end{aligned}$$

and integrating with respect to $t$ we find that $u_t^m$ is uniformly bounded in $L^2(0,T;H)$. For more details, see for example [7].

When $N = 3$, the situation is much more complicated. We can obtain results of the kind of theorem 14 only when the data are small (in appropriate norms). For instance, we have the following result:

**Theorem 15** *Assume that $N = 3$, $\Omega \subset \mathbb{R}^3$ is a bounded connected open set of class $C^{1,1}$, $f \in L^2(Q)^3$ and $u^0 \in V$. Then there exists $\varepsilon > 0$ (depending on $\Omega$) such that, whenever*

$$\|u^0\| + \|f\|_{L^2(Q)^3} \leq \varepsilon, \tag{49}$$

*the solution of problem II furnished by theorem 5 satisfies*

$$u \in L^2(0,T;H^2(\Omega)^3) \cap C^0([0,T];V), \quad u_t \in L^2(Q)^3 \tag{50}$$

*and is unique in this class.*

For the proof, we try to find the same estimates above for the Galerkin approximations $u^m$ and their time derivatives $u_t^m$. In this case, we find that

$$\begin{aligned} \frac{1}{2}\|u^m(t)\|^2 + \nu \int_0^t |Au^m(s)|^2\,ds \leq \frac{1}{2}\|u^0\|^2 + C\int_0^t |f(\cdot,s)|^2\,ds \\ + \frac{\nu}{2}\int_0^t |Au^m(s)|^2\,ds + C\int_0^t \|u^m(s)\|^6\,ds \end{aligned}$$

and, in order to conclude, the assumption (49) is needed (with $\varepsilon$ sufficiently small).

We refer to [13] and [7] for more details.

**Remark 10** Using appropriately lemma 13, we can deduce from (48) and (50) further regularity properties for $p$. In particular, under the assumptions of theorems 14 or 15, we find that $p \in L^2(0,T;H^1(\Omega))$. Accordingly, the PDE's in (11) are satisfied a.e. in $Q$. ■

**Remark 11** It is completely unknown whether "large" regular data $f$ and $u^0$ lead to regular solutions when $N = 3$. In fact, one of the one-million dollars open problems proposed by the Clay Institute in 2000 is the following:

> *Assume that $\Omega = \mathbb{R}^3$, $f \equiv 0$ and $u^0 \in \mathcal{V}$. Prove that problem II possesses a solution $u$ of class $C^\infty$.*

See `http://www.claymath.org/millennium/Navier-Stokes_Equations/` for more details. ■

## 5 Some other results and open questions

In this section, we will take $T = +\infty$, $Q = \Omega \times (0,+\infty)$ and $\Sigma = \partial\Omega \times (0,+\infty)$. Here, our aim is to recall very briefly some of the main known results concerning variants of the Navier-Stokes equations. We will only consider fluids modelled by the equations (2), (3), (4) and (5). We believe this is enough to get an idea of the variety and complexity of the subject.

### 5.1 The incompressible Euler equations

When viscous effects are negligible, it is admissible to take $\nu = 0$ in the motion equation. For example, this is the case when we are considering the flow of the air around an obstacle at high velocity and we observe the behavior of the fluid only at points located far from the obstacle. This leads to the incompressible Euler equations (2).

Now, the fluid is modelled by a system of nonlinear *first-order* equations. Accordingly, it is reasonable to look for solutions satisfying not so many complementary conditions as in (1). On the other hand, it is also realistic to expect that, in principle, the solution be less regular than in the Navier-Stokes case.

To fix ideas, we will consider the system

$$\begin{cases} u_t + (u \cdot \nabla)u + \nabla p = 0, \quad \nabla \cdot u = 0 & \text{in } Q, \\ u \cdot n = 0 & \text{on } \Sigma, \\ u(x,0) = u^0(x) & \text{in } \Omega, \end{cases} \tag{51}$$

where $u^0 = u^0(x)$ is given. It will be said that $u$ is (together with some $p$) a weak solution of (51) if $u \in L^2(0,+\infty;V) \cap L^\infty(0,+\infty;H)$,

$$\langle u_t, v\rangle_V + b(u,u,v) = 0 \quad \forall v \in V \tag{52}$$

and $u|_{t=0} = u^0$. It should be noticed that this is equivalent to

$$\iint_Q u \cdot (\varphi_t + (u \cdot \nabla)P\varphi)\; dx\, dt + \int_\Omega u^0(x) \cdot \varphi(x,0)\, dx = 0 \tag{53}$$

for any $\varphi \in \mathcal{D}(\overline{Q})^N$ (recall that $P : L^2(\Omega)^2 \mapsto H$ is the usual orthogonal projector) and slightly stronger than

$$\iint_Q u \cdot (\varphi_t + (u \cdot \nabla)\varphi)\; dx\, dt + \int_\Omega u^0(x) \cdot \varphi(x,0)\, dx = 0 \tag{54}$$

for any $\varphi \in \mathcal{D}(\overline{Q})^N$ satisfying $\nabla \cdot \varphi = 0$ in $Q$.

Then the following is known:

**Theorem 16** *Let us assume that $N = 2$, $u^0 \in H$ and $\nabla \times u^0 \in L^\infty(\Omega)$. Then* (51) *possesses exactly one weak solution $u$ furthermore satisfying*

$$u \in C^0([0,+\infty); W^{1,q}(\Omega)^2) \quad \forall q \in (1,+\infty). \tag{55}$$

For the proof, see [32]. A similar existence result can also be proved when $u^0 \in H$ and $\nabla \times u^0 \in L^r(\Omega)$ for some $r \in (1,+\infty)$, but the uniqueness of weak solution is unknown in this case; for more details, see [22].

When $N = 3$, the situation is much more complicated (and less understood). In general, only local in time existence results can be ensured for large initial data (even if they are smooth). A lot of appropriate numerical results and the analysis of some similar systems seem to indicate that regular solutions can blow-up at finite time; see for instance [12, 18] and the references therein. But the problem remains open at present.

For completeness, let us recall the following result, whose proof can be found in [1]:

**Theorem 17** *Let us assume that $N = 3$ and $u^0 \in H^s(\Omega)^3 \cap H$ for some $s > 5/2$. Then there exists $T^* > 0$ such that (51) possesses exactly one weak solution in $\Omega \times (0, T^*)$. This solution satisfies*

$$u \in C^0([0,T]; H^s(\Omega)^3) \quad \forall T \in (0, T^*).$$

*Furthermore, if $T^* < +\infty$, then*

$$\int_0^{T^*} \|(\nabla \times u)(\cdot, t)\|_{L^\infty} \, dt = +\infty. \tag{56}$$

### 5.2 The variable density Navier-Stokes equations

In practice, it is frequent to find fluids for which mass-density is not a constant but a function of space and time. This can be the case of a river or a portion of an ocean. The resulting equations are (3), where $\mu > 0$.

Observe that, in (3), the new variable $\rho$ satisfies a (first-order hyperbolic) *transport* equation governed by $u$ which is called the continuity equation:

$$\rho_t + u \cdot \nabla \rho = 0.$$

Accordingly, it may happen that $\rho(\cdot, t)$ be a piecewise regular discontinuous function and the discontinuities of $\rho(\cdot, t)$ be transported by $u$. This is readily understood by rewriting the continuity equation in the form

$$\frac{d}{dt} \rho(X(x,t), t) = 0,$$

where $X(x, \cdot)$ is the *trajectory* determined by $u$ and $x$, i.e.

$$\begin{cases} X_t(x,t) = u(X(x,t), t), & t \in (0, T), \\ X(x, 0) = x \end{cases} \tag{57}$$

(the components of $X$ are also known as the Lagrangian coordinates; in fact, $X(x,t)$ is the position at time $t$ of the particle located at $x$ at time $t = 0$).

For simplicity, let us consider the following system for (3):

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0 & \text{in } Q, \\ \rho\,(u_t + (u \cdot \nabla) u) - \mu \Delta u + \nabla p = 0, \quad \nabla \cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ (\rho u)(x, 0) = m^0(x), \quad \rho(x, 0) = \rho^0(x) & \text{in } \Omega, \end{cases} \tag{58}$$

where $\mu > 0$ is a constant and $m^0$ and $\rho^0$ are given. We will say that $\{\rho, u\}$ is (together with some $p$) a weak solution of (3) if

$$\left\{ \begin{array}{l} \rho \in L^\infty(Q) \cap C^0([0,+\infty); L^q(\Omega)) \quad \forall q \in [1,+\infty), \\ u \in L^2(0,+\infty; V), \quad \rho |u|^2 \in L^\infty(0,+\infty; L^1(\Omega)) \end{array} \right. \tag{59}$$

the continuity equation $\rho_t + \nabla \cdot (\rho u) = 0$ holds in $Q$ in the distributional sense, $\rho|_{t=0} = \rho^0$ in the $L^q$ sense for all $q \in [1,+\infty)$ and

$$\left\{ \begin{array}{l} \displaystyle\iint_Q (\rho u \cdot \varphi_t + \rho\, u_i\, u_j\, \partial_i \varphi_j - \mu\, \partial_i u_j\, \partial_i \varphi_j)\, dx\, dt \\ \displaystyle\qquad + \int_\Omega m^0(x) \cdot \varphi(x,0)\, dx = 0 \end{array} \right. \tag{60}$$

for any $\varphi \in \mathcal{D}(\Omega \times [0,+\infty))^N$ satisfying $\nabla \cdot \varphi = 0$ in $Q$.

Then the following result holds:

**Theorem 18** *Let us assume that $N = 2$ or $N = 3$, $\rho^0 \in L^\infty(\Omega)$, $\rho^0 \geq 0$ a.e. in $\Omega$, $m^0 \in L^2(\Omega)^N$, $m^0 = 0$ a.e. when $\rho^0 = 0$ and $|m^0|^2/\rho^0 \in L^1(\Omega)$. Then* (58) *possesses at least one weak solution $\{\rho, u\}$.*

For the proof, see for instance [30] and [22]. In this last reference, the result is proved in a more general case and it is found that the solution satisfies appropriate energy inequalities. It is also proved there that the distribution function of $\rho(\cdot, t)$ is independent of $t$; in other words, for any $\alpha, \beta \in \mathbb{R}$ and any $t > 0$, one has

$$\text{meas}\, \{\, x \in \Omega : \alpha \leq \rho(x,t) \leq \beta \,\} = \text{meas}\, \{\, x \in \Omega : \alpha \leq \rho^0(x) \leq \beta \,\}$$

(in fact, this property can be viewed as a reformulation of the mass conservation law).

In general, the uniqueness of weak solution of (58) is unknown even when $N = 2$. The same can be said for regularity results.

However, if $N = 2$ and the initial data also satisfy

$$\rho^0 \geq a > 0 \ \text{ a.e. in } \ \Omega, \qquad \frac{1}{\rho^0}\, m^0 \in V,$$

the regularity of the solutions (and therefore uniqueness) can be obtained. More precisely, under these assumptions one has

$$u \in L^2(0,+\infty; H^2(\Omega)^2) \cap C^0([0,+\infty); V), \quad u_t \in L^2(0,+\infty; H).$$

### 5.3 Some quasi-Newtonian fluids

For a general incompressible fluid of constant density $\rho = 1$, the motion equation states that

$$u_t + (u \cdot \nabla) u = \nabla \cdot \sigma + f, \tag{61}$$

where $\sigma = \sigma(x,t)$ is the *stress tensor*. This means that, for any regular open set $W \subset \Omega$, the resultant of the forces exerted on the particles in $W$ by the other fluid particles at time $t$ is given by

$$\mathcal{I}(W,t) = \int_{\partial W} \sigma(x,t)\cdot n(x)\,d\Gamma(x).$$

In the case of the Navier-Stokes equations, it is assumed that $\sigma$ is furnished by the so called *Stokes law*. This means that the stress tensor $\sigma$ is proportional to the *strain* or *deformation* tensor $Du = \frac{1}{2}(\nabla u + \nabla u^t)$. More precisely, we have

$$\sigma = 2\nu Du = \nu(\nabla u + \nabla u^t) \tag{62}$$

for some constant $\nu > 0$. Combining (62) and (61), we find at once the first equation in (1). The fluids satisfying this property are called *Newtonian*.

Sometimes, it is more accurate to assume a more general constitutive law for $\sigma$. Thus, instead of (62), we can assume that

$$\sigma = \nu(|Du|)(\nabla u + \nabla u^t), \tag{63}$$

where $\nu : \mathbb{R} \mapsto \mathbb{R}_+$ is a given function. This leads to the system (4). The fluids governed by (4) are called *quasi-Newtonian*.

In practical problems, many possible functions $\nu$ are found. Here, we will only consider the choice

$$\nu(s) = \alpha s^{r-2}, \tag{64}$$

where $r \geq 1$ and $\alpha$ is a positive constant. When $1 \leq r < 2$, $r = 2$ or $r > 2$, we are respectively considering a *visco-plastic*, *Newtonian* or *dilatant* fluid; in particular, in the limit $r = 1$, (63) must be understood as follows:

$$\sigma = \frac{2\alpha}{|Du|}\,Du \quad \text{if} \quad Du \neq 0; \qquad |\sigma| \leq 2\alpha \quad \text{if} \quad Du = 0.$$

In this case, we are considering a visco-plastic *Bingham* fluid (see [4, 5, 9] and the references therein).

There are many real phenomena in chemistry, glaciology, biology, etc. where the previous constitutive laws are appropriate; see [25] and the references therein.

For simplicity, let us consider the following initial-boundary value problem:

$$\begin{cases} u_t + (u\cdot\nabla)u - 2\nabla\cdot(\nu(|Du|)Du) + \nabla p = 0, \quad \nabla\cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = u^0(x) & \text{in } \Omega, \end{cases} \tag{65}$$

where $u^0$ is prescribed. Depending on the value of $r$, several different existence and/or uniqueness results can be established. In principle, as $r$ increases, better results are found. Thus, for very small $r$, it can only be proved that a local regular solution exists for regular initial data; for moderate $r$, global in time weak solutions exist. for larger $r$, the uniqueness of strong solution holds, etc. For a complete summary and a list of open questions, see [26].

In order to illustrate the situation, we will recall now one of these results.

We will assume that

$$\frac{3N}{N+2} < r < 2. \tag{66}$$

Let $V_r$ be the adherence of $\mathcal{V}$ in the Sobolev space $W_0^{1,r}(\Omega)^N$. Obviously, $V_r$ is a separable reflexive Banach space for the norm

$$\|v\|_r = \left( \int_\Omega |\nabla v|^r \, dx \right)^{1/r} \quad \forall v \in V_r.$$

It will be said that $u$ is (together with some $p$) a weak solution of (65) if $u \in L^r(0,+\infty; V_r) \cap L^\infty(0,+\infty; H)$ and

$$\left\{ \begin{array}{l} \displaystyle \iint_Q (\rho u \cdot \varphi_t + \rho\, u_i\, u_j\, \partial_i \varphi_j)\; dx\, dt \\ \displaystyle \quad -\frac{1}{2} \iint_Q \nu(|Du|)(\partial_i u_j + \partial_j u_i)(\partial_i \varphi_j + \partial_j \varphi_i)\, dx\, dt \\ \displaystyle \quad + \int_\Omega u^0(x) \cdot \varphi(x,0)\, dx = 0 \end{array} \right. \tag{67}$$

for any $\varphi \in \mathcal{D}(\Omega \times [0,+\infty))^N$ satisfying $\nabla \cdot \varphi = 0$ in $Q$.

**Theorem 19** *Assume that $\nu$ is given by* (64) *with $r$ satisfying* (66) *and $u^0 \in H$. Then* (65) *possesses at least one weak solution.*

It may be also meaningful to consider quasi-Newtonian viscous incompressible fluids with variable density. Roughly speaking, they lead to problems that need an analysis inspired by the arguments used in the proofs of theorems 18 and 19. See for instance [2, 10] for further details.

### 5.4 Viscoelastic Oldroyd models

Sometimes, the molecular structure of the fluid under consideration is so complicated that the constitutive law (63) does not suffice to provide a good description of the flow.

In particular, this is the case if elastic efforts among particles are relevant. Then, it has to be assumed that the stress tensor $\sigma$ is of the form

$$\sigma = \sigma_0 + \tau,$$

where $\sigma_0$ (the viscous-stress tensor) is given by Stokes law and $\tau$ (the elastic-stress tensor) satisfies an additional equation that is coupled to the conservation law (61) and serves to close the system.

The equation for $\tau$ can be a differential or an integral equation. Accordingly, it leads to a differential or to an integro-differential model. In differential models, $\tau$ is determined by $\nabla u$ through a system of PDE's. It is assumed that this system must satisfy the *material objectivity* or *frame indifference* principle (in other words, the law must be invariant under time-dependent proper rotations $Q = Q(t)$). As a consequence, the resulting system must involve *objective time derivatives*. By this, we mean first order operators of the form

$$\partial_t + p_i(x,t)\partial_i$$

such that we always have

$$Q \cdot (\partial_t + p_i\partial_i)\tau \cdot {}^tQ = (\partial_t + p_i\partial_i)\left(Q \cdot \tau \cdot {}^tQ\right).$$

The usual material derivative $\partial_t + u_i\partial_i$ does not satisfy the principle of material objectivity. On the contrary, the so called Oldroyd derivatives

$$\frac{\mathcal{D}_a\tau}{\mathcal{D}t} = \tau_t + (u\cdot\nabla)\tau + g_a(\nabla u, \tau) \tag{68}$$

are objective derivatives. Here,

$$g_a(\nabla u, \tau) = \tau W(u) - W(u)\tau - a(D(u)\tau + \tau D(u)) \tag{69}$$

($a \in [-1,1]$ is a constant). As before, $D(u) = \frac{1}{2}(\nabla u + \nabla u^t)$ and $W(u)$ is the *vorticity* tensor, i.e.

$$W(u) = \frac{1}{2}(\nabla u - \nabla u^t).$$

When $a = 0$, the corresponding derivative is known as the *Jaumann's* or *co-rotational* derivative. It is the following:

$$\frac{\mathcal{D}_0\tau}{\mathcal{D}t} = \tau_t + (u\cdot\nabla)\tau + \tau W(u) - W(u)\tau.$$

We will be concerned in this paragraph with the differential Oldroyd model. In dimensionless variables, this is (5), where $g_a(\nabla u, \tau)$ is given by (69). This provides a good description of the behaviour of some materials that have in part properties found for elastic solids and, also in part, properties similar to those of viscous fluids (this is why they are called viscoelastic). For a complete presentation and analysis, see for instance [15, 29].

Let us consider the problem

$$\begin{cases} u_t + (u\cdot\nabla)u - \nu\Delta u + \nabla p = \nabla\cdot\tau, \quad \nabla\cdot u = 0 & \text{in } Q, \\ \tau_t + (u\cdot\nabla)\tau + c\tau + g_a(\nabla u, \tau) = bDu & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = u^0(x), \quad \tau(x,0) = \tau^0(x) & \text{in } \Omega, \end{cases} \tag{70}$$

where $u^0$ and $\tau^0$ are prescribed. For convenience, let us denote by $L^2_{\mathrm{s}}$ the space of the symmetric tensors $\tau \in L^2(\Omega)^{N\times N}$. It will be said that $\{u,\tau\}$ is (together with some $p$) a weak solution of (70) if $u \in L^2(0,+\infty;V)\cap L^\infty(0,+\infty;H)$, $\tau \in L^\infty(0,+\infty;L^2_{\mathrm{s}})$,

$$\langle u_t, v\rangle_V + b(u,u,v) + \nu a(u,v) = \langle\nabla\cdot\tau, v\rangle \quad \forall v \in V, \tag{71}$$

$$\tau_t + (u \cdot \nabla)\tau + c\tau + g_a(\nabla u, \tau) = bDu \tag{72}$$

in the distributional sense, $u|_{t=0} = u^0$ and $\tau|_{t=0} = \tau^0$.

The system (70) is more difficult to analyze than (1) and (58). As the latter, it contains an equation of the Navier-Stokes kind coupled to a first-order hyperbolic equation for an additional variable. But now this additional variable and the associated first-order equation are nonscalar.

Furthermore, in general no a priori estimates are known for (70). In addition, even if we were able to find a family of approximated solutions $\{u^m, \tau^m\}$ a priori bounded in a *natural* energy space, it would be a difficult task to pass to the limit in the equations. Indeed, we may expect to get uniform bounds of $\nabla u^m$ and $\tau^m$ respectively in $L^2(Q)^N$ and $L^\infty(0, +\infty; L^2_{\mathrm{s}})$ but this is clearly insufficient, in view of the structure of $g_a(\nabla u^m, \tau^m)$.

To our knowledge, the unique global in time existence known result is the following:

**Theorem 20** *Let us assume that $N = 2$ or $N = 3$, $u^0 \in H$, $\tau^0 \in L^2_{\mathrm{s}}$ and $a = 0$, that is, the Oldroyd differential law for $\tau$ is*

$$\tau_t + (u \cdot \nabla)\tau + c\tau + \tau W(u) - W(u)\tau = bDu. \tag{73}$$

*Then* (70) *possesses at least one global in time weak solution.*

This has been proved by P.-L. Lions and N. Masmoudi in [24]. The argument they have used is very intricate and relies on the study of the behavior of the defect measure associated to a family of regular approximations. It would be interesting to know if the same result can be obtained adapting a compactness-Galerkin approach. When $N = 2$, it would also be interesting to know whether a simpler proof exists (notice that in the two-dimensional case to pass to the limit in $g_a(\nabla u^m, \tau^m)$ we just need strong convergence of the vorticity $\nabla \times u^m$).

In the general case, with $a \neq 0$, only local in time existence results can be established (at least if we do not impose additional geometric restrictions on the flow). Let us recall one of them, that has been taken from [11]:

**Theorem 21** *Let us assume that $N = 3$, $3 < r < +\infty$, $u^0 \in W^{2,r}(\Omega)^N \cap V_r$, $\tau^0 \in L^2_{\mathrm{s}}$ and $\tau^0_{ij} \in W^{1,r}(\Omega)$ for all $i, j$. Then there exist $T^* \in (0, +\infty)$ and exactly one strong local solution $\{u, p, \tau\}$ of* (70) *in* $[0, T^*]$ *($p$ is unique up to a function depending only on $t$).*

*This means that*

$$u \in L^s(0, T^*; W^{2,r}(\Omega)^N) \cap C^0([0, T^*]; V_r), \quad u_t \in L^s(0, T^*; L^r(\Omega)^N),$$
$$p \in L^s(0, T^*; W^{1,r}(\Omega)),$$
$$\tau \in C^0([0, T^*]; W^{1,r}(\Omega)^{N\times N}), \quad \tau_t \in L^s(0, T^*; L^r(\Omega)^{N\times N})$$

*for all finite $s > 1$ and the equations in* (70) *are satisfied a.e. in* $\Omega \times (0, T^*)$.

For more results concerning viscoelastic Oldroyd fluids and other related models, see [11] and the references therein. A very interesting approach to viscoelasticity is being developed recently with the introduction and analysis of the so called *micro-macro* models; see for instance [16, 17].

## References

[1] J. Beale, T. Kato, A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Comm. Math. Phys. 94 (1984), no. 1, 61–66.

[2] M. Böhm, *On a nonhomogeneous Bingham fluid*, J. Differ. Equations 60, p. 259–284, 1985.

[3] H. Brezis, *Análisis funcional. Teoría y aplicaciones*, Alianza Editorial, Madrid 1984.

[4] D. Cioranescu, *Sur une classe de fluides non-newtoniens*, Appl. Math. Optimiz. 3, p. 263–282, 1977.

[5] D. Cioranescu, *Quelques exemples de fluides newtoniens généralisés*, in "Mathematical topics in fluid mechanics", Pitman Res. Notes Math. Ser., 274, 1992.

[6] A.J. Chorin, J.E. Marsden, *A mathematical introduction to fluid mechanics*, Springer-Verlag, New-York 1979.

[7] P. Constantin, C. Foias, *Navier-Stokes equations*, The University of Chicago Press, Chicago 1988.

[8] R. Dautray & J.-L. Lions, *Analyse mathématique et calcul numérique pour les sciences et techniques, tome 3*, Masson, Paris 1985.

[9] G. Duvaut, J.-L. Lions, *Les inéquations en mécanique et en physique*, Dunod, Paris, 1972.

[10] E. Fernandez-Cara, F. Guillen, R.R. Ortega *Some theoretical results for visco-plastic and dilatant dluids with variable density*, Nonlinear Anal. T.M.A. 28, No. 6 (1997), p. 1079–1100.

[11] E. Fernandez-Cara, F. Guillen, R.R. Ortega, *Mathematical modeling and analysis of viscoelastic fluids of the Oldroyd kind*, Handbooks of Numerical Analysis, Part 2, Vol. VIII, p. 543–661, North-Holland, Amsterdam 2002.

[12] S. Friedlander, N. Pavlovic, *Blowup in a three-dimensional vector model for the Euler equations*, Comm. Pure Appl. Math. 57 (2004), no. 6, 705–725.

[13] J. Heywood, *The Navier-Stokes equations: on the existence, regularity and decay of solutions*, Ind. U. Math. J., 29 (1980), 639–681.

[14] J. Heywood, *Open problems in the theory of the Navier-Stokes equations for viscous incompressible flow*, Lecture Notes in Math. No. 1431, pp. 1–22, Springer-Verlag, Berlin 1988.

[15] D.D. Joseph, *Fluid dynamics of viscoelastic liquids*, Applied Mathematical Sciences, Vol. 84, Springer-Verlag, New York 1990.

[16] B. Jourdain, T. Lelievre, C. Le Bris, *Existence of solution for a micro-macro model of polymeric fluid: the FENE model*, J. Funct. Anal. 4 209 (2004) 162–193.

[17] B. Jourdain, T. Lelievre, C. Le Bris, *Numerical analysis of micro-macro simulations of polymeric fluid flows: a simple case*, Mat. Models and Methods in Appl. Sci. 12, No. 9 (2002) 1205–1243.

[18] R.M. Kerr, *Evidence for a singularity of the three-dimensional, incompressible Euler equations*, Phys. Fluids A 5 (1993), no. 7, 1725–1746.

[19] O.A. Ladyzhenskaja, *The mathematical theory of viscous incompresible flow*, Gordon and Breach, New York 1969.

[20] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthiers-Villars, Paris 1969.

[21] J.-L. Lions, J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Etudes, no. 19, Paris (1964), 5–68.

[22] P.-L. Lions, *Mathematical topics in fluid mechanics, Vol. 1: Incompressible models*, Clarendon Press, Oxford 1996.

[23] P.-L. Lions, N. Masmoudi, *Unicité des solutions faibles de Navier-Stokes dans $L^N(\Omega)$*, C.R. Acad. Sci. Paris, t. 327, Série I, p. 491–496, 1998.

[24] P.-L. Lions, N. Masmoudi, *Global solutions for some Oldroyd models of non-newtonian flows,* Chin. Ann. of Math. 21B:2 (2000), 131–146.

[25] J. Malek, K.R. Rajagopal, M. Ruzicka, *Existence and regularity of solutions and stability of the rest state for fluids with shear dependent viscosity*, Math. Models and Methods in Appl. Sci. 5, No. 6, 789–812.

[26] J. Malek, J. Necas, M. Rokyta, M. Ruzicka, *Weak and measure-valued solutions to evolutionary PDEs*, Chapman & Hall, London 1996.

[27] R.L. Panton, *Incompressible Flow*, Wiley Interscience, New York 1984.

[28] G. Prodi, *Un teorema de unicita oer le equazioni di Navier-Stokes*, Annali di Mat. 48, 173–182 (1959).

[29] M. Renardy, W.J. Hrusa, J.A. Nohel, *Mathematical problems in viscoelasticity*, Longman, London 1987.

[30] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Annali Mat. Pura ed Appl. (IV), Vol. CXLVI, 1987, pp. 65–96.

[31] R. Temam, *Navier-Stokes equations*, 3rd Edition, North-Holland, Amsterdam 1984.

[32] V.I. Yudovitch, *Uniquness theorem for the basic nonstationary problem in dynamocs of ideal incompressible fluid*, Math. Rev. Lett., 2 (1995), p. 27–38.

[33] E. Zeidler, *Nonlinear functional analysis and its applications, Vol. IV*, Springer-Verlag, New York, 1988.