EXACT CONTROLLABILITY TO TRAJECTORIES FOR SEMILINEAR HEAT EQUATIONS WITH DISCONTINUOUS DIFFUSION COEFFICIENTS

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Abstract. The results of this paper concern exact controllability to the trajectories for a coupled system of semilinear heat equations. We have transmission conditions on the interface and Dirichlet boundary conditions at the external part of the boundary so that the system can be viewed as a single equation with discontinuous coefficients in the principal part. Exact controllability to the trajectories is proved when we consider distributed controls supported in the part of the domain where the diffusion coefficient is the smaller and if the nonlinear term $f(y)$ grows slower than $|y| \log^{3/2}(1 + |y|)$ at infinity. In the proof we use null controllability results for the associate linear system and global Carleman estimates with explicit bounds or combinations of several of these estimates. In order to treat the terms appearing on the interface, we have to construct specific weight functions depending on geometry.

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1. Introduction and hypothesis

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be a bounded connected open set with boundary Γ of class C^2. Let $\omega \subset \Omega$ be a nonempty open subset and $T > 0$. We will use the following notation: $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$. For any $p \in [1, +\infty]$, we will denote by $|| \cdot ||_p$ the usual norm in $L^p(Q)$.

There are two different situations that will be analyzed in this paper. More precisely, let Ω_0 and Ω_1 be a partition of Ω in two non empty open sets such that

$$\textbf{Case 1:} \quad \overline{\Omega}_0 \subset\subset \Omega, \quad \Omega_1 = \Omega \backslash \overline{\Omega}_0 \quad \text{(see Fig. 1, left);} \tag{1}$$

$$\textbf{Case 2:} \quad \overline{\Omega}_1 \subset\subset \Omega, \quad \Omega_0 = \Omega \backslash \overline{\Omega}_1 \quad \text{(see Fig. 1, right).} \tag{2}$$

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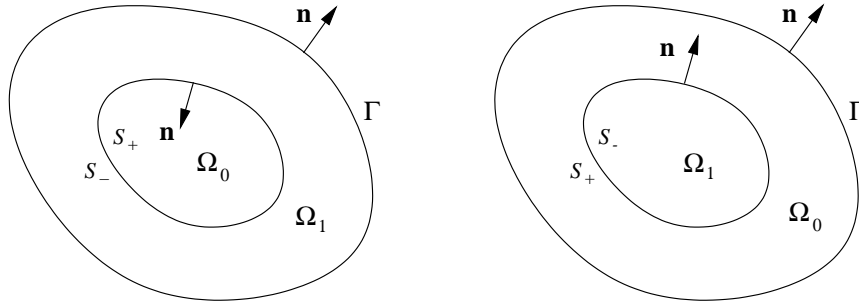
FIGURE 1. Two geometrical cases covered in this paper depending on $\Omega_0 \subset\subset \Omega$ or $\Omega_1 \subset\subset \Omega$.

We denote by $S = \overline{\Omega}_0 \cap \overline{\Omega}_1$ the interface, which will be supposed of class C^2 and by n the outward unit normal to Ω_1 at the points of S and also the outward unit normal to Ω at the points of Γ. Let S^+ (resp. S^-) be the part of S corresponding to the positive (resp. negative) direction of the normal n.

Remark 1.1. The two cases mentioned above are not exhaustive, we do not treat other possible geometrical situations in this paper.

In both cases mentioned above, we will consider the following transmission problem for semilinear heat equation

$$
\begin{cases}
\partial_t y - \mathrm{div}(a_0(x)\nabla y) + f(y) = v1_\omega + g_0 & \text{in} \quad \Omega_0 \times (0,T), \\
\partial_t y - \mathrm{div}(a_1(x)\nabla y) + f(y) = v1_\omega + g_1 & \text{in} \quad \Omega_1 \times (0,T), \\
y|_{S^+\times(0,T)} = y|_{S^-\times(0,T)}, \\
a_0(x)\,\partial_n y|_{S^+\times(0,T)} = a_1(x)\,\partial_n y|_{S^-\times(0,T)}, \\
y = 0, & \text{on} \quad \Sigma \\
y(x,0) = y_0 & \text{in} \quad \Omega.
\end{cases}
\tag{3}
$$

Here $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz-continuous function, $\partial_n y$ denotes the outward normal derivative to Ω_1, $y_0 \in L^2(\Omega)$ and $v \in L^r(0,T; L^r(\omega))$, $g_i \in L^r(0,T; L^r(\Omega_i))$, $i = 0, 1$ with r such that

$$
\begin{cases}
\dfrac{1}{r} + \dfrac{N}{2r} < 1 & \text{if } N \geq 2, \\
r = 2 & \text{if } N = 1.
\end{cases}
\tag{4}
$$

Remark 1.2. We could in fact consider $v \in L^p(0,T; L^q(\omega))$, $g_i \in L^p(0,T; L^q(\Omega_i))$, $i = 0, 1$ with $1/p + N/(2q) < 1$ in order to have L^∞ solutions, but in the sake of simplicity we take $p = q = r$.

Remark 1.3. Without loss of generality we can assume $y_0 \in L^\infty(\Omega)$. Otherwise, taking $v = 0$ for $t \in (0,\delta)$, $\delta > 0$ and thanks to the regularizing effect of parabolic equations, $y(\delta) \in L^\infty(\Omega)$ for some $\delta > 0$ [21, 22].

In (3), $y = y(x,t)$ is the state and $v = v(x,t)$ is the control which acts on the system through ω since 1_ω is the characteristic function of the set ω.

We will assume that the diffusion coefficient in (3) satisfies the following:

$$
\begin{aligned}
& a_i \in C^2(\overline{\Omega}_i) \quad \text{for } i = 0, 1, \\
& a_0|_{S^+} \neq a_1|_{S^-}.
\end{aligned}
\tag{5}
$$

System (3) represents the coupling between two parabolic semilinear equations whose diffusion coefficient has a jump. At the interface S, we impose the continuity of the solution y and also of the fluxes.

Let us set

$$a(x) = \begin{cases} a_0(x) & \text{if } x \in \Omega_0, \\ a_1(x) & \text{if } x \in \Omega_1. \end{cases} \tag{6}$$

We also set

$$g(x) = \begin{cases} g_0(x) & \text{if } x \in \Omega_0, \\ g_1(x) & \text{if } x \in \Omega_1. \end{cases} \tag{7}$$

Taking into account notations (6) and (7), problem (3) can be written in the divergence form (with discontinuous diffusion coefficients) as follows:

$$\begin{cases} \partial_t y - \operatorname{div}(a(x)\nabla y) + f(y) = v1_\omega + g & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0 & \text{in } \Omega. \end{cases} \tag{8}$$

We will require a to satisfy

$$a(x) \geq \alpha > 0 \quad \text{a.e. in } \Omega \tag{9}$$

and the following additional hypothesis:

$$a_0|_{S^+} \leq a_1|_{S^-} . \tag{10}$$

We assume that for each $\eta > 0$, there exists $C_\eta > 0$ such that

$$\left| \frac{f(s) - f(s')}{s - s'} \right|^{2/3} \leq C_\eta + \eta \log(1 + |s - s'|) \quad \forall\, s, s' \in \mathbb{R}. \tag{11}$$

Let us also consider an "ideal" trajectory y^*, solution of the problem (without control)

$$\begin{cases} \partial_t y^* - \operatorname{div}(a(x)\nabla y^*) + f(y^*) = g & \text{in } Q, \\ y^* = 0 & \text{on } \Sigma, \\ y(x,0)^* = y_0^* & \text{in } \Omega \end{cases} \tag{12}$$

where $y_0^* \in L^2(\Omega)$ and $g \in L^r(0,T;L^r(\Omega))$, with r as in (4). We know that under conditions (9) and (11), problem (12) possesses exactly one local solution in time (*cf.* [21] and [22]). Moreover, we can say that there exists a time $T^* > 0$, such that for $T < T^*$, the solution y^* of (12) satisfies $y^* \in C^0([0,T];L^2(\Omega)) \cap L^\infty(\delta,T;L^\infty(\Omega))$, for every $\delta > 0$.

The main goal of this paper is to analyze the controllability properties of (8).

Definition 1.1. We say that (8) *is exactly controllable to the trajectories* if, for any trajectory y^* solution of (12) and for any initial condition $y_0 \in L^2(\Omega)$, for every $T < T^*$, there exists a control $v \in L^r(0,T;L^r(\omega))$ such that (8) has a solution y on $(0,T)$ satisfying

$$y(x,T) = y^*(x,T) \quad \text{in } \Omega. \tag{13}$$

Definition 1.2. System (8) is said *null controllable* at time T if, for each $y_0 \in L^2(\Omega)$, there exists $v \in L^r(0,T; L^r(\omega))$ such that the corresponding initial boundary problem (8) admits a solution $y \in C^0([0,T]; L^2(\Omega))$ satisfying

$$y(x, T) = 0 \quad \text{in } \Omega. \tag{14}$$

For linear problems, it is easy to see that the notions of null controllability and exact controllability to the trajectories are equivalent, but this is not true for nonlinear systems.

Definition 1.3. It will be said that (8) is *approximately controllable* in $L^2(\Omega)$ at time T if, for any $y_0 \in L^2(\Omega)$, any $y_d \in L^2(\Omega)$ and any $\varepsilon > 0$, there exists a control $v \in L^r(0,T; L^r(\omega))$ such that the corresponding initial boundary problem (8) possesses a solution $y \in C^0([0,T]; L^2(\Omega))$, with

$$\|y(\cdot, T) - y_d\|_{L^2(\Omega)} \leq \varepsilon. \tag{15}$$

In the case in which the diffusion coefficients are sufficiently regular, the controllability of linear and semilinear parabolic systems has been analyzed in several recent papers. Among them, let us mention [1,5,11,13,15–17], and [8] concerning null controllability [9,12,13,25] and [8] for approximate controllability [16] and [13] for exact controllability to the trajectories.

2. MAIN RESULT

2.1. **Geometric hypothesis and main result**

In order to state the main result of this work, we need the following geometrical conditions.

Condition 2.1 (corresponding to case (1)). *We assume that there exists a vector field* $\zeta : \overline{\Omega}_1 \mapsto \mathbb{R}^N$, $\zeta \in C^1(\overline{\Omega}_1)$, *such that*

$$\zeta(x) \cdot n(x) < 0 \quad \forall\, x \in \Gamma, \tag{16}$$

$$\zeta(x) \cdot n(x) > 0 \quad \forall\, x \in S, \tag{17}$$

$$\zeta(x) \neq 0 \quad \forall\, x \in \Omega_1 \tag{18}$$

and if we consider the characteristics associated to ζ

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = \zeta(x(t)), & t > 0, \\ x(0) = x_0, \end{cases} \tag{19}$$

with $x_0 \in \Gamma$, *we also assume that for some time* $T_1 > 0$ *and for every* $x_0 \in \Gamma$, *there exists* $t_1(x_0) < T_1$ *such that the solution* $x(t)$ *of* (19) *verifies*

$$x(t) \in \Omega_1 \quad \text{for } 0 < t < t_1(x_0) \tag{20}$$

and

$$x(t_1(x_0)) \in S \quad \text{for } x_0 \in \Gamma. \tag{21}$$

Remark 2.1. Condition 2.1 implies that Γ and S are isotopic, but it is not clear whether isotopy is sufficient to ensure this condition.

Remark 2.2. Notice that Condition 2.1 is fulfilled for usual domains, see for example the cases of Figure 2.
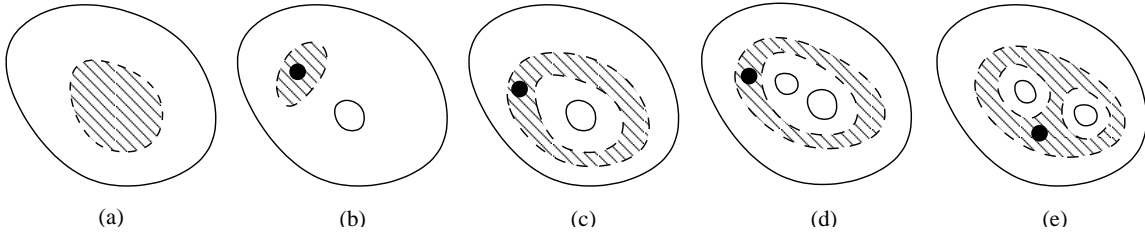
FIGURE 2. Condition 2.1 is fulfilled in situations (a, c, e) but not in (b) and (d). The boundary Γ is represented by a solid line and the interface S by a dashed line, the dashed region represents Ω_0 and the black dot the location of the control zone.
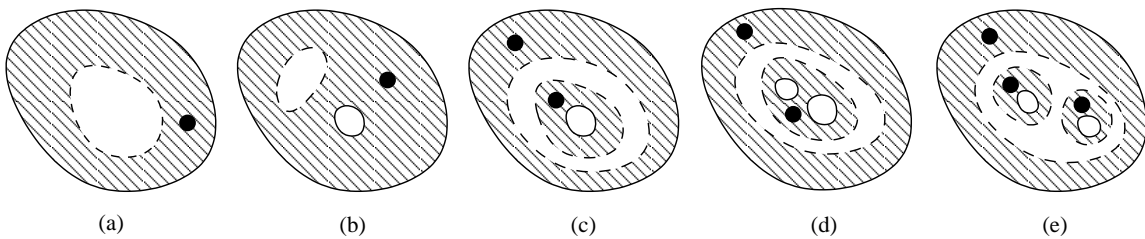

FIGURE 3. Condition 2.2 is fulfilled in situations (a–d) but not in (e) with the same notations as in the previous figure.

Condition 2.2 (corresponding to case (2)). *We assume that there exist two disjoint open sets $\mathcal{O}_1, \mathcal{O}_2 \subset\subset \Omega_1$ (with always a unit outward normal n) and vectors fields $\xi^i : \overline{\Omega}_1 \mapsto \mathbb{R}^N$, $\xi^i \in C^1(\overline{\Omega}_1)$, $i = 1, 2$, such that*

$$\begin{aligned}
\xi^i(x) \cdot n(x) &> 0 \quad \forall\, x \in S, \\
\xi^i(x) \cdot n(x) &> 0 \quad \forall\, x \in \partial\mathcal{O}_i, \quad i = 1, 2, \\
\xi^i(x) &\neq 0 \qquad \forall\, x \in \Omega_1 \backslash \mathcal{O}_i
\end{aligned} \tag{22}$$

and for the characteristics associated to ξ^i

$$\begin{cases}
\dfrac{\mathrm{d}x^i(t)}{\mathrm{d}t} = -\xi^i(x^i(t)), \quad t > 0, \\
x^i(0) = x_0^i,
\end{cases} \tag{23}$$

with $x_0^i \in S$, we assume also that for some time $T_2^i > 0$, and for all $x_0^i \in S$, there exists $t_2^i(x_0^i) < T_2^i$ such that the solution $x^i(t)$ of (23) verifies

$$x^i(t) \in \Omega_1 \backslash \mathcal{O}_i \quad \text{for } 0 < t < t_2^i(x_0^i)$$

and

$$x^i(t_2^i(x_0^i)) \in \partial\mathcal{O}_i \quad \text{for } x_0^i \in S, \ i = 1, 2.$$

Remark 2.3. Notice that this hypothesis is essentially Condition 2.1 written for the case (2). It is also fulfilled in usual geometrical cases, see for example the cases in Figure 3.

The aim of this paper is to prove the following theorem:

Theorem 2.1. *Assume that in problem (8) the coefficient a satisfies (5, 6, 9, 10), f is a locally Lipschitz-continuous function satisfying* (11) *or Condition* 2.1 *in case* (1) *or Condition 2.2 in case* (2) *are fulfilled. If $\omega \cap \Omega_0^i \neq \emptyset$, for each connected component Ω_0^i of Ω_0, then for each case (1) or (2, 8) is exactly controllable to the trajectories.*

The idea of the proof of Theorem 2.1 is the following. With a simple change of variables we reduce the problem of exact controllability to the trajectories for (8) to null controllability for a still nonlinear similar transmission problem. For this null controllability result we use approximate controllability to the zero state for an associated linear transmission problem with controls in $L^r(0, T; L^r(\omega))$ for r as in (4) and then we apply a fixed point method. For this we need explicit estimates on the cost of approximate controllability which is obtained from observability inequalities (see Props. 4.1 and 4.2). These estimates are deduced from global Carleman inequalities. In case (1), we use one single global Carleman inequality (see Th. 3.3) with a suitable weight function, whose construction is presented in Lemma 3.1. Case (2) is more complicated and we have to combine two different global Carleman inequalities (see Th. 3.4) with two appropriate weight functions whose construction are given in Lemma 3.2. The growth condition of the non linear term f is analyzed using the arguments of [13].

The idea of combining the controllability of a linearized system and a fixed point argument in the proof is rather general. It was introduced in [23] in the context of the boundary controllability of the semilinear wave equation. For other controllability results proved in a similar way, see for instance [9, 13, 15] and [8].

In the proofs we will suppose that Ω_0 and Ω_1 are connected sets and we assume the simpler hypothesis $\omega \cap \Omega_0 \neq \emptyset$. Otherwise the weight functions for Carleman inequalities are constructed analogously on each connected component of Ω_0 and Ω_1.

The paper is organized as follows. In Section 3 we deduce global Carleman inequalities, that we use for proving the main result. Section 4 is devoted to obtain some observability estimates. In Section 5, we prove Theorem 2.1. Finally, in Section 6, we give an explicit construction of suitable weight functions, needed for the global Carleman inequalities.

2.2. **Some consequences and extensions**

1. Observe that, the controllability result holds if the control acts in the part of the domain where the diffusion coefficient is smaller. To our knowledge, this result is the first one in the literature related to exact controllability to the trajectories when the diffusion coefficients are discontinuous.

2. In the case $s' = 0$ and $f(0) = 0$, notice that assumption (11) can be simply read as follows:

$$\lim_{|s| \to +\infty} \frac{f(s)}{|s| \log^{3/2}(1 + |s|)} = 0. \tag{24}$$

The proof of Theorem 2.1 also gives the result of null controllability for (8) with a suitable hypothesis on g under the hypothesis and the same geometrical cases considered in Theorem 2.1 by taking condition (24) instead of (11).

3. Notice that approximate controllability for a linear transmission problem is always true and it is independent of the choice of the part of the domain where the control acts as a consequence of unique continuation property. Nonlinear problem (8) with f growing as in (11) is still approximately controllable under the conditions of Theorem 2.1. This is due to the fact that approximate controllability in this case can be proved as a consequence of exact controllability to the trajectories. This idea is taken from [12], where approximate controllability for semilinear heat equations is obtained in such a way.

4. We can also consider in (8) the more general case in which the diffusion coefficients are represented by a real symmetric uniformly elliptic matrix A, *i.e.* there exists a constant $\alpha > 0$ such that

$$A(x, \xi, \xi) = \sum_{i,j=1}^{N} A_{ij} \xi_i \xi_j \geq \alpha |\xi|^2 \quad \forall \xi \in \mathbb{R}^N, \text{ for a.e. } x \in \Omega \tag{25}$$

and A is regular in each Ω_i, $i = 0, 1$. In this case, condition (10) has to be replaced by

$$\left[\frac{\det A}{An \cdot n} \right]_S \geq 0, \tag{26}$$

where $[\ ]_S$ denotes the jump across S.

Until now, null controllability for semilinear parabolic systems (in the divergence form) has been analyzed when the diffusion coefficients are sufficiently regular. More precisely, when $A = (A_{ij})$, $i, j = 1, \ldots, N$ with $A_{ij} \in C^{1,2}(\overline{Q})$ (see [15]).

2.3. **Open problems related to Theorem 2.1**

1. If $\omega \subset \Omega_1$ we do not know whether or not system (8) is exactly controllable to the trajectories. Is in this case null controllability also an open problem.

2. In [13], it is proved that even in the case of regular diffusion coefficients, for each $\beta > 2$, there exist functions $f = f(s)$ with $f(0) = 0$ and

$$\lim_{|s| \to \infty} \frac{|f(s)|}{|s| \log^\beta (1 + |s|)} = \alpha \quad \text{with } \alpha > 0, \tag{27}$$

such that the corresponding system (for the semilinear heat equation) is not null-controllable for any $T > 0$. In view of point 2 in Section 2.2, we see that, when f satisfies (27) with $3/2 \leq \beta \leq 2$, null-controllability of (8) is an open question.

3. On the other hand, it is proved in [13], that also in the case of regular diffusion coefficients, for each $\beta > 2$, there exist functions f satisfying (27) such that the corresponding system (for the semilinear heat equation) is not approximately controllable for all $T > 0$. Then, approximate controllability for the transmission problem (8) with $3/2 \leq \beta \leq 2$, is also an open question.

4. An abstract result due to Russell [20] shows that boundary exact controllability for the wave equation implies boundary exact null controllability for the heat equation with the same type of control and geometry. This result is proved in the case of smooth coefficients. If we consider this principle still true in the case of non smooth coefficients, the geometrical hypothesis that we consider here seems to be too restrictive in the case $N = 1$ (*cf.* [7] for the controllability of the corresponding wave equation) but not for $N \geq 2$.

5. In [14], it is proved null controllability result for the one-dimensional linear heat equation like $\rho(x) - (a(x)z_x)_x + m(x)z = 0$ with only BV coefficients without any assumption on the control zone. However, the proof is definitely strictly one dimensional relying on the corresponding one for the wave equation and null controllability result is true if the potential m depends only on space variable, but not on time. Then, even in the one-dimensional case it is not clear how to treat with this method a similar nonlinear problem.

3. GLOBAL CARLEMAN INEQUALITIES

In this section we will deduce two global Carleman inequalities, that we need for the proof of Theorem 2.1. For this purpose, we will introduce suitable weight functions. Let us first consider the situation of case (1) (see Fig. 1, left). The first weight function is given by the following result:

Lemma 3.1. *Assume that we have the geometrical situation of case* (1) *(see Fig. 1). Assume that the function a defined in (5, 6) satisfies (9, 10) and that Condition 2.1 holds. If $\omega \cap \Omega_0 \neq \emptyset$ then for every open set $\omega_0 \subset\subset \omega \cap \Omega_0$ there exists a function $\widetilde{\beta} \in C^0(\overline{\Omega})$, $\widetilde{\beta}_i = \widetilde{\beta}|_{\Omega_i} \in C^2(\overline{\Omega}_i)$, $i = 0, 1$, $\widetilde{\beta} > 0$ in Ω, such that*

$$\widetilde{\beta} = 0 \quad on \ \Gamma, \tag{28}$$

$$\partial_n \widetilde{\beta} < 0 \quad on \ \Gamma, \tag{29}$$

$$\widetilde{\beta} = 1 \quad on \ S, \tag{30}$$

$$\partial_n \widetilde{\beta}_0 > 0, \ \partial_n \widetilde{\beta}_1 > 0 \quad on \ S, \tag{31}$$

$$a_0 \partial_n \widetilde{\beta}_0 = a_1 \partial_n \widetilde{\beta}_1 \quad on \ S \tag{32}$$

and

$$|\nabla \widetilde{\beta}| > 0 \quad in \ \overline{\Omega} \backslash \omega_0. \tag{33}$$

The proof of Lemma 3.1 will be given in Section 6.

Now, we consider the geometrical case (2) (see Fig. 1, right). For the second Carleman inequality, which we will use to treat the situation 2, we need two suitable weight functions.

We have the following result:

Lemma 3.2. *Assume that we have the geometrical situation of case (2) (see Fig. 1). Assume that the function a defined in (5, 6) satisfies (9, 10) and that there exist two open disjoint sets $\mathcal{O}_1, \mathcal{O}_2 \subset\subset \Omega_1$ verifying Condition 2.2. Let B_i and $\widetilde{B}_i$, $i = 1, 2$ be balls such that $B_1 \subset\subset \widetilde{B}_1 \subset\subset \mathcal{O}_1$ and $B_2 \subset\subset \widetilde{B}_2 \subset\subset \mathcal{O}_2$. If $\omega \cap \Omega_0 \neq \emptyset$ then for every open set $\omega_0 \subset\subset \omega \cap \Omega_0$ there exist two functions $\widetilde{\beta}^1$ and $\widetilde{\beta}^2$ such that*

$$\widetilde{\beta}^1(x) = \begin{cases} \widetilde{\beta}_0(x) & if \ x \in \Omega_0, \\ \widetilde{\beta}_1^1(x) & if \ x \in \Omega_1, \end{cases} \qquad \widetilde{\beta}^2(x) = \begin{cases} \widetilde{\beta}_0(x) & if \ x \in \Omega_0, \\ \widetilde{\beta}_1^2(x) & if \ x \in \Omega_1, \end{cases} \tag{34}$$

with the following properties: $\widetilde{\beta}_0 \in C^2(\overline{\Omega}_0)$, $\widetilde{\beta}_0 > 0$ in Ω_0,

$$\widetilde{\beta}_0 = 0 \ on \ \Gamma, \quad \partial_n \widetilde{\beta}_0 < 0 \ on \ \Gamma, \tag{35}$$

$$\partial_n \widetilde{\beta}_0 > 0 \ on \ S, \quad \widetilde{\beta}_0 = 2 \ on \ S, \tag{36}$$

$$|\nabla \widetilde{\beta}_0| > 0 \quad in \quad \overline{\Omega}_0 \backslash \omega_0. \tag{37}$$

And for $i = 1, 2$, $\widetilde{\beta}_1^i \in C^2(\overline{\Omega}_1)$, $\widetilde{\beta}_1^i > 0$ in Ω_1,

$$\widetilde{\beta}_1^i = \widetilde{\beta}_0 = 2 \quad on \ S, \tag{38}$$

$$a_0 \partial_n \widetilde{\beta}_0 = a_1 \partial_n \widetilde{\beta}_1^i \quad on \ S, \quad i = 1, 2, \tag{39}$$

$$\widetilde{\beta}_1^1 \geq 2\widetilde{\beta}_1^2 \quad in \ \widetilde{B}_2, \tag{40}$$

$$\widetilde{\beta}_1^2 \geq 2\widetilde{\beta}_1^1 \quad in \ \widetilde{B}_1, \tag{41}$$

and

$$|\nabla \widetilde{\beta}_1^i| > 0 \quad in \ \overline{\Omega}_1 \backslash B_i, \quad i = 1, 2. \tag{42}$$

The proof of Lemma 3.2 will also be given in Section 6.

Remark 3.1. Notice that in geometrical case (2) ($\Omega_1 \subset\subset \Omega$) it is impossible to have a function $\widetilde{\beta}$ which is constant on S and such that $\nabla\widetilde{\beta} \neq 0$ in Ω_1.

Let us consider the functions

$$\beta = \widetilde{\beta} + K, \quad \overline{\beta} = \frac{5}{4} \max_{\overline{\Omega}} \beta, \tag{43}$$

with $K > 0$ such that $K \geq 5 \max_{\overline{\Omega}} \widetilde{\beta}$, and $\widetilde{\beta}$ is given by Lemma 3.1.

Let λ be a sufficiently large positive constant that only depends on Ω and ω. It will be fixed later on. For $t \in (0, T)$ and following [16] and [12], we introduce the following functions:

$$\varphi(x, t) = \frac{e^{\lambda\beta(x)}}{t(T - t)}, \quad \eta(x, t) = \frac{e^{\lambda\overline{\beta}} - e^{\lambda\beta}}{t(T - t)} \cdot \tag{44}$$

Notice that

$$\nabla\eta = -\lambda\varphi\nabla\beta, \quad \nabla\varphi = \lambda\varphi\nabla\beta. \tag{45}$$

Let us set

$$Z_0 = \{q \ : \ q \in C^2(\overline{\Omega}_i \times [0, T]), \quad i = 0, 1, \quad q|_{S^+ \times (0,T)} = q|_{S^- \times (0,T)},$$
$$a_0\partial_n q|_{S^+ \times (0,T)} = a_1\partial_n q|_{S^- \times (0,T)}, \quad q = 0 \quad \text{on} \quad \Sigma\}.$$

We have the following Carleman estimate:

Theorem 3.3. *Assume that $\omega \cap \Omega_0 \neq \emptyset$, a satisfies (5, 6, 9) and (10) and Condition 2.1 in case (1) is fulfilled. There exists $\lambda_1(\Omega, \omega, a) > 0$ such that for each $\lambda > \lambda_1$ there exists a positive constant C that only depends on Ω, ω and a, and $s_1(\lambda) > 0$ so that the following estimate holds*

$$s^3 \iint_Q e^{-2s\eta}t^{-3}(T - t)^{-3}|q|^2 \, dx \, dt + s \iint_Q e^{-2s\eta}t^{-1}(T - t)^{-1}|\nabla q|^2 \, dx \, dt$$

$$\leq C \left(s^3 \iint_{\omega \times (0,T)} e^{-2s\eta}t^{-3}(T - t)^{-3}|q|^2 \, dx \, dt \right. \tag{46}$$

$$\left. + \iint_Q e^{-2s\eta}|\partial_t q + \text{div}(a(x)\nabla q)|^2 \, dx \, dt \right)$$

for all $q \in Z_0$ and $s \geq s_1$. Moreover, s_1 is of the form $s_1 = \sigma_1(\Omega, \omega, a, \lambda)(T^2 + T)$, where σ_1 is a positive constant that only depends on Ω, ω, a and λ.

Proof of the Theorem 3.3. In the sequel, C will stand for a generic positive constant only depending on Ω, ω and a, whose value can change from line to line. We will also use the usual convention of repeated indices.

Let us assume $q \in Z_0$ and $s > 0$. We set

$$f = \partial_t q + \text{div}(a(x)\nabla q)$$

and

$$\psi = e^{-s\eta} q. \tag{47}$$

Notice that

$$\psi(0) = \psi(T) = 0. \tag{48}$$

We have the following equality:

$$e^{-s\eta}(\partial_t(e^{s\eta}\psi) + \operatorname{div}(a(x)\nabla(e^{s\eta}\psi))) = e^{-s\eta}f. \tag{49}$$

Using (45), we can write (49) in the form

$$M_1\psi + M_2\psi = e^{-s\eta}f + s\lambda\varphi\operatorname{div}(a(x)\nabla\beta)\psi - s\lambda^2\varphi a(x)|\nabla\beta|^2\psi, \tag{50}$$

where

$$M_1\psi = \operatorname{div}(a(x)\nabla\psi) + s^2\lambda^2\varphi^2|\nabla\beta|^2 a(x)\psi + s\partial_t\eta\psi \tag{51}$$

and

$$M_2\psi = \partial_t\psi - 2s\lambda\varphi a(x)\nabla\beta\nabla\psi - 2s\lambda^2\varphi a(x)|\nabla\beta|^2\psi. \tag{52}$$

Let set

$$f_s = e^{-s\eta}f + s\lambda\varphi\operatorname{div}(a(x)\nabla\beta)\psi - s\lambda^2\varphi a(x)|\nabla\beta|^2\psi. \tag{53}$$

From (50), we obtain

$$\|M_1\psi\|_2^2 + \|M_2\psi\|_2^2 + 2(M_1\psi, M_2\psi) = \|f_s\|_2^2, \tag{54}$$

where $(\cdot, \cdot)$ denotes the scalar product in $L^2(Q)$. Let us compute the scalar product in the left hand side of (54). We can write

$$(M_1\psi, M_2\psi) = I_{11'} + I_{12'} + I_{13'} + I_{21'} + I_{22'} + I_{23'} + I_{31'} + I_{32'} + I_{33'}. \tag{55}$$

In (55), all the integrals denote the respective scalar products for the terms of $M_1\psi$ and $M_2\psi$. For simplicity, in the sequel, we will write a instead of $a(x)$. We have

$$
\begin{aligned}
I_{11'} &= \iint_Q \operatorname{div}(a\nabla\psi)\partial_t\psi \,\mathrm{d}x\,\mathrm{d}t = -\iint_Q a\nabla\psi\,\partial_t(\nabla\psi)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^T\!\!\int_S a_1\nabla\psi \cdot n\,\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_S a_0\nabla\psi \cdot n\,\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t = -\frac{1}{2}\iint_Q a\partial_t(|\nabla\psi|^2)\,\mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned} \tag{56}
$$

Here we have used (48) which says that $\psi(0) = \psi(T) = 0$.

$$I_{12'} = -2s\lambda \sum_{l=0}^{1} \int_0^T \int_{\Omega_l} \varphi \operatorname{div}(a_l \nabla \psi) a_l \nabla \beta \nabla \psi \, \mathrm{d}x \, \mathrm{d}t = 2s\lambda \sum_{l=0}^{1} \int_0^T \int_{\Omega_l} a_l \partial_{x_i} \psi \partial_{x_i} (\varphi a_l \partial_{x_j} \beta \partial_{x_j} \psi) \, \mathrm{d}x \, \mathrm{d}t$$

$$-2s\lambda \int_0^T \int_{S^-} \varphi a_1^2 (\nabla \beta_1 \cdot \nabla \psi)(\nabla \psi \cdot n) \, \mathrm{d}\sigma \, \mathrm{d}t + 2s\lambda \int_0^T \int_{S^+} \varphi a_0^2 (\nabla \beta_0 \cdot \nabla \psi)(\nabla \psi \cdot n) \, \mathrm{d}\sigma \, \mathrm{d}t \qquad (57)$$

$$-2s\lambda \int_0^T \int_{\Gamma} \varphi a_1^2 (\nabla \beta \cdot \nabla \psi)(\nabla \psi \cdot n) \, \mathrm{d}\sigma \, \mathrm{d}t.$$

Let us consider the first term of (57). Also for simplicity, we will make the computation only for the integrals in Ω_1. We set

$$I_{12'}^1 = 2s\lambda \int_0^T \int_{\Omega_1} a_1 \partial_{x_i} \psi \, \partial_{x_i} (\varphi a_1 \partial_{x_j} \beta \partial_{x_j} \psi) \, \mathrm{d}x \, \mathrm{d}t.$$

We will use (45) and

$$\nabla \beta = (\nabla \beta \cdot n)n + \nabla_\tau \beta, \qquad (58)$$

$$\nabla \psi = (\nabla \psi \cdot n)n + \nabla_\tau \psi, \qquad (59)$$

where $\nabla_\tau \beta$ and $\nabla_\tau \psi$ denote the tangential gradients. Thanks to the choice of β we know from (30) and (43) that β is a constant on S, then $\nabla_\tau \beta = 0$ on S and we can write

$$I_{12'}^1 = 2s\lambda^2 \int_0^T \int_{\Omega_1} \varphi(a_1 \partial_{x_i} \psi \partial_{x_i} \beta)(a_1 \partial_{x_j} \beta \partial_{x_j} \psi) \, \mathrm{d}x \, \mathrm{d}t + 2s\lambda \int_0^T \int_{\Omega_1} \varphi a_1 \partial_{x_i} \psi \, \partial_{x_i}(a_1 \partial_{x_j} \beta) \partial_{x_j} \psi \, \mathrm{d}x \, \mathrm{d}t$$

$$+ s\lambda \int_0^T \int_{\Omega_1} \varphi a_1 (a_1 \partial_{x_j} \beta) \partial_{x_j} (|\nabla \psi|^2) \, \mathrm{d}x \, \mathrm{d}t. \qquad (60)$$

Integrating now by parts in the third term of (60) we obtain

$$I_{12'}^1 = 2s\lambda^2 \int_0^T \int_{\Omega_1} \varphi(a_1 \partial_{x_i} \psi \partial_{x_i} \beta)(a_1 \partial_{x_j} \beta \partial_{x_j} \psi) \, \mathrm{d}x \, \mathrm{d}t + 2s\lambda \int_0^T \int_{\Omega_1} \varphi a_1 \partial_{x_i} \psi \, \partial_{x_i}(a_1 \partial_{x_j} \beta) \partial_{x_j} \psi \, \mathrm{d}x \, \mathrm{d}t$$

$$-s\lambda^2 \int_0^T \int_{\Omega_1} \varphi |a_1 \nabla \beta|^2 |\nabla \psi|^2 \, \mathrm{d}x \, \mathrm{d}t - s\lambda \int_0^T \int_{\Omega_1} \varphi a_1 \partial_{x_j}(a_1 \partial_{x_j} \beta)|\nabla \psi|^2 \, \mathrm{d}x \, \mathrm{d}t$$

$$-s\lambda \int_0^T \int_{\Omega_1} \varphi(\partial_{x_j} a_1)(a_1 \partial_{x_j} \beta)|\nabla \psi|^2 \, \mathrm{d}x \, \mathrm{d}t + s\lambda \int_0^T \int_{S^-} \varphi a_1^2(\partial_n \beta_1)|\nabla \psi|^2 \, \mathrm{d}\sigma \, \mathrm{d}t \qquad (61)$$

$$+s\lambda \int_0^T \int_{\Gamma} \varphi a_1^2(\partial_n \beta)|\nabla \psi|^2 \, \mathrm{d}\sigma \, \mathrm{d}t.$$

For the integrals in Ω_0 it is sufficient to take into account that n is the outward unit normal to Ω_1 and replace in (61), n by $-n$, S^- by S^+ and Ω_1 by Ω_0.

Consequently, from (57) and (61), using again (59) we deduce

$$
\begin{aligned}
I_{12'} = &-s\lambda^2 \iint_Q \varphi |a\nabla\beta|^2 |\nabla\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + 2s\lambda^2 \iint_Q \varphi(a\nabla\psi \cdot \nabla\beta)^2 \,\mathrm{d}x\,\mathrm{d}t \\
&- s\lambda \int_0^T \int_{S^-} \varphi(\partial_n\beta_1)|a_1\partial_n\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}t + s\lambda \int_0^T \int_{S^+} \varphi(\partial_n\beta_0)|a_0\partial_n\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}t \\
&+ s\lambda \int_0^T \int_{S^-} \varphi a_1(a_1\partial_n\beta_1)|\nabla_\tau\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}t - s\lambda \int_0^T \int_{S^+} \varphi a_0(a_0\partial_n\beta_0)|\nabla_\tau\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}t \\
&- s\lambda \int_0^T \int_\Gamma \varphi a_1^2(\partial_n\beta)|\partial_n\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}t + X_1
\end{aligned}
\tag{62}
$$

with

$$
\begin{aligned}
X_1 = &\, 2s\lambda \iint_Q \varphi a \partial_{x_i}\psi \, \partial_{x_i}(a\partial_{x_j}\beta)\partial_{x_j}\psi \,\mathrm{d}x\,\mathrm{d}t - s\lambda \iint_Q \varphi a \partial_{x_j}(a\partial_{x_j}\beta)|\nabla\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&- s\lambda \iint_Q \varphi(\partial_{x_j}a)(a\partial_{x_j}\beta)|\nabla\psi|^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{63}
$$

Finally, we get

$$
\begin{aligned}
I_{12'} = &-s\lambda^2 \iint_Q \varphi |a\nabla\beta|^2 |\nabla\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + 2s\lambda^2 \iint_Q \varphi(a\nabla\psi \cdot \nabla\beta)^2 \,\mathrm{d}x\,\mathrm{d}t \\
&+ s\lambda \int_0^T \int_S \varphi |a\partial_n\psi|^2 \, [\partial_n\beta]_S \,\mathrm{d}\sigma\,\mathrm{d}t - s\lambda \int_0^T \int_S \varphi |\nabla_\tau\psi|^2 (a\partial_n\beta)[a]_S \,\mathrm{d}\sigma\,\mathrm{d}t \\
&- s\lambda \int_0^T \int_\Gamma \varphi |a\partial_n\psi|^2 (\partial_n\beta) \,\mathrm{d}\sigma\,\mathrm{d}t + X_1,
\end{aligned}
\tag{64}
$$

where X_1 is given by (63), and $[\cdot]_S$ denote the jump on S. Notice that in (64), due to the choice of $\widetilde{\beta}$ the boundary integrals are nonnegative and this is essential. In fact, from (9) and (10) we have

$$
[a]_S = a_0 - a_1 \leq 0 \quad \text{on} \quad S.
\tag{65}
$$

On the other hand, from (10, 31, 32) and (43), we deduce that

$$
[\partial_n\beta]_S = \partial_n\beta_0 - \partial_n\beta_1 \geq 0 \quad \text{on} \quad S,
\tag{66}
$$

since n is the outward unit normal to Ω_1. Moreover, thanks to (29) and (43), we have $\partial_n\beta \leq 0$ on Γ. This justifies the above statement.

Let us compute the scalar product of the first term of $M_1\psi$ and the third one of $M_2\psi$.

$$
\begin{aligned}
I_{13'} &= -2s\lambda^2 \sum_{l=0}^{1} \int_0^T \int_{\Omega_l} \operatorname{div}(a_l\nabla\psi)\varphi a_l|\nabla\beta|^2\psi \, \mathrm{d}x \, \mathrm{d}t = 2s\lambda^2 \iint_Q a\nabla\psi\nabla(\varphi a|\nabla\beta|^2\psi) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad -2s\lambda^2 \int_0^T \int_{S^-} \varphi a_1(\partial_n\psi)a_1|\nabla\beta_1|^2\psi \, \mathrm{d}\sigma \, \mathrm{d}t + 2s\lambda^2 \int_0^T \int_{S^+} \varphi a_0(\partial_n\psi)a_0|\nabla\beta_0|^2\psi \, \mathrm{d}\sigma \, \mathrm{d}t \\
&= 2s\lambda^2 \iint_Q \varphi|a\nabla\beta|^2|\nabla\psi|^2 \, \mathrm{d}x \, \mathrm{d}t + 2s\lambda^2 \sum_{l=0}^{1} \int_0^T \int_{\Omega_l} \varphi a_l\nabla(a_l|\nabla\beta|^2)\nabla\psi\psi \, \mathrm{d}x \, \mathrm{d}t \\
&\quad +2s\lambda^3 \iint_Q \varphi a^2\nabla\beta|\nabla\beta|^2\nabla\psi\psi \, \mathrm{d}x \, \mathrm{d}t \\
&\quad -2s\lambda^2 \int_0^T \int_{S^-} \varphi a_1(\partial_n\psi)a_1|\partial_n\beta_1|^2\psi \, \mathrm{d}\sigma \, \mathrm{d}t + 2s\lambda^2 \int_0^T \int_{S^+} \varphi a_0(\partial_n\psi)a_0|\partial_n\beta_0|^2\psi \, \mathrm{d}\sigma \, \mathrm{d}t.
\end{aligned}
\tag{67}
$$

In (67), again we have used (58) and the fact that $\nabla_\tau\beta = 0$ on S. Finally, we have

$$
I_{13'} = 2s\lambda^2 \iint_Q \varphi|a\nabla\beta|^2|\nabla\psi|^2 \, \mathrm{d}x \, \mathrm{d}t + X_2,
\tag{68}
$$

where

$$
\begin{aligned}
X_2 &= 2s\lambda^2 \iint_Q \varphi a\nabla(a|\nabla\beta|^2)\nabla\psi\psi \, \mathrm{d}x \, \mathrm{d}t + 2s\lambda^3 \iint_Q \varphi a^2\nabla\beta|\nabla\beta|^2\nabla\psi\psi \, \mathrm{d}x \, \mathrm{d}t \\
&\quad +2s\lambda^2 \int_0^T \int_S \varphi(a\partial_n\psi)(a\partial_n\beta)\left[\partial_n\beta\right]_S \psi \, \mathrm{d}\sigma \, \mathrm{d}t.
\end{aligned}
\tag{69}
$$

The scalar product of the second term of $M_1\psi$ with the first one of $M_2\psi$ gives

$$
I_{21'} = s^2\lambda^2 \iint_Q \varphi^2|\nabla\beta|^2 a\partial_t\psi\psi \, \mathrm{d}x \, \mathrm{d}t = -s^2\lambda^2 \iint_Q \varphi\partial_t\varphi|\nabla\beta|^2 a|\psi|^2 \, \mathrm{d}x \, \mathrm{d}t.
\tag{70}
$$

We now consider the scalar product between the second term of $M_1\psi$ with the second one of $M_2\psi$. The following holds:

$$
\begin{aligned}
I_{22'} &= -2s^3\lambda^3 \sum_{l=0}^{1} \int_0^T \int_{\Omega_l} \varphi^3 a_l^2|\nabla\beta|^2(\nabla\beta)\nabla\psi\psi \, \mathrm{d}x \, \mathrm{d}t \\
&= 3s^3\lambda^4 \iint_Q \varphi^3 a^2|\nabla\beta|^4|\psi|^2 \, \mathrm{d}x \, \mathrm{d}t + s^3\lambda^3 \sum_{l=0}^{1} \int_0^T \int_{\Omega_l} \varphi^3\operatorname{div}(a_l^2|\nabla\beta|^2\nabla\beta)|\psi|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad -s^3\lambda^3 \int_0^T \int_{S^-} \varphi^3|a_1\partial_n\beta_1|^2(\partial_n\beta_1)|\psi|^2 \, \mathrm{d}\sigma \, \mathrm{d}t + s^3\lambda^3 \int_0^T \int_{S^+} \varphi^3|a_0\partial_n\beta_0|^2(\partial_n\beta_0)|\psi|^2 \, \mathrm{d}\sigma \, \mathrm{d}t \\
&= 3s^3\lambda^4 \iint_Q \varphi^3 a^2|\nabla\beta|^4|\psi|^2 \, \mathrm{d}x \, \mathrm{d}t + s^3\lambda^3 \int_0^T \int_S \varphi^3|a\partial_n\beta|^2\left[\partial_n\beta\right]_S |\psi|^2 \, \mathrm{d}\sigma \, \mathrm{d}t + X_3,
\end{aligned}
\tag{71}
$$

where

$$X_3 = s^3\lambda^3 \sum_{l=0}^{1} \int_0^T\!\!\int_{\Omega_l} \varphi^3 \mathrm{div}(a_l^2|\nabla\beta|^2\nabla\beta)|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{72}$$

Observe again that the last boundary integral in (71) is nonnegative because (66) holds.

Now we consider the third and the second terms of $M_1\psi$ and $M_2\psi$, respectively. Using (45) we can write

$$
\begin{aligned}
I_{32'} = -2s^2\lambda \sum_{l=0}^{1}\int_0^T\!\!\int_{\Omega_l} \varphi\partial_t\eta a_l\nabla\beta\cdot\nabla\psi\psi \,\mathrm{d}x\,\mathrm{d}t &= s^2\lambda\iint_Q \varphi\partial_t\eta\,\mathrm{div}(a\nabla\beta)|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+ s^2\lambda^2\iint_Q \varphi(\partial_t\eta)a|\nabla\beta|^2|\psi|^2\,\mathrm{d}x\,\mathrm{d}t - s^2\lambda^2\iint_Q \varphi(\partial_t\varphi)a|\nabla\beta|^2|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&- s^2\lambda\int_0^T\!\!\int_{S^-}\varphi\partial_t\eta(a_1\partial_n\beta_1)|\psi|^2\,\mathrm{d}\sigma\,\mathrm{d}t + s^2\lambda\int_0^T\!\!\int_{S^+}\varphi\partial_t\eta(a_0\partial_n\beta_0)|\psi|^2\,\mathrm{d}\sigma.
\end{aligned}
\tag{73}
$$

Since (32) holds we know that $a_0\partial_n\beta_0|_{S^+} - a_1\partial_n\beta_1|_{S^-} = 0$, then we get

$$I_{32'} = s^2\lambda\iint_Q \varphi\partial_t\eta\,\mathrm{div}(a\nabla\beta)|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + s^2\lambda^2\iint_Q \varphi(\partial_t\eta)a|\nabla\beta|^2|\psi|^2\,\mathrm{d}x\,\mathrm{d}t - s^2\lambda^2\iint_Q \varphi(\partial_t\varphi)a|\nabla\beta|^2|\psi|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{74}$$

The last integrals give

$$I_{23'} = -2s^3\lambda^4\iint_Q \varphi^3 a^2|\nabla\beta|^4|\psi|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{75}$$

$$I_{31'} = s\iint_Q \partial_t\eta\psi\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t = -\frac{1}{2}s\iint_Q \partial_t^2\eta|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \tag{76}$$

and

$$I_{33'} = -2s^2\lambda^2\iint_Q \varphi a\partial_t\eta|\nabla\beta|^2|\psi|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{77}$$

Finally from (55), taking into account (56, 64, 67, 70, 71, 74–76) and (77) we deduce that

$$
\begin{aligned}
(M_1\psi, M_2\psi) = s\lambda^2\iint_Q \varphi|a\nabla\beta|^2|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t &+ 2s\lambda^2\iint_Q \varphi(a\nabla\psi\cdot\nabla\beta)^2\,\mathrm{d}x\,\mathrm{d}t\\
&+ s^3\lambda^4\iint_Q \varphi^3 a^2|\nabla\beta|^4|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^3\int_0^T\!\!\int_S \varphi^3|a\partial_n\beta|^2\,[\partial_n\beta]_S\,|\psi|^2\,\mathrm{d}\sigma\,\mathrm{d}t\\
&+ s\lambda\int_0^T\!\!\int_S \varphi|a\partial_n\psi|^2\,[\partial_n\beta]_S\,\mathrm{d}\sigma\,\mathrm{d}t - s\lambda\int_0^T\!\!\int_S \varphi|\nabla_\tau\psi|^2(a\partial_n\beta)[a]_S\,\mathrm{d}\sigma\,\mathrm{d}t\\
&- s\lambda\int_0^T\!\!\int_\Gamma \varphi|a\partial_n\psi|^2(\partial_n\beta)\,\mathrm{d}\sigma\,\mathrm{d}t + X_1 + X_2 + I_{21'} + X_3 + I_{31'} + I_{32'} + I_{33'},
\end{aligned}
\tag{78}
$$

where X_1, X_2, $I_{21'}$, X_3, $I_{31'}$, $I_{32'}$, and $I_{33'}$, are given by (63, 69, 70, 72, 74, 76) and (77) respectively. Thus, we have from (54) the following identity:

$$
\|M_1\psi\|_2^2 + \|M_2\psi\|_2^2 + 2s\lambda^2 \iint_Q \varphi|a\nabla\beta|^2|\nabla\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + 4s\lambda^2 \iint_Q \varphi(a\nabla\psi\cdot\nabla\beta)^2 \,\mathrm{d}x\,\mathrm{d}t
$$

$$
+ 2s^3\lambda^4 \iint_Q \varphi^3 a^2 |\nabla\beta|^4 |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + 2s^3\lambda^3 \int_0^T\!\!\int_S \varphi^3 |a\partial_n\beta|^2 \left[\partial_n\beta\right]_S |\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}t
$$

$$
+ 2s\lambda \int_0^T\!\!\int_S \varphi|a\partial_n\psi|^2 \left[\partial_n\beta\right]_S \,\mathrm{d}\sigma\,\mathrm{d}t - 2s\lambda \int_0^T\!\!\int_S \varphi|\nabla_\tau\psi|^2 (a\partial_n\beta)[a]_S \,\mathrm{d}\sigma\,\mathrm{d}t
$$

$$
- 2s\lambda \int_0^T\!\!\int_\Gamma \varphi|a\partial_n\psi|^2 (\partial_n\beta) \,\mathrm{d}\sigma\,\mathrm{d}t = \|f_s\|_2^2 - 2(X_1 + X_2 + I_{21'} + X_3 + I_{31'} + I_{32'} + I_{33'}). \tag{79}
$$

As we mentioned above, since $\partial_n\beta \leq 0$ on Γ and thanks to (66) and (65), all the boundary integrals in the left hand side of (79) are nonnegative. Moreover, we know that (9) and (33) hold. Then, for some $\lambda_0(\Omega,\omega,a) \geq 1$ we have

$$
\begin{aligned}
\lambda^2\varphi|a\nabla\beta|^2 &\geq C(\Omega,\omega,a)\lambda^2\varphi &&\text{in} \quad (\Omega\backslash\omega_0)\times(0,T) \quad \forall\lambda \geq \lambda_0(\Omega,\omega,a), \\
\lambda^4\varphi^3 a^2|\nabla\beta|^4 &\geq C(\Omega,\omega,a)\lambda^4\varphi^3 &&\text{in} \quad (\Omega\backslash\omega_0)\times(0,T) \quad \forall\lambda \geq \lambda_0(\Omega,\omega,a).
\end{aligned} \tag{80}
$$

On the other hand, we use that

$$
|a\partial_{x_j}(a\partial_{x_j}\beta)| \leq C(\Omega,\omega,a), \tag{81}
$$

$$
a|\nabla\beta|^2 \leq C(\Omega,\omega,a), \tag{82}
$$

$$
|\nabla(a_l|\nabla\beta|^2)| \leq C(\Omega,\omega,a), \quad l=0,1, \tag{83}
$$

$$
\mathrm{div}(a_l^2|\nabla\beta|^2\nabla\beta) \leq C(\Omega,\omega,a), \quad l=0,1, \tag{84}
$$

$$
\mathrm{div}(a\nabla\beta) \leq C(\Omega,\omega,a). \tag{85}
$$

Then, it is not difficult to check that for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$
|X_1| \leq C(\Omega,\omega,a)s\lambda \iint_Q \varphi|\nabla\psi|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{86}
$$

$$
|X_2| \leq C(\Omega,\omega,a)\left(s\lambda \iint_Q \varphi|\nabla\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + s\lambda^3 T^4 \iint_Q \varphi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t\right) \tag{87}
$$

$$
+ \varepsilon s\lambda^2 \iint_Q \varphi|\nabla\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + C_\varepsilon s\lambda^4 T^4 \iint_Q \varphi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t
$$

$$
+ \varepsilon s\lambda \int_0^T\!\!\int_S \varphi|a\partial_n\psi|^2 \left[\partial_n\beta\right]_S \,\mathrm{d}\sigma\,\mathrm{d}t + C_\varepsilon s\lambda^3 T^4 \int_0^T\!\!\int_S \varphi^3|a\partial_n\beta|^2 \left[\partial_n\beta\right]_S |\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}t,
$$

$$
|X_3| \leq C(\Omega,\omega,a)s^3\lambda^3 \iint_Q \varphi^3|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{88}
$$

In the estimate (87), we have used Young inequalities and the fact that

$$
\varphi \leq C\varphi^3 t^2(T-t)^2 \leq CT^4\varphi^3
$$

and we will chose $\varepsilon > 0$ sufficiently small. Moreover, we have

$$|\partial_t \varphi| = \frac{|T - 2t|}{t^2(T-t)^2} e^{\lambda \beta} \leq C(\Omega, \omega) T \varphi^2, \tag{89}$$

$$|\partial_t \eta| = \frac{|T - 2t|(e^{\lambda \overline{\beta}} - e^{\lambda \beta})}{t^2(T-t)^2} \leq C(\Omega, \omega) \frac{T e^{\lambda \overline{\beta}}}{t^2(T-t)^2} \leq C(\Omega, \omega) \frac{T e^{2\lambda \beta}}{t^2(T-t)^2} \leq C(\Omega, \omega) T \varphi^2, \tag{90}$$

$$|\partial_{tt}^2 \eta| = \frac{2|T^2 - 3Tt + 3t^2|(e^{\lambda \overline{\beta}} - e^{\lambda \beta})}{t^3(T-t)^3} \leq \frac{14 T^2 e^{\lambda \overline{\beta}}}{t^3(T-t)^3} \leq C(\Omega, \omega) \frac{T^2 e^{2\lambda \beta}}{t^3(T-t)^3} \leq C(\Omega, \omega) T^2 \varphi^3. \tag{91}$$

In (90) and (91), we have used that $e^{\lambda \overline{\beta}} \leq e^{2\lambda \beta}$. This is implied by the fact that

$$\overline{\beta} = \frac{5}{4} \max_{\overline{\Omega}} \beta \leq 2 \min_{\overline{\Omega}} \beta, \tag{92}$$

which is a consequence of the choice of K in (43). Taking into account, equations (82, 85, 89, 90) and (91) we deduce

$$|I_{21'}| \leq C(\Omega, \omega, a) T s^2 \lambda^2 \iint_Q \varphi^3 |\psi|^2 \, dx \, dt, \tag{93}$$

$$|I_{31'}| \leq C(\Omega, \omega, a) T^2 s \iint_Q \varphi^3 |\psi|^2 \, dx \, dt, \tag{94}$$

$$|I_{32'}| \leq C(\Omega, \omega, a) T s^2 \lambda^2 \iint_Q \varphi^3 |\psi|^2 \, dx \, dt, \tag{95}$$

$$|I_{33'}| \leq C(\Omega, \omega, a) T s^2 \lambda^2 \iint_Q \varphi^3 |\psi|^2 \, dx \, dt. \tag{96}$$

On the other hand, from (53) we can write that

$$\|f_s\|_2^2 \leq \|e^{-s\eta} f\|_2^2 + C(\Omega, \omega, a) s^2 \lambda^4 \iint_Q \varphi^2 |\psi|^2 \leq \|e^{-s\eta} f\|_2^2 + C(\Omega, \omega, a) s^2 \lambda^4 T^2 \iint_Q \varphi^3 |\psi|^2. \tag{97}$$

Using (80, 86–88, 93–96) and (97) in (79), we obtain:

$$\|M_1 \psi\|_2^2 + \|M_2 \psi\|_2^2 + C s^3 \lambda^4 \int_0^T \int_{\Omega \setminus \omega_0} \varphi^3 |\psi|^2 \, dx \, dt + C s \lambda^2 \int_0^T \int_{\Omega \setminus \omega_0} \varphi |\nabla \psi|^2 \, dx \, dt$$

$$+ 2 s^3 \lambda^3 \int_0^T \int_S \varphi^3 |a \partial_n \beta|^2 \, [\partial_n \beta]_S \, |\psi|^2 \, d\sigma \, dt + 2 s \lambda \int_0^T \int_S \varphi |a \partial_n \psi|^2 \, [\partial_n \beta]_S \, d\sigma \, dt$$

$$\leq \|e^{-s\eta} f\|_2^2 + C s \lambda^3 T^4 \int_0^T \int_S \varphi^3 |a \partial_n \beta|^2 \, [\partial_n \beta]_S \, |\psi|^2 \, d\sigma \, dt + \varepsilon s \lambda \int_0^T \int_S \varphi |a \partial_n \psi|^2 \, [\partial_n \beta]_S \, d\sigma \, dt$$

$$+ C s \lambda \iint_Q \varphi |\nabla \psi|^2 \, dx \, dt + \varepsilon s \lambda^2 \iint_Q \varphi |\nabla \psi|^2 \, dx \, dt + C s^3 \lambda^3 \iint_Q \varphi^3 |\psi|^2 \, dx \, dt$$

$$+ C s^2 (\lambda^4 T^2 + \lambda^2 T) \iint_Q \varphi^3 |\psi|^2 \, dx \, dt + C s (T^2 + \lambda^4 T^4) \iint_Q \varphi^3 |\psi|^2 \, dx \, dt.$$

From this, for $\lambda \geq \lambda_1(\Omega, \omega, a) \geq \lambda_0(\Omega, \omega, a)$, with λ_1 not depending on T, and for ε small enough, we can write

$$\|M_1\psi\|_2^2 + \|M_2\psi\|_2^2 + s^3\lambda^4 \iint_Q \varphi^3|\psi|^2 \, \mathrm{d}x\,\mathrm{d}t + s\lambda^2 \iint_Q \varphi|\nabla\psi|^2 \, \mathrm{d}x\,\mathrm{d}t$$

$$+2s^3\lambda^3 \int_0^T\!\!\int_S \varphi^3|a\partial_n\beta|^2\,[\partial_n\beta]_S\,|\psi|^2\,\mathrm{d}\sigma\,\mathrm{d}t \leq C\left[\|\mathrm{e}^{-s\eta}f\|_2^2\right.$$

$$+s^3\lambda^4 \int_0^T\!\!\int_{\omega_0} \varphi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + s\lambda^2 \int_0^T\!\!\int_{\omega_0} \varphi|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t + s(T^2+\lambda^4T^4)\iint_Q \varphi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\left.+s^2(\lambda^4T^2+\lambda^2T)\iint_Q \varphi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + s\lambda^3T^4 \int_0^T\!\!\int_S \varphi^3|a\partial_n\beta|^2\,[\partial_n\beta]_S\,|\psi|^2\,\mathrm{d}\sigma\,\mathrm{d}t\right].$$

We take now $s \geq \sigma_0(\Omega, \omega, a, \lambda)(T^2 + T)$, then we also have

$$\|M_1\psi\|_2^2 + \|M_2\psi\|_2^2 + s^3\lambda^4 \iint_Q \varphi^3|\psi|^2 \, \mathrm{d}x\,\mathrm{d}t + s\lambda^2 \iint_Q \varphi|\nabla\psi|^2 \, \mathrm{d}x\,\mathrm{d}t$$

$$\leq C\left[\|\mathrm{e}^{-s\eta}f\|_2^2 + s^3\lambda^4 \int_0^T\!\!\int_{\omega_0} \varphi^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + s\lambda^2 \int_0^T\!\!\int_{\omega_0} \varphi|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right]. \quad (98)$$

Let us deduce from (98) that (46) holds for all $s \geq s_1$ where $s_1 = \sigma_1(\Omega, \omega, a, \lambda)(T^2 + T)$.
    Recall that $\psi = \mathrm{e}^{-s\eta}q$. Then,

$$\partial_{x_i}\psi = \mathrm{e}^{-s\eta}(\partial_{x_i}q - s\partial_{x_i}\eta q) = \mathrm{e}^{-s\eta}(\partial_{x_i}q + s\lambda\varphi\partial_{x_i}\beta\,q).$$

So we can write that

$$\mathrm{e}^{-s\eta}\partial_{x_i}q = \partial_{x_i}\psi - s\lambda\mathrm{e}^{-s\eta}\varphi\partial_{x_i}\beta\,q.$$

Consequently, we find the following:

$$s\lambda^2 \iint_Q \mathrm{e}^{-2s\eta}\varphi|\nabla q|^2 \, \mathrm{d}x\,\mathrm{d}t = s\lambda^2 \iint_Q \varphi|\nabla\psi - \mathrm{e}^{-s\eta}s\lambda\varphi\nabla\beta q|^2 \, \mathrm{d}x\,\mathrm{d}t$$

$$\leq Cs\lambda^2 \iint_Q \varphi|\nabla\psi|^2 \, \mathrm{d}x\,\mathrm{d}t + C(\Omega, \omega)s^3\lambda^4 \iint_Q \mathrm{e}^{-2s\eta}\varphi^3|q|^2 \, \mathrm{d}x\,\mathrm{d}t.$$

Then, from (98) we have

$$s^3\lambda^4 \iint_Q \mathrm{e}^{-2s\eta}\varphi^3|q|^2 \, \mathrm{d}x\,\mathrm{d}t + s\lambda^2 \iint_Q \mathrm{e}^{-2s\eta}\varphi|\nabla q|^2 \, \mathrm{d}x\,\mathrm{d}t$$

$$\leq C(\Omega, \omega, a)\left[\|\mathrm{e}^{-s\eta}f\|_2^2 + s^3\lambda^4 \int_0^T\!\!\int_\omega \mathrm{e}^{-2s\eta}\varphi^3|q|^2\,\mathrm{d}x\,\mathrm{d}t + s\lambda^2 \int_0^T\!\!\int_{\omega_0} \mathrm{e}^{-2s\eta}\varphi|\nabla q|^2\,\mathrm{d}x\,\mathrm{d}t\right]. \quad (99)$$

In order to conclude the proof of the Carleman inequality (46) it is sufficient to prove that

$$s\lambda^2 \int_0^T\!\!\int_{\omega_0} \mathrm{e}^{-2s\eta}\varphi|\nabla q|^2 \, \mathrm{d}x\,\mathrm{d}t \leq C\left[\|\mathrm{e}^{-s\eta}f\|_2^2 + s^3\lambda^4 \int_0^T\!\!\int_\omega \mathrm{e}^{-2s\eta}\varphi^3|q|^2\,\mathrm{d}x\,\mathrm{d}t\right.$$

$$\left.+ s^2(\lambda^4T^2+\lambda^2T)\int_0^T\!\!\int_\omega \mathrm{e}^{-2s\eta}\varphi^3|q|^2\,\mathrm{d}x\,\mathrm{d}t + s(\lambda^3T^4+\lambda^3T^2+\lambda^2T^3)\int_0^T\!\!\int_\omega \mathrm{e}^{-2s\eta}\varphi^3|q|^2\,\mathrm{d}x\,\mathrm{d}t\right]. \quad (100)$$

In fact, combining (99) and (100), we deduce that the global Carleman estimate (46) is true for $s \geq \sigma_1(\Omega, \omega, a, \lambda)$ $(T^2 + T^{3/2} + T)$. Notice that it is possible to drop the term in $T^{3/2}$ since $T^{3/2} \leq 1/2(T^2 + T)$.

In order to obtain (100), we consider a function $\rho \in C_0^\infty(\omega)$ such that $\rho \equiv 1$ in ω_0 and $\rho \geq 0$. We consider $\omega \subset \Omega_0$ and the estimates obtained below remain true for larger ω. Multiplying by $s\lambda e^{-2s\eta}\rho\varphi q$ the equation

$$\partial_t q + \operatorname{div}(a\nabla q) = f$$

and integrating in $\omega \times (0, T)$, we obtain

$$\frac{s\lambda^2}{2} \int_0^T \!\!\int_\omega e^{-2s\eta} \rho\varphi\partial_t|q|^2 + s\lambda^2 \int_0^T \!\!\int_\omega e^{-2s\eta} \rho\varphi \operatorname{div}(a\nabla q)q \, \mathrm{d}x \, \mathrm{d}t = s\lambda^2 \int_0^T \!\!\int_\omega e^{-2s\eta} \rho\varphi f q \, \mathrm{d}x \, \mathrm{d}t. \tag{101}$$

In (101), the second term, can be written after integration by parts as follows:

$$s\lambda^2 \int_0^T \!\!\int_\omega e^{-2s\eta} \rho\varphi \, \partial_{x_i}(a\partial_{x_i}q)q \, \mathrm{d}x \, \mathrm{d}t = -s\lambda^2 \int_0^T \!\!\int_\omega e^{-2s\eta} \rho\varphi a|\nabla q|^2 \, \mathrm{d}x \, \mathrm{d}t - \frac{s\lambda^2}{2} \int_0^T \!\!\int_\omega \partial_{x_i}(e^{-2s\eta}\rho\varphi)a \, \partial_{x_i}|q|^2 \, \mathrm{d}x \, \mathrm{d}t.$$

Then, from (101) we deduce

$$s\lambda^2 \int_0^T \!\!\int_{\omega_0} e^{-2s\eta} \varphi|\nabla q|^2 \, \mathrm{d}x \, \mathrm{d}t \leq C(a)\frac{s\lambda^2}{2}\left|\int_0^T \!\!\int_\omega e^{-2s\eta} \rho\varphi\partial_t|q|^2 \, \mathrm{d}x \, \mathrm{d}t\right| + C(a)s\lambda^2 \left|\int_0^T \!\!\int_\omega e^{-2s\eta} \rho\varphi f q \, \mathrm{d}x \, \mathrm{d}t\right|$$

$$+ C(a)s\lambda^2 \left|\int_0^T \!\!\int_\omega e^{-2s\eta} \partial^2_{x_i x_i}(e^{-2s\eta}\rho\varphi)|q|^2 \, \mathrm{d}x \, \mathrm{d}t\right|. \tag{102}$$

Let us consider the first term of the right hand side of (102). We have

$$X_4 = \frac{s\lambda^2}{2} \int_0^T \!\!\int_\omega e^{-2s\eta} \rho\varphi\partial_t|q|^2 \, \mathrm{d}x \, \mathrm{d}t = -\frac{s\lambda^2}{2} \int_0^T \!\!\int_\omega \partial_t(e^{-2s\eta}\rho\varphi)|q|^2 \, \mathrm{d}x \, \mathrm{d}t$$

$$= s^2\lambda^2 \int_0^T \!\!\int_\omega e^{-2s\eta} \rho\varphi\partial_t\eta|q|^2 \, \mathrm{d}x \, \mathrm{d}t - \frac{s\lambda^2}{2} \int_0^T \!\!\int_\omega e^{-2s\eta} \rho\partial_t\varphi|q|^2 \, \mathrm{d}x \, \mathrm{d}t. \tag{103}$$

Using now (89) and (90) in (103) we obtain

$$|X_4| \leq Cs^2\lambda^2 T \int_0^T \!\!\int_\omega e^{-2s\eta} \rho\varphi^3|q|^2 \, \mathrm{d}x \, \mathrm{d}t + Cs\lambda^2 T^3 \int_0^T \!\!\int_\omega e^{-2s\eta} \rho\varphi^3|q|^2 \, \mathrm{d}x \, \mathrm{d}t. \tag{104}$$

On the other hand, for the third term of the right hand side of (102), we can write

$$s\lambda^2 \left|\int_0^T \!\!\int_\omega \partial^2_{x_i x_i}(e^{-2s\eta}\rho\varphi)|q|^2 \, \mathrm{d}x \, \mathrm{d}t\right| \leq Cs^3\lambda^4 \int_0^T \!\!\int_\omega e^{-2s\eta}\varphi^3\rho|q|^2 \, \mathrm{d}x \, \mathrm{d}t$$

$$+ C(s^2\lambda^4 T^2 + s\lambda^3 T^4 + s\lambda^3 T^2) \int_0^T \!\!\int_\omega e^{-2s\eta}\varphi^3\rho|q|^2 \, \mathrm{d}x \, \mathrm{d}t. \tag{105}$$

This is a consequence of the particular form of η and φ. Indeed, after the following calculation

$$\partial_{x_i}(\mathrm{e}^{-2s\eta}\rho\varphi) = \mathrm{e}^{-2s\eta}\left[\partial_{x_i}\rho\varphi + \rho\partial_{x_i}\varphi - 2s\rho\varphi\partial_{x_i}\eta\right],$$

$$\partial^2_{x_i x_j}(\mathrm{e}^{-2s\eta}\rho\varphi) = \mathrm{e}^{-2s\eta}\left[-2s\varphi\partial_{x_j}\eta\partial_{x_i}\rho - 2s\rho\partial_{x_j}\eta\partial_{x_i}\varphi + 4s^2\varphi\rho\partial_{x_j}\eta\partial_{x_i}\eta\right.$$
$$\left. + \varphi\partial^2_{x_i x_j}\rho + \partial_{x_i}\rho\partial_{x_j}\varphi + \partial_{x_j}\rho\partial_{x_i}\varphi + \rho\partial^2_{x_i x_j}\varphi - 2s\varphi\partial_{x_j}\rho\partial_{x_i}\eta - 2s\rho\partial_{x_j}\varphi\partial_{x_i}\eta - 2s\rho\varphi\partial^2_{x_i x_j}\eta\right]$$

and using the following straightforward estimates

$$|\varphi\partial_{x_j}\eta| \quad \leq \lambda\varphi^2|\nabla\beta| \leq C\lambda\varphi^2 \leq CT^2\lambda\varphi^3,$$

$$|\partial_{x_j}\eta\partial_{x_i}\varphi| \quad \leq \lambda^2\varphi^2|\nabla\beta|^2 \leq C\lambda^2\varphi^2 \leq CT^2\lambda^2\varphi^3,$$

$$|\varphi\partial_{x_j}\eta\partial_{x_i}\eta| \leq \lambda^2\varphi^3|\nabla\beta|^2 \leq C\lambda^2\varphi^3,$$

$$|\partial_{x_j}\varphi| \quad \leq \lambda\varphi|\nabla\beta| \leq C\lambda\varphi \leq CT^4\lambda\varphi^3$$

$$|\partial^2_{x_i x_j}\varphi| \quad \leq \lambda\varphi|\Delta\beta| + \lambda^2\varphi^2|\nabla\beta|^2 \leq C(\lambda\varphi + \lambda^2\varphi^2) \leq C(T^4\lambda + T^2\lambda^2)\varphi^3$$

$$|\varphi\partial^2_{x_i x_j}\eta| \quad \leq \lambda\varphi^2|\Delta\beta| + \lambda^2\varphi^3|\nabla\beta|^2 \leq CT^2\lambda\varphi^3 + C\lambda^2\varphi^3$$

it is not difficult to see that (105) holds. On the other hand we have

$$s\lambda^2\left|\int_0^T\int_\omega \mathrm{e}^{-2s\eta}\rho\varphi fq\,\mathrm{d}x\,\mathrm{d}t\right| \leq C\|\mathrm{e}^{-s\eta}f\|_2^2 + Cs^2\lambda^4\int_0^T\int_\omega \mathrm{e}^{-2s\eta}\varphi^2\rho^2|q|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq C\|\mathrm{e}^{-s\eta}f\|_2^2 + Cs^2\lambda^4 T^2\int_0^T\int_\omega \mathrm{e}^{-2s\eta}\varphi^3|q|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{106}$$

Using (104, 105) and (106) in (102) we get (100). As we mentioned above, this ends the proof of Carleman inequality (46) of Theorem 3.3.

The situation of Case 2 is quite different. Let us consider the functions

$$\beta^i = \widetilde{\beta}^i + K^i, \quad \overline{\beta}^i = \frac{5}{4}\max_{\overline{\Omega}}\beta^i, \quad \text{for } i = 1, 2, \tag{107}$$

with $K^i > 0$ such that $K^i \geq 5\max_{\overline{\Omega}}\widetilde{\beta}^i$, and $\widetilde{\beta}^i$ is given by the Lemma 3.2. We also introduce the following weight functions:

$$\varphi^i(x,t) = \frac{\mathrm{e}^{\lambda\beta^i(x)}}{t(T-t)}, \quad \eta^i(x,t) = \frac{\mathrm{e}^{\lambda\overline{\beta}^i} - \mathrm{e}^{\lambda\beta^i}}{t(T-t)}, \quad i = 1, 2. \tag{108}$$

Our second Carleman estimate is the following:

Theorem 3.4. *Assume that $\omega\cap\Omega_0 \neq \emptyset$, a satisfies (5, 6, 9) and (10) and Condition 2.2 in case (2) is fulfilled. There exists $\lambda_2(\Omega,\omega,a) > 0$ so that for each $\lambda > \lambda_2$ there exists a positive constant C that only depends on Ω,*

ω, $\mathcal{O}_1$, $\mathcal{O}_2$ *and* a, *and* $s_6(\lambda) > 0$ *so that the following estimate holds*

$$s^3 \iint_Q (e^{-2s\eta^1} + e^{-2s\eta^2})t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt + s \iint_Q (e^{-2s\eta^1} + e^{-2s\eta^2})t^{-1}(T-t)^{-1}|\nabla q|^2 \, dx \, dt$$

$$\leq C s^3 \iint_{\omega \times (0,T)} (e^{-2s\eta^1} + e^{-2s\eta^2})t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt \tag{109}$$

$$+ C \iint_Q (e^{-2s\eta^1} + e^{-2s\eta^2})|\partial_t q + \text{div}(a(x)\nabla q)|^2 \, dx \, dt$$

for all $q \in Z_0$ *and* $s \geq s_6$. *Moreover,* s_6 *is of the form* $s_6 = \sigma_6(\Omega, \omega, \mathcal{O}_1, \mathcal{O}_2, a, \lambda)(T^2 + T)$, *where* σ_6 *is a positive constant that only depends on* Ω, ω, $\mathcal{O}_1$, $\mathcal{O}_2$, a *and* λ.

Proof of Theorem 3.4. In order to obtain (109), we will apply the global Carleman inequality (46) from Theorem 3.3 and the properties of weight functions (108).

We observe, that from Lemma 3.2, we know that $\nabla \beta^i$ can vanish only in ω_0 and B_i for $i = 1, 2$, where the open subsets B_1 and B_2 are fixed balls defined in Lemma 3.2. Taking into account these statements, we can use two weight functions given by (108) and write two Carleman estimates like (46). More precisely, there exist a positive constant C and s_1 that only depends on Ω ω and a, such that

$$s^3 \iint_Q e^{-2s\eta^i} t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt + s \iint_Q e^{-2s\eta^i} t^{-1}(T-t)^{-1}|\nabla q|^2 \, dx \, dt$$

$$\leq C \left(s^3 \iint_{\omega \times (0,T)} e^{-2s\eta^i} t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt + s^3 \iint_{\widetilde{B}_i \times (0,T)} e^{-2s\eta^i} t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt \right. \tag{110}$$

$$\left. + \iint_Q e^{-2s\eta^i} |\partial_t q + \text{div}(a(x)\nabla q)|^2 \, dx \, dt \right)$$

for $i = 1, 2$, for all $q \in Z_0$ and $s \geq s_1$. Moreover, s_1 is of the form $s_1 = \sigma_1(\Omega, \omega, a, \lambda)(T^2 + T)$.

Let us show that from (110), using the properties of the functions β^1 and β^2, we can deduce the Carleman estimate (109). For this, it will be sufficient to see that for each $C > 0$, there exists s_4 such that

$$e^{-2s\eta^2} \geq 2C e^{-2s\eta^1} \quad \text{in } \widetilde{B}_1, \tag{111}$$

$$e^{-2s\eta^1} \geq 2C e^{-2s\eta^2} \quad \text{in } \widetilde{B}_2 \tag{112}$$

for $s \geq s_4 = \sigma_4(\Omega, \omega, \mathcal{O}_1, \mathcal{O}_2, \lambda)T^2$.

Indeed, adding the two Carleman inequality (110), we deduce

$$s^3 \iint_Q (e^{-2s\eta^1} + e^{-2s\eta^2})t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt + s \iint_Q (e^{-2s\eta^1} + e^{-2s\eta^2})t^{-1}(T-t)^{-1}|\nabla q|^2 \, dx \, dt$$

$$\leq C s^3 \iint_{\omega \times (0,T)} (e^{-2s\eta^1} + e^{-2s\eta^2})t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt \tag{113}$$

$$+ C \left(s^3 \iint_{\widetilde{B}_1 \times (0,T)} e^{-2s\eta^1} t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt + s^3 \iint_{\widetilde{B}_2 \times (0,T)} e^{-2s\eta^2} t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt \right)$$

$$+ C \iint_Q (e^{-2s\eta^1} + e^{-2s\eta^2})|\partial_t q + \text{div}(a(x)\nabla q)|^2 \, dx \, dt$$

for $s \geq s_5 = \max(s_1, s_4)$.

On the other hand, according to (111) and (112), we obtain that

$$
Cs^3 \iint_{\widetilde{B}_1 \times (0,T)} \mathrm{e}^{-2s\eta^1} t^{-3}(T-t)^{-3}|q|^2 \,\mathrm{d}x\,\mathrm{d}t + Cs^3 \iint_{\widetilde{B}_2 \times (0,T)} \mathrm{e}^{-2s\eta^2} t^{-3}(T-t)^{-3}|q|^2 \,\mathrm{d}x\,\mathrm{d}t
$$

$$
\leq \frac{s^3}{2} \iint_{\widetilde{B}_1 \times (0,T)} \mathrm{e}^{-2s\eta^2} t^{-3}(T-t)^{-3}|q|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{s^3}{2} \iint_{\widetilde{B}_2 \times (0,T)} \mathrm{e}^{-2s\eta^1} t^{-3}(T-t)^{-3}|q|^2 \,\mathrm{d}x\,\mathrm{d}t \tag{114}
$$

$$
\leq \frac{s^3}{2} \iint_Q (\mathrm{e}^{-2s\eta^1} + \mathrm{e}^{-2s\eta^2}) t^{-3}(T-t)^{-3}|q|^2 \,\mathrm{d}x\,\mathrm{d}t
$$

for $s \geq s_6 = \sigma_6(\Omega, \omega, \mathcal{O}_1, \mathcal{O}_2, \lambda)T^2$. Combining (113) and (114), we find (109).

To conclude the proof, let us justify (111) and (112). By construction, we have that

$$
\beta_1^2 \geq 2\beta_1^1 \quad \text{in } \widetilde{B}_1. \tag{115}
$$

Using (115), we can deduce that for all $\lambda \geq 1$ there exists a positive constant α, which only depends on Ω, ω, $\mathcal{O}_1$, $\mathcal{O}_2$ such that

$$
\eta^1 - \eta^2 \geq \alpha \eta^1 \quad \text{in } \widetilde{B}_1. \tag{116}
$$

Indeed, equation (116) is a consequence of the following:

$$
\eta^1 - \eta^2 = \frac{\mathrm{e}^{\lambda \beta_1^2} - \mathrm{e}^{\lambda \beta_1^1}}{t(T-t)} \geq \frac{\mathrm{e}^{2\lambda \beta_1^1} - \mathrm{e}^{\lambda \beta_1^1}}{t(T-t)} \geq \alpha \frac{\mathrm{e}^{\lambda \overline{\beta}^1} - \mathrm{e}^{\lambda \beta_1^1}}{t(T-t)} = \alpha \eta^1 \quad \text{in } \widetilde{B}_1.
$$

Then, from (116) we obtain that, for each $C > 0$, there exists s_2 such that

$$
\frac{\mathrm{e}^{-2s\eta^2}}{\mathrm{e}^{-2s\eta^1}} \geq \mathrm{e}^{2s\alpha \eta^1} \geq \mathrm{e}^{2s\alpha \min \eta^1} \geq 2C \quad \text{in } \widetilde{B}_1
$$

for $s \geq s_2 = \sigma_2(\Omega, \omega, \mathcal{O}_1, \mathcal{O}_2)T^2$. This is exactly the inequality (111) for $s \geq s_2$.

By similar arguments, using the fact that

$$
\beta_1^1 \geq 2\beta_1^2 \quad \text{in } \mathcal{O}_2, \tag{117}
$$

it is easy to see that, for each $C > 0$, there exists s_3 such that

$$
\frac{\mathrm{e}^{-2s\eta^1}}{\mathrm{e}^{-2s\eta^2}} \geq 2C \quad \text{in } \widetilde{B}_2
$$

for $s \geq s_3 = \sigma_3(\Omega, \omega, \mathcal{O}_1, \mathcal{O}_2)T^2$. Then, equation (112) holds for $s \geq s_3$. Taking now $s_4 = \max(s_2, s_3)$ we get (111) and (112) for $s \geq s_4$. This ends the proof of Theorem 3.4.

4. Observability inequalities and technical results

In this section we will deduce some observability estimates as a consequence of appropriate global Carleman inequalities and regularizing effect of the heat equation. This will be needed to prove the null controllability result for a linear transmission problem with controls in $L^r(0,T; L^r(\omega))$ with r sufficiently large, such that (4) holds.

Let us consider the following linear (adjoint) transmission problem:

$$\begin{cases} -\partial_t q - \operatorname{div}(a(x)\nabla q) + bq = 0 & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(x,T) = q_T & \text{in } \Omega, \end{cases} \tag{118}$$

where a satisfies (5, 6, 9) and (10), $b \in L^\infty(Q)$, and $q_T \in L^2(\Omega)$. First of all, let us prove the observability estimate with $L^2(0,T;L^2(\omega))$-norm in the right hand side. This can be used to deduce null controllability result (and estimates) for linear transmition problem with bounded potential, with controls in $L^2(0,T;L^2(\omega))$. We have the following:

Proposition 4.1. *Assume that $\omega \cap \Omega_0 \neq \emptyset$ and that Condition 2.1 (resp. Condition 2.2) in case (1) (resp. in case (2)) are fulfilled. Then for any a satisfying (5, 6, 9, 10), $b \in L^\infty(Q)$ and $q_T \in L^2(\Omega)$, there exists a positive constant C that only depends on Ω, ω and a (resp. Ω, ω $\mathcal{O}_1$, $\mathcal{O}_2$ and a), such that*

$$||q(\cdot,0)||^2_{L^2(\Omega)} \leq \exp\left[C\left(1 + \frac{1}{T} + T||b||_\infty + ||b||^{2/3}_\infty\right)\right] \iint_{\omega \times (0,T)} |q|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{119}$$

where q is the solution to the corresponding system (118).

For simplicity, we only present the proof of Proposition 4.1 for the situation (1). We just note, that the proof corresponding to the situation (2) is similar, it suffices to take into account the different estimates (in space) for the weight functions that appear in the global Carleman inequality (109). Thus we also obtain the constants depending on $\mathcal{O}_1$ and $\mathcal{O}_2$.

Proof of Proposition 4.1. We will use global Carleman inequality (46) from Theorem 3.3 and some estimates for the weight functions. Let b and q_T be given and let q be the solution to (118).

Step 1: We will first see that

$$\iint_{\Omega \times (T/4,3T/4)} |q|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \exp\left[C\left(1 + \frac{1}{T} + ||b||^{2/3}_\infty\right)\right] \iint_{\omega \times (0,T)} |q|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{120}$$

By density, we can write (46) for q being the solution of (118). This gives

$$s^3 \iint_Q \mathrm{e}^{-2s\eta} t^{-3}(T-t)^{-3}|q|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C\left(s^3 \iint_{\omega \times (0,T)} \mathrm{e}^{-2s\eta} t^{-3}(T-t)^{-3}|q|^2 \,\mathrm{d}x\,\mathrm{d}t \right.$$

$$\left. + \iint_Q \mathrm{e}^{-2s\eta}|bq|^2 \,\mathrm{d}x\,\mathrm{d}t\right) \tag{121}$$

for all $s \geq s_1$. We can estimate the second term in the right as follows:

$$\iint_Q \mathrm{e}^{-2s\eta}|bq|^2 \,\mathrm{d}x\,\mathrm{d}t \leq 2^{-6}T^6||b||^2_\infty \iint_Q \mathrm{e}^{-2s\eta} t^{-3}(T-t)^{-3}|q|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Thus, we deduce from (121) that

$$\iint_Q \mathrm{e}^{-2s\eta} t^{-3}(T-t)^{-3}|q|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C \iint_{\omega \times (0,T)} \mathrm{e}^{-2s\eta} t^{-3}(T-t)^{-3}|q|^2 \,\mathrm{d}x\,\mathrm{d}t \tag{122}$$

provided

$$s \geq s_7 = \max\left(s_1, CT^2 ||b||_\infty^{2/3}\right). \tag{123}$$

On the other hand, it can be easily verified that

$$e^{-2s\eta}t^{-3}(t - T)^{-3} \leq 2^6 T^{-6} \exp\left(-CsT^{-2}\right) \quad \forall (x,t) \in \overline{Q} \tag{124}$$

and

$$e^{-2s\eta}t^{-3}(t - T)^{-3} \geq \left(\frac{16}{3}\right)^3 T^{-6} \exp\left(-CsT^{-2}\right) \quad \forall (x,t) \in \overline{\Omega} \times [T/4, 3T/4], \tag{125}$$

whenever

$$s \geq s_8 = \max\left(s_7, CT^2\right)$$

(constants C in (124) and (125) may be different). If we analyze the structure of the constants s_7 and s_8, we see that $s_8 \leq s_9$, where s_9 is of the form

$$s_9 = \sigma_9(\Omega, \omega, a)\left(T + T^2 + T^2 ||b||_\infty^{2/3}\right). \tag{126}$$

Let us fix the constant $s = s_9$. We write (122) for $s = s_9$ taking into account (124) and (125) and we deduce that (120) is satisfied for any solution q of (118).

Step 2: Let us now prove that

$$||q(x,0)||_2^2 \leq \exp\left[C\left(\frac{1}{T} + T||b||_\infty\right)\right] \iint_{\Omega \times (T/4, 3T/4)} |q|^2 \, dx \, dt. \tag{127}$$

The estimate (127) together with (120) leads to the desired observability inequality (119).

Let $\theta_0 \in C^1[0,1]$ be a function such that, $0 \leq \theta_0 \leq 1$, $\theta_0 = 1$ in $[0, 1/4]$, $\theta_0 = 0$ in $[3/4, 1]$. Now, we consider a function $\theta(t) = \theta_0(t/T)$ and we write (118) for $\theta(t)q$. We obtain

$$\begin{cases} -\partial_t(\theta q) - \operatorname{div}(a(x)\nabla(\theta q)) + b(\theta q) = -q\partial_t\theta & \text{in} \quad \Omega \times (0, 3T/4), \\ \theta q = 0 & \text{on} \quad \partial\Omega \times (0, 3T/4), \\ \theta q(x, 3T/4) = 0 & \text{in} \quad \Omega. \end{cases} \tag{128}$$

Multiplying (128) by θq and integrating in Ω, we have

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\theta q|^2 \, dx + \int_\Omega a|\nabla(\theta q)|^2 \, dx = -\int_\Omega b|\theta q|^2 \, dx - \int_\Omega \theta(\partial_t\theta)|q|^2 \, dx \quad \forall \, t \geq 0.$$

Thus,

$$-\frac{d}{dt}\int_\Omega |\theta q|^2 \, dx + 2\int_\Omega a|\nabla(\theta q)|^2 \, dx \leq 2||b||_\infty \int_\Omega |\theta q|^2 \, dx + 2\int_\Omega \theta|\partial_t\theta||q|^2 \, dx$$

and

$$-\frac{d}{dt}\left(\exp\left(2||b||_\infty t\right)\int_\Omega |\theta q|^2 \, dx\right) \leq 2\exp\left(2||b||_\infty t\right)\int_\Omega \theta|\partial_t\theta||q|^2 \, dx \tag{129}$$

for all $t \geq 0$. Integrating this inequality with respect to time in $[0, t]$ with $t \in [3T/4, T]$, we obtain

$$
\begin{aligned}
\int_{\Omega} |q(x,0)|^2 \, \mathrm{d}x &\leq \int_0^t \exp\left(2||b||_{\infty} t\right) 2 \int_{\Omega} |q|^2 \theta \partial_t \theta \, \mathrm{d}x \\
&\leq \frac{2C}{T} \exp\left(\frac{3T}{2}||b||_{\infty}\right) \iint_{\Omega \times (T/4, 3T/4)} |q|^2 \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
\tag{130}
$$

In (130), we have used the fact that $\theta \leq 1$ and $|\partial_t \theta| = |\partial_t \theta_0(t/T)|/T \leq C/T$. This justifies the estimate (127) and ends the proof of Proposition 4.1.

As we mentioned above, to analyze the controllability for nonlinear problem (8) we need the controls in $L^r(0, T; L^r(\omega))$ for r sufficiently large $(1/r + N/(2r) < 1)$. For this we are going to prove a refined version of the observability inequality (119), *i.e.* with $L^{r'}(0, T; L^{r'}(\omega))$-norm in the right hand side, where r' is the dual exponent to r. We have:

Proposition 4.2. *Assume that $\omega \cap \Omega_0 \neq \emptyset$ and that Condition 2.1 (resp. Condition 2.2) in case (1) (resp. in case (2)) are fulfilled. Then for any a satisfying (5, 6, 9) and (10), $b \in L^{\infty}(Q)$, $q_T \in L^2(\Omega)$ and any r' sufficiently small, there exist a positive constant C that only depends on Ω, ω, a, r' and N (resp. Ω, ω $\mathcal{O}_1$, $\mathcal{O}_2$, a, r' and N) and a positive constant $\widetilde{C}$ depending on Ω, ω, a (resp. Ω, ω $\mathcal{O}_1$, $\mathcal{O}_2$ and a) such that*

$$
||q(\cdot,0)||_{L^2(\Omega)}^2 + \iint_Q \mathrm{e}^{-2s\widetilde{C}T^{-1}/(T-t)}(T-t)^{-3}|q|^2 \, \mathrm{d}x\mathrm{d}t \leq \exp\left[CH(T,||b||_{\infty})\right]\left(\iint_{\omega \times (0,T)} |q|^{r'} \mathrm{d}x \, \mathrm{d}t\right)^{2/r'}
\tag{131}
$$

for all $s \geq \sigma(\Omega, \omega, a)(T^2 + T + T^2||b||_{\infty}^{2/3})$, where σ is a positive constant depending on Ω, ω and a (resp. Ω, ω, a, $\mathcal{O}_1$, $\mathcal{O}_2$), $H(T, ||b||_{\infty})$ is given by

$$
H(T, ||b||_{\infty}) = 1 + \frac{1}{T} + T + (T + T^{1/2})||b||_{\infty} + ||b||_{\infty}^{2/3}
\tag{132}
$$

and q is the solution to the corresponding system (118).

As before, for simplicity, we only present the proof corresponding to case (1). We take into account the estimates for the weight functions in (109) for treatment of case (2). In the sequel, $\sigma(\Omega, \omega, a)$ will stand for a generic positive constant only depending on Ω, ω and a, whose value can change from line to line. Let us first prove the following technical lemma:

Lemma 4.3. *Let $\widetilde{\omega}$ be a nonempty open set such that $\widetilde{\omega} \subset\subset \omega$. Then, for any a satisfying (5, 6) and (9), $b \in L^{\infty}(Q)$, $q_T \in L^2(\Omega)$ and any r' sufficiently small, there exists $C = C(\Omega, \omega, a) > 0$ such that*

$$
\iint_{\widetilde{\omega} \times (0,T)} \mathrm{e}^{-2s\eta}t^{-3}(T-t)^{-3}|q|^2 \, \mathrm{d}x \, \mathrm{d}t \leq CT^{-3}T^{\alpha}K(T, ||b||_{\infty})^{\gamma}\mathrm{e}^{-CsT^{-2}}\left(\iint_{\omega \times (0,T)} |q|^{r'} \, \mathrm{d}x \, \mathrm{d}t\right)^{2/r'}
\tag{133}
$$

for all $s \geq \sigma(\Omega, \omega, a)T^2$, where σ is a positive constant depending on Ω, ω and a, α, γ are positive numbers only depending of N, $K(T, ||b||_{\infty})$ is given by

$$
K(T, ||b||_{\infty}) = 1 + T^{1/2}(1 + ||b||_{\infty}) + T^{-5/2}(s + T^2)
\tag{134}
$$

and q is the solution to the corresponding system (118).

Proof of Lemma 4.3. Let $\widetilde{\omega}$ be a nonempty open set such that $\widetilde{\omega} \subset\subset \omega$. Notice that without loss of generality we can consider $\omega \subset\subset \Omega_0$ or $\omega \subset\subset \Omega_1$ with a smooth boundary and the estimates obtained below remain true for a larger ω. Let us consider a function $\theta \in \mathcal{D}(\omega)$, such that $\theta = 1$ in $\widetilde{\omega}$. We set

$$w(x,t) = \theta(x)\varphi(x,t)q(x,t),$$

where q is the solution of (118) and φ is given by

$$\varphi(x,t) = \frac{e^{-s\eta}}{t^{3/2}(T-t)^{3/2}} \cdot \tag{135}$$

Notice that $w(T) = w(0) = 0$. Taking into account (118), we deduce that w satisfy the following problem:

$$\begin{cases} -\partial_t w - \operatorname{div}(a(x)\nabla w) = -b\theta q\varphi - \theta q\partial_t\varphi - 2a\nabla(\theta\varphi) \cdot \nabla q - \operatorname{div}(a\nabla(\theta\varphi))q & \text{in} \quad Q, \\ w = 0 & \text{on} \quad \Sigma, \\ w(x,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{136}$$

For simplicity of the computations, we put $\widetilde{w}(x,t) = \widetilde{w}(x,T-t)$ for $(x,t) \in Q$. In a similar way, we introduce the functions $\widetilde{a}$, $\widetilde{b}$, $\widetilde{\varphi}$ and $\widetilde{q}$. Then we have

$$\begin{cases} \partial_t \widetilde{w} - \operatorname{div}(\widetilde{a}(x)\nabla\widetilde{w}) = -\widetilde{b}\theta\widetilde{q}\widetilde{\varphi} + \theta\widetilde{q}\partial_t\widetilde{\varphi} - 2\widetilde{a}\nabla(\theta\widetilde{\varphi}) \cdot \nabla\widetilde{q} - \operatorname{div}(\widetilde{a}\nabla(\theta\widetilde{\varphi}))\widetilde{q} & \text{in } Q, \\ \widetilde{w} = 0 & \text{on } \Sigma, \\ \widetilde{w}(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{137}$$

On the other hand, let z be the solution of the problem

$$\begin{cases} -\partial_\tau z - \operatorname{div}(a(x)\nabla z) = 0 & \text{in } \omega \times (0,t), \\ z = 0 & \text{on } \partial\omega \times (0,t), \\ z(x,t) = \psi & \text{in } \omega, \end{cases} \tag{138}$$

where $\psi \in L^2(\Omega)$ is given. Multiplying (137) by z and integrating in ω and in $\tau \in (0,t)$, we obtain the following for $t \in (0,T)$:

$$\begin{aligned} (\widetilde{w}(t), z(t)) &= \int_0^t \int_\omega (-\widetilde{b}\theta\widetilde{\varphi} + \theta\partial_t\widetilde{\varphi} - \operatorname{div}(\widetilde{a}\nabla(\theta\widetilde{\varphi})))\widetilde{q}z \, dx \, d\tau - 2\int_0^t \int_\omega \widetilde{a}\nabla(\theta\widetilde{\varphi}) \cdot \nabla\widetilde{q}\, z \, dx \, d\tau \\ &\leq C(1 + \|b\|_\infty) \int_0^t |\widetilde{\varphi}| \, \|\widetilde{q}\|_{L^{r'}(\omega)} \|z\|_{L^r(\omega)} \, d\tau + C \int_0^t |\widetilde{\varphi}| \, \|\widetilde{q}\|_{L^{r'}(\omega)} \|\nabla z\|_{L^r(\omega)} \, d\tau \\ &\quad + \int_0^t |\partial_t\widetilde{\varphi}| \, \|\widetilde{q}\|_{L^{r'}(\omega)} \|z\|_{L^r(\omega)} \, d\tau, \end{aligned} \tag{139}$$

where C is a positive constant depending on ω, $\widetilde{\omega}$ (*i.e.* on ω) and a. In (139), we have used that $|\nabla\widetilde{\varphi}| \leq C|\widetilde{\varphi}|$ and that $|\Delta\widetilde{\varphi}| \leq C|\widetilde{\varphi}|$.

Notice that, since the diffusion coefficients are sufficiently regular in ω and thanks to the regularizing effect of the heat equation (*cf.* [19] and [6]), we know that for all $t > 0$ and $1 \leq p, q \leq +\infty$ the following holds:

$$\|S(t)u\|_{L^p(\omega)} \leq Ct^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u\|_{L^q(\omega)} \ \forall \, u \in L^q(\omega), \tag{140}$$

$$\|S(t)u\|_{W^{1,p}(\omega)} \leq Ct^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}}\|u\|_{L^q(\omega)} \ \forall \, u \in L^q(\omega), \tag{141}$$

where $\{S(t) : t \geq 0\}$ denotes the semigroup generated by the heat equation with Dirichlet boundary conditions.

We apply to z (solution of (138)) the estimates (140) and (141) with $q = 2$ and $p = r$, so we can write from (139) that

$$
\begin{aligned}
(\widetilde{w}(t), z(t)) &\le C(1 + \|b\|_\infty) \int_0^t |\widetilde{\varphi}| \, (t-\tau)^{-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{r}\right)} \|\widetilde{q}(\cdot,\tau)\|_{L^{r'}(\omega)} \|\psi\|_{L^2(\omega)} \, \mathrm{d}\tau \\
&+ C \int_0^t |\widetilde{\varphi}| \, (t-\tau)^{-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{r}\right)-\frac{1}{2}} \|\widetilde{q}(\cdot,\tau)\|_{L^{r'}(\omega)} \|\psi\|_{L^2(\omega)} \, \mathrm{d}\tau \\
&+ C \int_0^t |\partial_t \widetilde{\varphi}| \, (t-\tau)^{-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{r}\right)} \|\widetilde{q}(\cdot,\tau)\|_{L^{r'}(\omega)} \|\psi\|_{L^2(\omega)} \, \mathrm{d}\tau
\end{aligned}
\tag{142}
$$

for all $t \in [0, T]$. Thus, from (142) we have

$$
\begin{aligned}
\|\widetilde{w}(\cdot,t)\|_{L^2(\widetilde{\omega})} &= \sup_{\|\psi\|_{L^2(\omega)}=1} (\widetilde{w}(\cdot,t), \psi) \le C(1 + \|b\|_\infty) \int_0^t |\widetilde{\varphi}| \, (t-\tau)^{-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{r}\right)} \|\widetilde{q}(\cdot,\tau)\|_{L^{r'}(\omega)} \, \mathrm{d}\tau \\
&+ C \int_0^t |\widetilde{\varphi}| \, (t-\tau)^{-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{r}\right)-\frac{1}{2}} \|\widetilde{q}(\cdot,\tau)\|_{L^{r'}(\omega)} \, \mathrm{d}\tau + C \int_0^t |\partial_t \widetilde{\varphi}| \, (t-\tau)^{-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{r}\right)} \|\widetilde{q}(\cdot,\tau)\|_{L^{r'}(\omega)} \, \mathrm{d}\tau
\end{aligned}
$$

for all $t \in [0, T]$. This gives,

$$
\begin{aligned}
\|\widetilde{w}(\cdot,t)\|_{L^2(\widetilde{\omega})} &\le C(1 + T^{1/2} + T^{1/2}\|b\|_\infty) \int_0^t |\widetilde{\varphi}| \, (t-\tau)^{-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{r}\right)-\frac{1}{2}} \|\widetilde{q}(\cdot,\tau)\|_{L^{r'}(\omega)} \, \mathrm{d}\tau \\
&+ CT^{1/2} \int_0^t |\partial_t \widetilde{\varphi}| \, (t-\tau)^{-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{r}\right)-\frac{1}{2}} \|\widetilde{q}(\cdot,\tau)\|_{L^{r'}(\omega)} \, \mathrm{d}\tau.
\end{aligned}
\tag{143}
$$

Using the definition of η, it is not difficult to see that

$$
|\widetilde{\varphi}| \le CT^{-3} \mathrm{e}^{-CsT^{-2}} \quad \forall (x,t) \in Q
\tag{144}
$$

and

$$
\frac{\mathrm{e}^{-s\eta}}{t^{7/2}(T-t)^{7/2}} \le CT^{-7} \mathrm{e}^{-CsT^{-2}} \quad \forall (x,t) \in Q
\tag{145}
$$

for $s \ge \sigma(\Omega, \omega, a)T^2$. Moreover, using (135) and (145), we can write that

$$
|\partial_t \widetilde{\varphi}| = \left| \frac{\mathrm{e}^{-s\eta}(T-2t)}{t^{7/2}(T-t)^{7/2}} \left(s + \frac{3}{2}tT - \frac{3}{2}T^2 \right) \right| \le CT^{-6}(s + T^2)\mathrm{e}^{-CsT^{-2}}
\tag{146}
$$

for $s \ge \sigma(\Omega, \omega, a)T^2$. Thanks to (145) and (146), from (143) we have

$$
\begin{aligned}
\|\widetilde{w}(\cdot,t)\|_{L^2(\widetilde{\omega})} &\le CT^{-3}(1 + T^{1/2} + T^{1/2}\|b\|_\infty)\mathrm{e}^{-CsT^{-2}} \int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{r}\right)-\frac{1}{2}} \|\widetilde{q}(\cdot,\tau)\|_{L^{r'}(\omega)} \, \mathrm{d}\tau \\
&+ CT^{1/2}T^{-6}(s + T^2)\mathrm{e}^{-CsT^{-2}} \int_0^t (t-\tau)^{-\frac{N}{2}\left(\frac{1}{2}-\frac{1}{r}\right)-\frac{1}{2}} \|\widetilde{q}(\cdot,\tau)\|_{L^{r'}(\omega)} \, \mathrm{d}\tau
\end{aligned}
\tag{147}
$$

for $s \ge \sigma(\Omega, \omega, a)T^2$. If r' is such that

$$
\frac{N}{2}\left(\frac{1}{2} - \frac{1}{r}\right) + \frac{1}{2} + \frac{1}{r'} < \frac{3}{2},
$$

that is to say

$$r' > \frac{2(N+2)}{N+4}, \tag{148}$$

then, we can apply Young's inequality to (143) and estimate the $L^2(0,T;L^2(\widetilde{\omega}))$-norm of $\widetilde{w}$ as follows:

$$
\begin{aligned}
||\widetilde{w}||_{L^2(0,T;L^2(\widetilde{\omega}))} &\leq CT^{\kappa}T^{-3}(1+T^{1/2}+T^{1/2}||b||_{\infty})\mathrm{e}^{-CsT^{-2}}\left(\iint_{\omega\times(0,T)}|q|^{r'}\,\mathrm{d}x\,\mathrm{d}t\right)^{1/r'} \\
&\quad + CT^{\kappa}T^{1/2}T^{-6}(s+T^2)\mathrm{e}^{-CsT^{-2}}\left(\iint_{\omega\times(0,T)}|q|^{r'}\,\mathrm{d}x\,\mathrm{d}t\right)^{1/r'},
\end{aligned}
\tag{149}
$$

where C is a new positive constant depending on Ω, ω, $\widetilde{\omega}$ r' and N, κ depends on N and r and $s \geq \sigma(\Omega,\omega,a)T^2$. Notice that for $N < 4$ and r' as in (4) the condition (148) is satisfied. For $N \geq 4$ we apply again this process in order to obtain the inequality (133) for r' sufficiently small, in such a way that (4) holds for $s \geq \sigma(\Omega,\omega,a)T^2$ and with α and γ only depending on N (*cf.* [13]). This ends the proof of Lemma 4.3.

Proof of Proposition 4.2. We will use a global Carleman inequality (46) from Theorem 3.3, some estimates for the weight functions, classical parabolic estimates and technical result of Lemma 4.3.

Step 1: Let b and q_T be given and let q be the solution to (118). We will first see that

$$\iint_{\Omega\times(T/4,3T/4)}|q|^2\,\mathrm{d}x\,\mathrm{d}t \leq T^6\exp\left[C\left(1+\frac{1}{T}+||b||_{\infty}^{2/3}\right)\right]\iint_{\omega\times(0,T)}\mathrm{e}^{-2s\eta}t^{-3}(T-t)^{-3}|q|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{150}$$

Let us consider (122), that is true for $s \geq s_7$ with s_7 given by (123). Then, using (125), we obtain that

$$\iint_{\Omega\times(T/4,3T/4)}|q|^2\,\mathrm{d}x\,\mathrm{d}t \leq CT^6\exp\left(CsT^{-2}\right)\iint_{\omega\times(0,T)}\mathrm{e}^{-2s\eta}t^{-3}(T-t)^{-3}|q|^2\,\mathrm{d}x\,\mathrm{d}t \tag{151}$$

for all $s \geq s_8$. Thanks to the structure of the constant s_9, given by (126), the inequality (151) written for $s = s_9$ implies that (150) is satisfied for any solution q of (118).

Step 2: Let us now prove the first part of desired observability estimate (131), *i.e.* the following inequality:

$$\|q(x,0)\|_{L^2(\Omega)}^2 \leq \exp\left[CH(T,\|b\|_{\infty})\right]\left(\iint_{\omega\times(0,T)}|q|^{r'}\,\mathrm{d}x\,\mathrm{d}t\right)^{2/r'} \tag{152}$$

with $H(T,\|b\|_{\infty})$ given by (132). Let $\widetilde{\omega}$ be a nonempty open set such that $\widetilde{\omega} \subset\subset \omega \cap \Omega_0$. We consider (130), that is to say

$$\|q(x,0)\|_{L^2(\Omega)}^2 \leq \frac{2C}{T}\exp\left(\frac{3T}{2}\|b\|_{\infty}\right)\iint_{\Omega\times(T/4,3T/4)}|q|^2\,\mathrm{d}x\,\mathrm{d}t.$$

We apply to the right hand side of this inequality, the estimates (150) written for $\widetilde{\omega}$ and we obtain

$$\|q(x,0)\|_{L^2(\Omega)}^2 \leq CT^5\exp\left(1+\frac{1}{T}+T\|b\|_{\infty}+\|b\|_{\infty}^{2/3}\right)\iint_{\widetilde{\omega}\times(0,T)}\mathrm{e}^{-2s\eta}t^{-3}(T-t)^{-3}|q|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{153}$$

On the other hand, we notice that the constant $\sigma(\Omega, \omega, a)T^2$ appearing in Lemma 4.3 is less then the constant s_9 given by (126). So, we can write technical estimate (133) for $s = s_9$. This gives

$$\iint_{\widetilde{\omega}\times(0,T)} e^{-2s\eta}t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt \leq CT^{-3}K_1(T, \|b\|_\infty)\left(\iint_{\omega\times(0,T)} |q|^{r'} \, dx \, dt\right)^{2/r'}, \qquad (154)$$

where

$$K_1(T, \|b\|_\infty) = \exp\left(1 + \frac{1}{T} + T\|b\|_\infty + \|b\|_\infty^{2/3}\right)T^\alpha\left[1 + T^{1/2}\left(1 + \|b\|_\infty + \frac{1}{T^2} + \frac{1}{T}\|b\|_\infty^{2/3}\right)\right]^\gamma. \qquad (155)$$

Combining (153) and (154), we deduce that

$$\|q(x,0)\|_{L^2(\Omega)}^2 \leq CT^2 K_1(T, \|b\|_\infty)\left(\iint_{\omega\times(0,T)} |q|^{r'} \, dx \, dt\right)^{2/r'}. \qquad (156)$$

Consequently, we also have (152).

Step 3: Let us finally deduce that

$$\iint_{\Omega\times(0,T)} e^{-2s\widetilde{C}T^{-1}/(T-t)}(T-t)^{-3}|q|^2 \, dx \, dt \leq \exp\left[CH(T, \|b\|_\infty)\right]\left(\iint_{\omega\times(0,T)} |q|^{r'} \, dx \, dt\right)^{2/r'}, \qquad (157)$$

where $H(T, \|b\|_\infty)$ is done by (132). Integrating (129) with respect to time and using that $\theta = 1$ in $[0, T/4]$, we obtain for all $t \in [0, 3T/4]$

$$\exp\left(2\|b\|_\infty t\right)\int_\Omega |\theta q(x,t)|^2 \, dx \leq \int_{T/4}^{3T/4} 2\exp\left(2\|b\|_\infty \tau\right)\int_\Omega \theta|\partial_t\theta||q|^2 \, dx \, d\tau. \qquad (158)$$

Since $\theta \leq 1$, $|\partial_t\theta| = |\partial_t\theta_0(t/T)/T| \leq C/T$, from (158) we obtain

$$\int_\Omega |q(x,t)|^2 \, dx \leq \frac{2}{T}\exp\left(\frac{3T}{2}\|b\|_\infty\right)\iint_{\Omega\times(T/4,3T/4)} |q|^2 \, dx \, dt \qquad (159)$$

for all $t \in [0, T/4]$. Then, we have

$$\iint_{\Omega\times(0,T/4)} |q(x,t)|^2 \, dx \leq \exp\left(\frac{3T}{2}\|b\|_\infty\right)\iint_{\Omega\times(T/4,3T/4)} |q|^2 \, dx \, dt. \qquad (160)$$

On the other hand, using the definition (44) of the function η, we can say that there exist a positive constants C and $\widetilde{C}$, depending on Ω, ω and a, such that

$$e^{-2s\eta}t^{-3}(T-t)^{-3} \geq CT^{-3}(T-t)^{-3}e^{-2s\widetilde{C}T^{-1}/(T-t)} \quad \forall x \in \Omega, \quad \forall t \in [T/4, T] \qquad (161)$$

whenever $s \geq \sigma(\Omega, \omega, a)T^2$. We can write

$$\iint_{\Omega \times (0,T)} e^{-2s\widetilde{C}T^{-1}/(T-t)}(T-t)^{-3}|q|^2 \, dx \, dt$$

$$\leq \iint_{\Omega \times (0,T/4)} e^{-2s\widetilde{C}T^{-1}/(T-t)}(T-t)^{-3}|q|^2 \, dx \, dt + CT^3 \iint_{\Omega \times (T/4,T)} e^{-2s\eta}t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt$$

$$\leq T^{-3}e^{-2CsT^{-2}} \iint_{\Omega \times (0,T/4)} |q|^2 \, dx \, dt + CT^3 \iint_{\Omega \times (T/4,T)} e^{-2s\eta}t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt$$

$$\leq T^{-3}e^{C(-sT^{-2}+T\|b\|_\infty)} \iint_{\Omega \times (T/4,3T/4)} |q|^2 \, dx \, dt$$

$$+ CT^3 \iint_{\Omega \times (0,T)} e^{-2s\eta}t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt. \tag{162}$$

Here, we have used (161) and (160). Applying (151), written for $\widetilde{\omega}$ to the first term of the right hand side of (161) and Carleman inequatily (122) (written also for $\widetilde{\omega}$) to the second one, we deduce that for $s = s_9$ the following holds:

$$\iint_Q e^{-2s\widetilde{C}T^{-1}/(T-t)}(T-t)^{-3}|q|^2 \, dx \, dt$$

$$\leq CT^3 \left(1 + \exp\left[C\left(sT^{-2} + T\|b\|_\infty\right)\right]\right) \iint_{\widetilde{\omega} \times (0,T)} e^{-2s\eta}t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt \tag{163}$$

$$\leq CT^3 \exp\left[C\left(1 + \frac{1}{T} + T\|b\|_\infty + \|b\|_\infty^{2/3}\right)\right] \iint_{\widetilde{\omega} \times (0,T)} e^{-2s\eta}t^{-3}(T-t)^{-3}|q|^2 \, dx \, dt.$$

Using (154) in (163), we get (157). This ends the proof of Proposition 4.2.

5. Proof of Theorem 2.1

We will first prove a null controllability result for a corresponding linear problem which is interesting for itself. For this proof we use a result of approximate controllability to the zero state and then apply an observability inequality to obtain estimates which enable us to pass to the limit. We could then try to use this null controllability result and apply the fixed point method to treat the nonlinear problem. For technical reasons, we prefer here to apply the fixed point method to the approximate controllability problem to the zero state and then use the observability inequality to obtain the desired result of exact controllability to the trajectories for the nonlinear system.

5.1. A null controllability result for a linear problem

We will consider the linear system

$$\begin{cases} \partial_t y - \operatorname{div}(a(x)\nabla y) + by = v1_\omega + k & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y_0 & \text{in} \quad \Omega, \end{cases} \tag{164}$$

where a satisfies (5, 6, 9), $b \in L^\infty(Q)$, $k \in L^r(0, T; L^r(\Omega))$, and $y_0 \in L^2(\Omega)$ are given. The following holds:

Theorem 5.1. *Let $T > 0$. Assume that $\omega \cap \Omega_0 \neq \emptyset$, Condition 2.1 (resp. Condition 2.2) in case (1) (resp. in case (2)) are fulfilled and a satisfies (5, 6, 9) and (10). We take data $b \in L^\infty(Q)$, $y_0 \in L^2(\Omega)$, $k \in L^2(Q)$ and such that*

$$\iint_Q e^{2s\widetilde{C}T^{-1}/(T-t)}(T-t)^3|k|^2 \, dx \, dt < +\infty \qquad (165)$$

for $s \geq \sigma(\Omega, \omega, a)(T + T^2 + T^2\|b\|_\infty^{2/3})$. Then for each case (1) or (2), there exists a control $\widehat{v} \in L^r(\mathcal{O} \times (0, T))$ such that the corresponding solution $\widehat{y}$ of (164) verifies

$$\widehat{y}(x, T) = 0 \quad in \; \Omega. \qquad (166)$$

Moreover, $\widehat{v}$ can be chosen satisfying the estimate

$$||\widehat{v}||_{L^r(\omega \times (0,T))} \leq \widehat{H}(\Omega, \omega, a, T, ||b||_\infty) \left(||y_0||_{L^2(\Omega)} + ||e^{s\widetilde{C}T^{-1}/(T-t)}(T-t)^{3/2}k||_{L^2(Q)} \right), \qquad (167)$$

with

$$\widehat{H}(\Omega, \omega, a, T, ||b||_\infty) = \exp\left[C \left(1 + \frac{1}{T} + T + (T + T^{1/2})||b||_\infty + ||b||_\infty^{2/3} \right) \right]. \qquad (168)$$

Remark 5.1. Notice that if $k \in L^r(0, T; L^r(\Omega))$ such that (165) holds, then k vanishes exponentially at $t = T$. If $k = 0$ in a neighbourhood of $t = T$ and $k \in L^r(0, T; L^r(\Omega))$ then (165) is satisfied.

Proof of Theorem 5.1. Let us fix $T > 0$, $b \in L^\infty(Q)$, $y_0 \in L^2(\Omega)$, $k \in L^r(0, T; L^r(\Omega))$, with r verifying (4) and such that (165) holds. For every $\varepsilon > 0$, let us consider the functional J_ε definite by

$$J_\varepsilon(q_T) = \frac{1}{2} \left(\iint_{\omega \times (0,T)} |q|^{r'} \, dx \, dt \right)^{2/r'} + \varepsilon||q_T||_{L^2(\Omega)} + \int_\Omega q(x, 0)\, y_0(x) \, dx + \iint_Q kq\, dx \, dt \quad \forall q_T \in L^2(\Omega), \quad (169)$$

where q is the solution of (118) associated to $q_T \in L^2(\Omega)$ and r' is the dual exponent of r. It is easy to see that J_ε is a continuous and strictly convex functional. Furthermore, from (122), it is immediate to deduce the following *unique continuation property* for the adjoint problem (118):

$$If \; q = 0 \; in \; \omega \times (0, T), \; then \; q \equiv 0.$$

Then, arguing as in [9] and [10], we see that

$$\liminf_{\|q_T\|_{L^2} \to \infty} \frac{J_\varepsilon(q_T)}{\|q_T\|_{L^2}} \geq \varepsilon$$

and, therefore, J_ε achieves its minimum at a unique point $\widehat{q}_{T,\varepsilon} \in L^2(\Omega)$. Let $\widehat{q}_\varepsilon$ be the solution of (118) associated to $\widehat{q}_{T,\varepsilon}$. Arguing again as in [9], we take in (164) $v = \widehat{v}_\varepsilon$, where

$$\widehat{v}_\varepsilon = sgn(\widehat{q}_\varepsilon)|\widehat{q}_\varepsilon|^{r'-1}\|\widehat{q}_\varepsilon\|_{L^{r'}(\omega \times (0,T))}\mathbf{1}_\omega, \qquad (170)$$

then, we find a solution $\widehat{y}_\varepsilon$ satisfying

$$||\widehat{y}_\varepsilon(\cdot, T)||_{L^2(\Omega)} \leq \varepsilon. \qquad (171)$$

Moreover, we can obtain the estimates of the $L^r(\omega \times (0,T))$-norm of the control independent of ε. We claim that for a suitable $C = C(\Omega, \omega, a) > 0$ we have:

$$||\widehat{v}_\varepsilon||_{L^r(\omega \times (0,T))} \leq \widehat{H}(\Omega, \omega, T, a, ||b||_\infty) \left(||y_0||_{L^2(\Omega)} + ||e^{s\widetilde{C}T^{-1}/(T-t)}(T-t)^{3/2}k||_{L^2(Q)} \right) \tag{172}$$

for all $\varepsilon > 0$ and with $\widehat{H}$ given by (168). Indeed, from (170) we can write that

$$||\widehat{v}_\varepsilon||_{L^r(\omega \times (0,T))} = \left(\iint_{\omega \times (0,T)} |\widehat{q}_\varepsilon|^{r'} \, dx \, dt \right)^{1/r'}. \tag{173}$$

On the other hand, at the minimum $\widehat{q}_{T,\varepsilon}$, we have

$$J_\varepsilon(\widehat{q}_{T,\varepsilon}) \leq J_\varepsilon(0) = 0.$$

Taking (169) into account, we see that

$$\frac{1}{2} \left(\iint_{\omega \times (0,T)} |\widehat{q}_\varepsilon|^{r'} \, dx \, dt \right)^{2/r'} \leq - \int_\Omega \widehat{q}_\varepsilon(x,0)y_0(x) \, dx - \iint_Q kq \, dx \, dt \leq ||\widehat{q}_\varepsilon(\cdot,0)||_{L^2(\Omega)}||y_0(x)||_{L^2(\Omega)}$$

$$+ \left(\iint_Q e^{2s\widetilde{C}T^{-1}/(T-t)}(T-t)^3|k|^2 \, dx \, dt \right)^{1/2} \left(\iint_Q e^{-2s\widetilde{C}T^{-1}/(T-t)}(T-t)^{-3}|q|^2 \, dx \, dt \right)^{1/2}. \tag{174}$$

In view of (131) and (165), the estimate (172) holds.

Since $\widehat{v}_\varepsilon$ is uniformly bounded in $L^r(\omega \times (0,T))$, for an appropriate subsequence, we deduce that

$$\widehat{v}_\varepsilon \to \widehat{v} \quad \text{weakly in } L^r(\omega \times (0,T)), \tag{175}$$

where $\widehat{v} \in L^r(\omega \times (0,T))$ and satisfies (167). Accordingly,

$$\widehat{y}_\varepsilon(T) \to \widehat{y}(T) \quad \text{in } L^2(\Omega),$$

where $\widehat{y}$ is the solution of (164) associated to $\widehat{v}$. Since we have (171) for all $\varepsilon > 0$, we also have (166). This ends the proof of Theorem 5.1.

Remark 5.2. Notice that, using the argument of [13] and [9], it is also possible to obtain the controls in $L^\infty(0,T; L^\infty(\omega))$.

5.2. The fixed point method. Conclusion

We are now ready to complete the proof of Theorem 2.1. We will apply a fixed point method to obtain ε-approximate controllability to the zero state for the nonlinear transmission problem, using the ideas of [9] and [10]. Then the main ingredients of our proof are the explicit estimates of the controls needed for controllability of a linear transmission problem, which have been obtained in the previous section and the choice of the control time depending on the size of the potential. This will allow us to pass to the limit in the nonlinear problem when ε tends to zero and obtain the complete results of Theorem 2.1.

Step 1: Let us consider a trajectory y^*, solution of the problem (12) without control. We introduce the change of variable $p = y - y^*$, where y is a solution of (8). Then, we obtain that

$$\begin{cases} \partial_t p - \operatorname{div}(a(x)\nabla p) + f(y^* + p) - f(y^*) = v 1_\omega & \text{in} \quad Q, \\ p = 0 & \text{on} \quad \Sigma, \\ p(x,0) = p_0 & \text{in} \quad \Omega, \end{cases} \tag{176}$$

where $p_0 = y_0 - y_0^*$. Theorem 2.1 will be proved if we show that, for each $p_0 \in L^2(\Omega)$, there exists $v \in L^r(\omega \times (0,T))$ such that

$$p(x,T) = 0 \quad \text{in } \Omega. \tag{177}$$

We will first consider the case in which $p_0 \in L^\infty$ and $f \in C^1$ in $\mathbb{R}$. We denote by h the following function:

$$h(a,s) = \begin{cases} \dfrac{f(a+s) - f(a)}{s} & \text{if } s \neq 0, \\ f'(a) & \text{if } s = 0. \end{cases}$$

Then h is continuous. Thanks to hypothesis (11) we know that for each $\eta > 0$, there exists $C_\eta > 0$ (depending only of η and the function f) such that

$$|h(y^*(x,t),s)|^{2/3} \leq C_\eta + \eta \log(1 + |s|) \quad \forall s \in \mathbb{R}, \quad \forall (x,t) \in Q. \tag{178}$$

Step 2: Let $\eta > 0$ and $R > 0$ be given positive constants whose values will be fixed later on.

Let us fix a time

$$T^R = \min\left\{ T, \|h\|_{L^\infty(-R,R)}^{-2/3}, \|h\|_{L^\infty(-R,R)}^{-1/3} \right\}. \tag{179}$$

For simplicity, in the sequel, we will refer only to the case (1), but we also take into account the dependence of the constants corresponding to the case (2).

Step 3: **(a)** We consider the truncation function $\mathbf{T}_R : \mathbb{R} \mapsto \mathbb{R}$, which is given as follows:

$$\mathbf{T}_R(s) = \begin{cases} s & \text{if } |s| \leq R, \\ R\,\operatorname{sgn}(s) & \text{otherwise.} \end{cases}$$

For each $z \in L^2(Q)$, we consider the linear system

$$\begin{cases} \partial_t p - \operatorname{div}(a(x)\nabla p) + h(y^*(x,t), \mathbf{T}_R(z))p = v 1_\omega & \text{in} \quad \Omega \times (0,T^R), \\ p = 0 & \text{on} \quad \Gamma \times (0,T^R), \\ p(x,0) = p_0 & \text{in} \quad \Omega. \end{cases} \tag{180}$$

Notice that (180) is of the form (164), with $b = h(y^*, \mathbf{T}_R(z)) \in L^\infty(Q)$. Then we can apply the arguments of the proof of Theorem 5.1 to (180). In fact, we will apply this result in a time interval $(0, T^R)$, where T^R is given by (179). This is a key point in this proof that will drive to appropriate estimates (the idea is taken from [13] and it has been applied later in [8]).

(b) More precisely, for every $\varepsilon > 0$, let us consider the functional J_ε of the form (169). Arguing as in the proof of Theorem 5.1, we obtain the existence of a control $v_z^\varepsilon \in L^r(\omega \times (0,T^R))$, minimizing the $L^r(\omega \times (0,T^R))$-norm of the form $v_z^\varepsilon = sgn(q_z^\varepsilon)|q_z^\varepsilon|^{r'-1}\|q_z^\varepsilon\|_{L^{r'}(\omega \times (0,T))} 1_\omega$, $(r' > 1)$ with q_z^ε the solution of the corresponding

problem (118), such that the solution p_z^ε of (180) with $v = v_z^\varepsilon$ satisfies

$$\|p_z^\varepsilon(\cdot, T^R)\|_{L^2(\Omega)} \le \varepsilon. \tag{181}$$

Then, from (172) we have

$$\|v_z^\varepsilon\|_{L^r(\omega \times (0, T^R))} \le C_1(\Omega, \omega, a, T^R, \|h\|_{L^\infty(-R,R)})\|p_0\|_{L^2(\Omega)} \tag{182}$$

where

$$C_1\left(\Omega, \omega, a, T^R, \|h\|_{L^\infty(-R,R)}\right) = \mathrm{e}^{C(\Omega, \omega, a)\left(1 + \frac{1}{T^R} + T^R + (T^R + T^{R\,1/2})\|h\|_{L^\infty(-R,R)} + \|h\|_{L^\infty(-R,R)}^{2/3}\right)}. \tag{183}$$

Using now the definition of T^R, we deduce

$$\|v_z^\varepsilon\|_{L^r(\omega \times (0, T^R))} \le \mathrm{e}^{C_2(\Omega, \omega, a, T)\left(1 + \|h\|_{L^\infty(-R,R)}^{2/3}\right)} \|p_0\|_{L^2(\Omega)}. \tag{184}$$

Moreover, thanks to (178), from (184) for each $z \in L^2(Q)$ we obtain the following:

$$\|v_z^\varepsilon\|_{L^r(\omega \times (0, T^R))} \le \mathrm{e}^{C_2(\Omega, \omega, a, T)(1 + C_\eta + \eta \log(1+R))}\|p_0\|_{L^2(\Omega)}$$
$$= C_3(\Omega, \omega, a, T, \eta, p_0)(1 + R)^{\eta C_2(\Omega, \omega, a, T)}. \tag{185}$$

Here we have used the fact that from (178) we can easily write that

$$\|h\|_{L^\infty(-R,R)}^{2/3} \le C_\eta + \eta \log(1 + |R|). \tag{186}$$

(**c**) For each $\varepsilon > 0$ and $z \in L^2(Q)$ we introduce the mapping $\Lambda : L^2(Q) \mapsto L^2(Q)$ defined as follows: for each $z \in L^2(Q)$, $\Lambda(z) = p_z^\varepsilon$, where p_z^ε is the the solution of (180) satisfying (181) with $v = v_z^\varepsilon$ constructed in the point (b) of this step. In fact, Λ is of the following form

$$z \in L^2(Q) \mapsto \mathbf{T}_R(z) \in L^\infty(Q) \mapsto h(y^*, \mathbf{T}_R(z)) \in L^\infty(Q) \mapsto v_z^\varepsilon \in L^r(\omega \times (0, T^R)) \mapsto p_z^\varepsilon \in L^2(Q).$$

Arguing as in [9], we apply Schauder's theorem and we deduce for each $\varepsilon > 0$ the existence of a fixed point p^ε (associated to v^ε) of Λ which verifies

$$\|p_z(\cdot, T^R)\|_{L^2(\Omega)} \le \varepsilon. \tag{187}$$

Notice that we have used that the solution p_z^ε of (180) is bounded (uniformly in z) in $L^2(0, T^R; H_0^1(\Omega))$ and its time derivative $\partial_t p_z^\varepsilon$ is bounded in $L^2(0, T^R; H^{-1}(\Omega))$.

(**d**) Let p^ε be a fixed point of Λ associated to the control v^ε constructed as above. Since (185) holds for v^ε, then v^ε is bounded in $L^r(\omega \times (0, T^R))$ uniformly in ε, p^ε is bounded in $L^2(0, T^R; H_0^1(\Omega))$ and $\partial_t p_z^\varepsilon$ is bounded in $L^2(0, T^R; H^{-1}(\Omega))$. For an appropriate subsequence, we deduce that as $\varepsilon \to 0$

$$v^\varepsilon \to \bar{v}^R \quad \text{weakly in } L^r(\omega \times (0, T^R)), \tag{188}$$

where $\bar{v}^R \in L^r(\omega \times (0, T^R))$ also satisfies (185), and

$$p^\varepsilon \to \bar{p}^R \quad \text{weakly in } L^2(0, T^R; H_0^1(\Omega)),$$
$$\partial_t p^\varepsilon \to \partial_t \bar{p}^R \quad \text{weakly in } L^2(0, T^R; H^{-1}(\Omega)),$$

where $\bar{p}^R$ is the solution of the following problem:

$$\begin{cases} \partial_t \bar{p}^R - \mathrm{div}(a(x)\nabla\bar{p}^R) + h(y^*(x,t), \mathbf{T}_R(\bar{p}^R))\bar{p}^R = \bar{v}^R 1_\omega & \text{in} \quad \Omega \times (0, T^R), \\ \bar{p}^R = 0 & \text{on} \quad \Gamma \times (0, T^R), \\ \bar{p}^R(x, 0) = p_0 & \text{in} \quad \Omega. \end{cases} \quad (189)$$

Then

$$p^\varepsilon(T^R) \to \bar{p}^R(T^R) \quad \text{in } L^2(\Omega)$$

and since we have (187) for all $\varepsilon > 0$, we also have

$$\bar{p}^R(T^R) = 0. \quad (190)$$

On the other hand, since $\bar{v}^R \in L^r(0, T^R; L^r(\omega))$, with r as in (4), we can write (*cf.* for example [2] and [3]) that

$$||\bar{p}^R||_\infty \leq \mathrm{e}^{T^R ||h(y^*, \mathbf{T}_R(\bar{p}^R))||_\infty} ||p_0||_\infty + T^R \mathrm{e}^{T^R ||h(y^*, \mathbf{T}_R(\bar{p}^R))||_\infty} ||\bar{v}^R||_{L^r(0,T^R;L^r(\omega))}. \quad (191)$$

Using again the definition of T^R and also taking into account (185) and (186), we deduce from (191) that $\bar{p}^R$ verifies

$$||\bar{p}^R||_\infty \leq \mathrm{e}^{C_4(\Omega,\omega,a,T)\left(1 + ||h||^{2/3}_{L^\infty(-R,R)}\right)} \left(||p_0||_\infty + ||\bar{v}^R||_{L^r(0,T^R;L^r(\omega))} \right)$$
$$\leq C_5(\Omega, \omega, a, T, \eta, p_0)(1 + R)^{\eta C_6(\Omega,\omega,a,T)}. \quad (192)$$

Notice that in (192) the constant C_5 is independent of R and the constant C_6 is independent of η and R. Let us extend by zero $\bar{p}^R$ and $\bar{v}^R$ to the whole cylinder $Q = \Omega \times (0, T)$ and for simplicity, we still call them $\bar{p}^R$ and $\bar{v}^R$. It is clear that (192) holds and that $\bar{v}^R$ is such that

$$\bar{p}^R(T) = 0.$$

(**e**) In order to conclude the proof of this theorem, it is sufficient to check that for η and R suitably chosen, $\bar{p}^R$ (defined on $\Omega \times (0, T)$) satisfies

$$||\bar{p}^R||_\infty \leq R. \quad (193)$$

Then we can say that $\mathbf{T}_R(\bar{p}^R) = \bar{p}^R$. Of course, this implies the existence of a control $v \in L^r(0, T; L^r(\omega))$ such that the solution of (176) satisfies (177). Indeed, from (192) we can choose $\eta = 1/(2C_6)$ and $R > 0$ such that

$$C_5(\Omega, \omega, a, T, p_0)(1 + R)^{\eta C_6(\Omega,\omega,a,T)} < R.$$

Then we obtain (193). This proves Theorem 2.1 when $p_0 \in L^\infty(\Omega)$ and $f \in C^1(\mathbb{R})$. We just mention that we treat the case in which f is only locally Lipschitz continuous as for example in [9] and [13] using approximations of f by C^1 functions. Then, in this case we deduce the existence of a control $v \in L^r(0, T; L^r(\omega))$ such that the corresponding solution to (176) verifies (177).

Finally, if $p_0 \in L^2(\Omega)$, for $\delta > 0$ sufficiently small we set $v \equiv 0$ for $t \in (0, \delta)$. Using the regularizing effect of the heat equation (see, for example [21] and [22]), we deduce that the corresponding (local) solution p of (176) satisfies $p(\cdot, \delta) \in L^\infty(\Omega)$. Then, we argue as above for p in the interval $[\delta, T]$ and we obtain a control $v \in L^r(0, T; L^r(\omega))$ such that (177) holds. This ends the proof of Theorem 2.1.

6. Proofs of Lemma 3.1 and Lemma 3.2

In this section we will present the construction of the weight functions we used for our global Carleman inequalities.

Proof of the Lemma 3.1. We will proceed in several steps.

Step 1: Let ζ and x be vector fields verifying Condition 2.1. First, we will construct a function $\widetilde{\beta}_1 \in C^1(\overline{\Omega}_1)$, such that

$$\widetilde{\beta}_1 > 0 \text{ in } \Omega_1, \tag{194}$$

$$\widetilde{\beta}_1 = 0 \text{ on } \Gamma, \tag{195}$$

$$\partial_n \widetilde{\beta}_1 < 0 \text{ on } \Gamma, \tag{196}$$

$$\widetilde{\beta}_1 = 1 \text{ on } S, \tag{197}$$

$$\partial_n \widetilde{\beta}_1 > 0 \text{ on } S, \tag{198}$$

$$\nabla \widetilde{\beta}_1 \neq 0 \text{ in } \overline{\Omega}_1. \tag{199}$$

For $t \in [0, t_1(x_0)]$, let us introduce the following change of variables:

$$\tau = \frac{t}{t_1(x_0)}, \quad \tau \in [0,1], \quad \widetilde{x}(\tau) = x(t).$$

Observe that, if we take $x \in \overline{\Omega}_1$, then there exists $x_0 \in \Gamma$ such that $x = x(t, x_0)$ for $t \in [0, t_1(x_0)]$ or, in other words, there exists $\tau \in [0,1]$ such that $\widetilde{x}(0) = x_0 \in \Gamma$, $\widetilde{x}(\tau) = x(t)$ and $\widetilde{x}(1) = x(t_1(x_0)) \in S$. Moreover

$$\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}t}\, t_1(x_0) = \zeta(x(t)) t_1(x_0). \tag{200}$$

Let us set

$$\widetilde{\beta}_1(x(t)) = \widetilde{\beta}_1(\widetilde{x}(\tau)) = \tau, \quad \tau \in [0,1]. \tag{201}$$

This function verifies the following properties:

$$\frac{\mathrm{d}\widetilde{\beta}_1}{\mathrm{d}\tau}(x) = 1 \quad \text{for all } x \in \Omega_1, \tag{202}$$

$$0 < \widetilde{\beta}_1(x) < 1 \quad \text{for all } x \in \Omega_1, \tag{203}$$

$$\widetilde{\beta}_1(x) = 0 \quad \text{for all } x \in \Gamma, \tag{204}$$

and

$$\widetilde{\beta}_1(x) = \widetilde{\beta}_1(\widetilde{x}(1)) = 1 \quad \text{for all } x \in S. \tag{205}$$

On the other hand, from (200) and (202), for all $x \in \Omega_1$, we have

$$\frac{\mathrm{d}\widetilde{\beta}_1}{\mathrm{d}\tau}(x(\tau)) = \nabla\widetilde{\beta}_1(\widetilde{x}(\tau))\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}\tau}(\tau) = \nabla\widetilde{\beta}_1(x)\zeta(x(t))t_1(x_0) = 1, \tag{206}$$

therefore

$$\nabla\widetilde{\beta}_1(x) \neq 0 \quad \text{for all } x \in \Omega_1\,.$$

Moreover, using (200, 205) and (206), we can write that

$$\nabla\widetilde{\beta}_1(x)\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}\tau} = \left(\nabla\widetilde{\beta}_1(x)\cdot n\right)\left(\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}\tau}\cdot n\right) = \left(\nabla\widetilde{\beta}_1(x)\cdot n\right)(\zeta(x)\cdot n)t_1(x_0) = 1 \quad \text{for all } x \in S.$$

Then, taking into account (17), we deduce that

$$\nabla\widetilde{\beta}_1(x)\cdot n > 0 \quad \text{for all } x \in S.$$

This means (198). By the similar way, it is easy to check (196). Thus we also have (199).

In order to obtain $\widetilde{\beta}_1 \in C^2(\overline{\Omega}_1)$, we just notice that we can approximate the function of class C^1, which we have constructed above by an other function of class C^2 (that we keep calling $\widetilde{\beta}_1$), such that it still satisfies the properties (194–199).

Let us now consider the diffusion coefficients a_i, $i = 0, 1$ such that (5) holds and let $\omega_0 \subset\subset \omega \cap \Omega_0$.

Step 2: For $\varepsilon > 0$ small enough, we set

$$U_\varepsilon(S) = \{x : x \in \Omega_0, \quad \text{dist}(x, S) < \varepsilon\}.$$

We can construct a function α_0 in $\overline{U_\varepsilon(S)}$, such that $\alpha_0 \in C^2(\overline{U_\varepsilon(S)})$ and

$$\begin{aligned} \alpha_0 = 1 \quad &\text{on } S, \quad \partial_n\alpha_0 > 0 \quad \text{on } S, \\ \alpha_0 > 0 \quad &\text{in } \overline{U_\varepsilon(S)}, \nabla\alpha_0 \neq 0 \quad \text{in } \overline{U_\varepsilon(S)} \end{aligned} \tag{207}$$

and

$$a_0\partial_n\alpha_0 = a_1\partial_n\widetilde{\beta}_1 \quad \text{on } S. \tag{208}$$

Now, we extend this function in Ω_0 to a function that we call again α_0, with $\alpha_0 \in C^2(\overline{\Omega}_0)$ and $\alpha_0 > 0$ in Ω_0.

Step 3: Thanks to the Morse theorem, we deduce that there exists a sequence of Morse functions θ_k, $k \geq 1$ (functions with isolated critical points *i.e.* their gradient vanishes only in a finite number of points), such that

$$\theta_k \to \alpha_0 \quad \text{in } C^2(\overline{\Omega}_0) \quad \text{if } k \to +\infty. \tag{209}$$

If θ_k is close enough to α_0, the points where $\nabla\theta_k$ vanishes can not be in $\overline{U_\varepsilon(S)}$. Moreover, we can assume that for some $\delta > 0$ we have

$$|\nabla\alpha_0| \geq \delta > 0 \quad \text{in } \overline{U_\varepsilon(S)}. \tag{210}$$

Let us construct a Morse function $\mu \in C^2(\overline{\Omega}_0)$, such that

$$\mu = 1 \quad \text{on } S, \quad \partial_n\mu > 0 \quad \text{on } S \tag{211}$$

$$a_0\partial_n\mu = a_1\partial_n\widetilde{\beta}_1 \qquad \text{on } S \tag{212}$$

and

$$\nabla\mu \neq 0 \quad \text{in } \overline{U_\varepsilon(S)}. \tag{213}$$

For this, we consider $\varphi \in \mathcal{D}(\overline{U_\varepsilon(S)})$ and $\varphi = 1$ in $\overline{U_{\varepsilon_0}(S)}$, with $0 < \varepsilon_0 < \varepsilon$. We set

$$\mu_k(x) = \theta_k(x) + \varphi(x)(\alpha_0(x) - \theta_k(x)).$$

It is clear that

$$\mu_k = \alpha_0 \quad \text{in } \overline{U_{\varepsilon_0}(S)}. \tag{214}$$

Then, the function μ_k satisfies (211) and (212). Moreover, we have

$$\nabla\mu_k = \nabla\theta_k \quad \text{in } \overline{\Omega_0 \backslash U_\varepsilon(S)}$$

and

$$\nabla\mu_k = \nabla\theta_k + \varphi(\nabla\alpha_0 - \nabla\theta_k) + \nabla\varphi(\alpha_0 - \theta_k) \quad \text{in } \overline{U_\varepsilon(S)}. \tag{215}$$

Then, using (209) and (210) in (215), we deduce that there exist a positive number $k_0 = k_0(\delta)$ such that, if $k \geq k_0$ we have

$$|\nabla\mu_k| \geq |\nabla\theta_k| - 2\|\varphi\|_{C^1}\|\alpha_0 - \theta_k\|_{C^1} \geq \frac{\delta}{2} \quad \text{in } \Omega_0 \cap U_\varepsilon(S).$$

We choose $k \geq k_0$ and we set $\mu(x) = \mu_k(x)$. Then, μ is a Morse function whose gradient vanishes only in the set of points where the gradient of θ_k vanishes. This, together with (214), implies that (211) and (213) hold.

Step 4: On the other hand, arguing as in [15], we can deduce that there exists a mapping $g : \Omega_0 \mapsto \Omega_0$ which is a diffeomorphism on Ω_0, which leaves invariant $\overline{U_\varepsilon(S)}$ and transports the points where the gradient of μ vanishes in ω_0. We set

$$\widetilde{\beta}_0(x) = \mu(g(x)).$$

Then, equation (33) holds. Thanks to the properties (211) of the function μ, we also have (30, 31) and (32).

This ends the proof of Lemma 3.1.

Proof of the Lemma 3.2. Assume that we are in the situation of Case (2). Let $\omega_0 \subset\subset \omega \cap \Omega_0$ be an arbitrary fixed open subset of Ω_0.

Step 1: We assume that there exist $\mathcal{O}_1, \mathcal{O}_2 \subset\subset \Omega_1$ two open disjoint subsets, such that Condition 2.2 holds between Ω_1 and each one of two sets $\mathcal{O}_1$ and $\mathcal{O}_2$. Then, as in the first part of the proof of Lemma 3.1, we construct two functions $\beta_1^i \in C^1(\overline{\Omega}_1 \backslash \mathcal{O}_i)$, $\beta_1^i > 0$ in $\overline{\Omega}_1 \backslash \mathcal{O}_i$, $i = 1, 2$, such that

$$\begin{aligned}
\beta_1^i &= 2 \quad \text{on } S, & i &= 1, 2, \\
\partial_n\beta_1^i &> 0 \text{ on } S, & i &= 1, 2, \\
\partial_n\beta_1^i &> 0 \text{ on } \partial\mathcal{O}_i, & i &= 1, 2, \\
\beta_1^i &= 1 \quad \text{on } \partial\mathcal{O}_i, & i &= 1, 2, \\
\nabla\beta_1^i &\neq 0 \text{ in } \overline{\Omega}_1 \backslash \mathcal{O}_i, & i &= 1, 2,
\end{aligned} \tag{216}$$

where n stands for the unit exterior normal to Ω_1 and $\mathcal{O}_i$, $i = 1, 2$.

Step 2: Let $\widetilde{B}_i$ and B_i, $i = 1, 2$ be balls such that $B_1 \subset\subset \widetilde{B}_1 \subset\subset \mathcal{O}_1$ and $B_2 \subset\subset \widetilde{B}_2 \subset\subset \mathcal{O}_2$. We will present only the construction of $\widetilde{\beta}_1^1$. The second function will be obtained by the same arguments. Let us set

$$W_\varepsilon = \{x : x \in \mathcal{O}_1, \quad \text{dist}(x, \partial\mathcal{O}_1) < \varepsilon\}.$$

First, we observe that since $\partial_n\beta_1^1 > 0$ on $\partial\mathcal{O}_1$ and $\beta_1^1 = 1$ on $\partial\mathcal{O}_1$, we construct a function $\beta_1^1 \in C^1(\overline{W_\varepsilon})$, such that there exists $\delta > 0$ such that

$$\beta_1^1 \leq 1 \quad \text{in } W_\varepsilon, \quad 0 < \beta_1^1 \leq 1 - 4\delta \quad \text{on } \partial W_\varepsilon \backslash \partial\mathcal{O}_1 \quad \text{and} \quad \nabla\beta_1^1 \neq 0 \text{ in } \overline{W_\varepsilon}.$$

Then, we can extend this function by a function still called $\beta_1^1 \in C^1(\overline{\mathcal{O}}_1)$, such that

$$0 < \beta_1^1 \leq 1 - 3\delta \quad \text{in } \mathcal{O}_1 \backslash W_\varepsilon \quad \text{and} \quad \nabla \beta_1^1 \neq 0 \quad \text{in } \overline{W_\varepsilon}.$$

Now, we approximate β_1^1 by Morse functions in such a way that

$$0 < \beta_1^1 \leq 1 - 2\delta \quad \text{in } \mathcal{O}_1 \backslash W_\varepsilon, \tag{217}$$

where we keep the name β_1^1 for this approximation. The gradient of this function vanishes only in a finite number of points. As we already mentioned in the Step 3 of this proof, we can deduce the existence of a diffeomorphism on $\mathcal{O}_1$, which keeps invariant $\overline{W_\varepsilon}$ and transports the points where the gradient of β_1^1 vanishes in B_1. We obtain then a new function (that we keep on calling β_1^1) such that $\beta_1^1 \in C^1(\overline{\Omega}_1)$, $\beta_1^1 > 0$ in Ω_1 and

$$\nabla \beta_1^1 \neq 0 \quad \text{in } \overline{\Omega}_1 \backslash B_1. \tag{218}$$

Moreover, from (217) we obtain that for $\delta > 0$ we have

$$\beta_1^1 \leq 1 - \delta \quad \text{in } \overline{B}_1. \tag{219}$$

Analogously, we construct a function $\beta_1^2 \in C^1(\overline{\Omega}_1)$, $\beta_1^2 > 0$ in Ω_1, which verifies

$$\nabla \beta_1^2 \neq 0 \quad \text{in } \overline{\Omega}_1 \backslash B_2 \tag{220}$$

and

$$\beta_1^2 \leq 1 - \delta \quad \text{in } \overline{B}_2. \tag{221}$$

Step 3: Let us finally prove that the properties (40) and (41) are satisfied. For this, we will see that it is possible to modify β_1^1 (resp. β_1^2) in B_1 (resp. B_2) in order to obtain the conditions (40) and (40). We will be able to do this without changing the values of these functions in $\overline{\mathcal{O}}_1 \backslash B_1$ and $\overline{\mathcal{O}}_2 \backslash B_2$. For simplicity, we will present the details of the construction of only one of such a function, because the same arguments will be valid for the other one.

Let us define a new function $\widetilde{\beta}_1^1$ as follows:

$$\widetilde{\beta}_1^1(x) = \begin{cases} \beta_1^1(x) & \text{if} \quad x \in \overline{\Omega}_1 \backslash \mathcal{O}_1, \\ \left(\beta_1^1(x)\right)^{n(x)} & \text{if} \quad x \in \overline{\mathcal{O}}_1, \end{cases} \tag{222}$$

with

$$n(x) = \left(\frac{1}{\beta_1^1(x)}\right)^p, \tag{223}$$

where $p \in \mathbb{N}$ will be fixed later on. We can write that

$$\widetilde{\beta}_1^1 = \left(\beta_1^1\right)^n = e^{n \log \beta_1^1} \quad \text{in } \overline{\mathcal{O}}_1. \tag{224}$$

Since $\beta_1^1 = 1$ on $\partial \mathcal{O}_1$, we have that $n = 1$ on $\partial \mathcal{O}_1$ and then, from (224) we deduce that

$$\widetilde{\beta}_1^1 = \beta_1^1 = 1 \quad \text{on } \partial \mathcal{O}_1. \tag{225}$$

Moreover, we have that

$$\nabla \widetilde{\beta}_1^1 = \nabla \beta_1^1 \quad \text{on } \partial \mathcal{O}_1 \tag{226}$$

and that the gradient of $\widetilde{\beta}_1^1$ vanishes only in B_1, where the gradient of β_1^1 is zero, *i.e.*

$$\nabla \widetilde{\beta}_1^1 \neq 0 \quad \text{in } \overline{\Omega}_1 \backslash B_1. \tag{227}$$

Indeed, using (224) and (223), we have

$$\nabla \widetilde{\beta}_1^1 = \left(\beta_1^1\right)^n \left(\nabla n \log \beta_1^1 + \frac{n \nabla \beta_1^1}{\beta_1^1}\right) = \left(\beta_1^1\right)^n \nabla \beta_1^1 \left[\frac{-p \log \beta_1^1}{(\beta_1^1)^{p+1}} + \frac{1}{(\beta_1^1)^{p+1}}\right]. \tag{228}$$

Taking into account (217), it is easy to deduce from (228), that (226) and (227) hold.

In order to modify the values of $\widetilde{\beta}_1^1$ in B_1, we first use (219) and we obtain that

$$n \geq \left(\frac{1}{1-\delta}\right)^p.$$

Next, from (224) we deduce that

$$\widetilde{\beta}_1^1 \leq (1-\delta)^{(1/(1-\delta))^p}. \tag{229}$$

On the other hand, we know that the second function that we constructed in the Step 5 of this proof satisfies

$$0 < \beta_1^2 \leq \bar{\beta}_1^2 = \max_{\overline{B}_1} \beta_1^2 > 0 \quad \text{in } B_1.$$

Choosing now p large enough, we can deduce from the estimate (229) the following:

$$\widetilde{\beta}_1^1 \leq \frac{1}{2} \bar{\beta}_1^2 \quad \text{in } B_1.$$

This gives (41). The same arguments applied to the function β_1^2 lead to the existence of a new function $\widetilde{\beta}_1^2 \in C^1(\overline{\Omega}_1)$, $\widetilde{\beta}_1^2 > 0$ in Ω_1, such that satisfies (38, 40) and (42) for $i = 2$.

To conclude this step, we observe that we can approximate the functions of class C^1 already constructed by functions of class C^2, preserving the properties of the functions $\widetilde{\beta}_1^1$ and $\widetilde{\beta}_1^2$.

Step 4: For $\varepsilon, \varepsilon' > 0$ small enough, we set

$$V_\varepsilon(\Gamma) = \{x : x \in \Omega_0, \quad \text{dist}(x, \Gamma) < \varepsilon\}$$

and

$$V_{\varepsilon'}(S) = \{x : x \in \Omega_0, \quad \text{dist}(x, S) < \varepsilon'\}.$$

We can locally construct a function α_0 in $\overline{V_\varepsilon(\Gamma)}$, such that $\alpha_0 \in C^2(\overline{V_\varepsilon(\Gamma)})$ and

$$\begin{aligned} &\alpha_0 = 0 \quad \text{on } \Gamma, \quad \partial_n \alpha_0 > 0 \quad \text{on } \Gamma, \\ &\alpha_0 > 0 \quad \text{in } \overline{V_\varepsilon(\Gamma)}, \nabla \alpha \neq 0 \quad \text{in } \overline{V_\varepsilon(\Gamma)}. \end{aligned} \tag{230}$$

On the other hand, in $\overline{V_{\varepsilon'}(S)}$ we construct another function, which for simplicity, we also will denote by α_0, such that $\alpha_0 \in C^2(\overline{V_{\varepsilon'}(S)})$ and

$$\begin{aligned}
\alpha_0 &= 2 \quad \text{on } S, \quad \partial_n \alpha_0 > 0 \quad \text{on } S, \\
\alpha_0 &> 0 \quad \text{in } \overline{V_{\varepsilon'}(S)}, \ \nabla \alpha_0 \neq 0 \quad \text{in } \overline{V_{\varepsilon'}(S)}
\end{aligned} \tag{231}$$

and

$$a_0 \partial_n \alpha_0 = a_1 \partial_n \beta_1^i \quad \text{on } S, \quad i = 1, 2. \tag{232}$$

Now, we extend both functions in Ω_0, to a function which we keep on calling α_0, with the following properties:

$$\begin{aligned}
&\alpha_0 \in C^2(\overline{\Omega}_0), \quad \alpha_0 > 0 \quad \text{in } \Omega_0, \\
&\alpha_0 = 0 \quad \text{on } \Gamma, \ \partial_n \alpha_0 > 0 \text{ on } \Gamma, \\
&\alpha_0 = 2 \quad \text{on } S, \ \partial_n \alpha_0 > 0 \text{ on } S.
\end{aligned} \tag{233}$$

Step 5: In the sequel, we will use same arguments as for the proof of Lemma 3.1.

Thanks to the Morse theorem, we deduce that there exists a sequence of Morse functions θ_k, $k \geq 1$ (functions with isolated critical points *i.e.* their gradient vanishes only in a finite number of points), such that

$$\theta_k \to \alpha_0 \quad \text{in } C^2(\overline{\Omega}_0) \quad \text{if } k \to +\infty. \tag{234}$$

If θ_k is close enough to α_0, the points where $\nabla \theta_k$ vanishes can not be in $\overline{V_\varepsilon(\Gamma)} \cup \overline{V_{\varepsilon'}(S)}$. Moreover, we can assume that for some $\delta > 0$ we have

$$|\nabla \alpha_0| \geq \delta > 0 \quad \text{in } \overline{V_\varepsilon(\Gamma)} \cup \overline{V_{\varepsilon'}(S)}. \tag{235}$$

We can construct a Morse function $\mu \in C^2(\overline{\Omega}_0)$, such that

$$\mu = 0 \quad \text{on } \Gamma, \quad \partial_n \mu < 0 \quad \text{on } \Gamma, \tag{236}$$

$$\mu = 2 \quad \text{on } S, \quad \partial_n \mu > 0 \quad \text{on } S \tag{237}$$

and

$$\nabla \mu \neq 0 \quad \text{in } \overline{V_\varepsilon(\Gamma)} \cup \overline{V_{\varepsilon'}(S)}. \tag{238}$$

Indeed, it suffices to consider $\varphi \in \mathcal{D}(\overline{V_\varepsilon(\Gamma)} \cup \overline{V_{\varepsilon'}(S)})$ and $\varphi = 1$ in a neighborhood of $\Gamma \cup S$ and to define

$$\mu_k(x) = \theta_k(x) + \varphi(x)(\alpha_0(x) - \theta_k(x)).$$

Arguing as in the proof of Lemma 3.1, we can choose $k \geq k_0$ and $\mu(x) = \mu_k(x)$ in such a way that μ is a Morse function with gradient vanishing only in the points contained in the set of points where the gradient of θ_k vanishes and satisfying the previous properties.

Step 6: Finally, we can deduce that there exists a mapping $g : \Omega \mapsto \Omega$ which is a diffeomorphism on Ω, which leaves invariant $\overline{V_\varepsilon(\Gamma)} \cup \overline{V_{\varepsilon'}(S)}$ and transports the points where the gradient of μ vanishes in ω_0. We set

$$\widetilde{\beta}_0(x) = \mu(g(x)).$$

This ends the proof of Lemma 3.2.

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