On the singular times of fluids with nonlinear viscosity

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*research parcially financed by C.I.C.Y.T project MAR98-0486

†research financed by Patronato de la Fundación Cámara de la Universidad de Sevilla

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1 Introduction.

Many fluids in Nature are modeled by the Navier-Stokes equations. However, many experiences demonstrate there exist other type of fluids which cannot be modelated by these equations. These fluids are known as **Non-newtonian fluids**. The stress tensor σ is decomposed as $\sigma = \pi Id - \tau$, where π is the pressure and τ the extra stress tensor. In the case of incompressible fluids, the pressure is an unknown. Respect to τ, two main kinds of fluids are considered:

1) Shear-dependent viscosity fluids. In these fluids, τ is a given nonlinear function, depending on $e(u) = \frac{1}{2}\left(\nabla u +^t \nabla u\right)$, the symmetric gradient of the velocity u, as follows:

$$\tau = 2\mu(|e(u)|^2)e(u),$$

where $\mu : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is the generalized viscosity function and $|e(u)|^2 = e_{i,j}(u)e_{i,j}e(u)$ (the summation convention of repeated indices is used). Some examples are biological fluids of small molecular weight (blood, white of an egg, ...), polymer very dissolved in a base of newtonian liquid, etc. Applications in Glaciology (glacier ice slide) and Geology (dynamics on the Earth's mantle) are also important.

2) Viscoelastic fluids, which have intermediate properties between viscous fluids and elastic materials. They are **"fluids with memory "**, i.e., fluids whose extra stress tensor in a instant t depends on the fluid dynamic in t and also on the behaviour previous to t. This property is expressed by either integral or differential (constitutive) laws. Polymer mixtures and high density polymers are important examples of this kind of fluids.

We will focus on the first kind of fluids, assuming that, for simplicity, the extra stress tensor τ is given by either a power law or a Carreau's law, i.e.:

$$\tau = 2\{\mu_\infty + \mu_0|e(u)|^{p-2}\}e(u) \quad \textbf{(power law)}$$

or

$$\left.\begin{array}{r}\tau = 2\{\mu_\infty + \mu_0\,(1+|e(u)|)^{p-2}\}e(u)\\ \tau = 2\{\mu_\infty + \mu_0\left(1+|e(u)|^2\right)^{(p-2)/2}\}e(u)\end{array}\right\}\textbf{(Carreau's laws)},$$

where $p > 1$, $\mu_\infty \geq 0$ and $\mu_0 > 0$. When $p = 2$, we are in the newtonian case.

In this paper, an important simplification will be made; we are going to consider periodic boundary conditions. Let us define $(0, T)$ a time interval $(T > 0)$ and $\Omega = (0, L)^d$, $d = 2$ or 3, the spatial domain of periodicity, denoting his boundary $(\partial\Omega)$ as:

$$\Gamma_j = \partial\Omega \cap \{x_j = 0\}; \quad \Gamma_{j+d} = \partial\Omega \cap \{x_j = L\} \quad (j = 1, ..., d).$$

Then, we consider the following model of space periodic flows of incompressible non-newtonian fluids. Given f (external force) and u_0 (initial velocity), the problem is to find u (velocity) and π (pressure) such that:

$$(NS)^p_{per}\left\{\begin{array}{rl}\dfrac{\partial u}{\partial t} + u\cdot\nabla u + \nabla\pi - \nabla\cdot\tau(e(u)) = f & \text{in } (0,T)\times\Omega\\ \nabla\cdot u = 0 & \text{in } (0,T)\times\Omega\\ u(0,\cdot) = u_0 & \text{in } \Omega\\ u|_{\Gamma_j} = u|_{\Gamma_{j+d}} \quad \nabla u|_{\Gamma_j} = \nabla u|_{\Gamma_{j+d}} & \pi|_{\Gamma_j} = \pi|_{\Gamma_{j+d}} \quad (j = 1, .., d).\end{array}\right.$$

Remark: *In a general way, we may suppose the extra stress tensor τ given by:*

$$\tau = 2\mu_\infty e(u) + \tau^p(e(u)),$$

where $\mu_\infty \geq 0$ (the newtonian viscosity) and τ^p is the purely non-newtonian tensor, assuming that there exists a function $\mathcal{U}^p \in C^2(\mathbb{R}^{d\times d})$ such that (we denote $i, j, k, l \in \{1, ..., d\}$ and $\eta,\ \xi \in \mathbb{R}^{d\times d}$):

$$(H1) \qquad \frac{\partial\mathcal{U}^p}{\partial\eta_{i,j}}(\eta) = \tau^p_{i,j}(\eta), \quad \forall\, i, j, \quad \forall\,\eta$$

$$(H2) \qquad \mathcal{U}^p(0) = \frac{\partial\mathcal{U}^p}{\partial\eta_{i,j}}(0) = 0 \quad \forall\, i, j$$

$$(H3) \qquad \frac{\partial^2\mathcal{U}^p}{\partial\eta_{i,j}\partial\eta_{k,l}}(\eta)\xi_{i,j}\xi_{k,l} \geq C_1\left\{\begin{array}{ll}|\eta|^{p-2}|\xi|^2 & \textit{(power law)}\\ (1+|\eta|)^{p-2}|\xi|^2 & \textit{(Carreau's laws)}\end{array}\right. \qquad \forall\,\eta,\xi$$

$$(H4) \qquad \left|\frac{\partial^2 \mathcal{U}^p}{\partial \eta_{i,j}\eta_{k,l}}(\eta)\right| \leq C_2(1+|\eta|)^{p-2}, \quad \forall\, i,j,k,l, \quad \forall\, \eta,$$

where C_1, $C_2 > 0$ are two constants. $\mathcal{U}^p$ is called a potential function of τ^p. We understand $(H3)$ and $(H4)$ in the sense that only one of the two conditions is considered; either $(H3)_1$ and $(H4)_1$, which play the role of a power law, or $(H3)_2$ and $(H4)_2$ in the role of a Carreau's law.

Existence and uniqueness results of weak and strong solutions of $(NS)^p_{per}$ are known, which depending on the data, the boundary conditions and, mainly, the power p (see Section 2 for a definition of weak and strong solution and for a review of these results).

Under the conditions of existence and uniqueness of a global strong solution, it is proved in [7] the existence of a global attractor set of finite fractal dimension, applying the standard semigroup theory. Moreover, if there exists a unique solution which is not continuous in time, it is also possible to construct an attractor in another way; basically, a "short trajectory" plays the role of an instant of time t in the standard theory. In this sense, the solutions are no continuous over each point, but they are continuous over each short trajectory. On the other hand, when homogeneous Dirichlet conditions ($u = 0$ on $\partial\Omega$) are imposed, the asymptotic behaviour of solutions is studied in [1], [12]. For example, it is proved in [12], that for a Carreau's law with $p \geq 2$ or a power law with $6/5 < p < 2$, the solution associated to the data $u_0 \in H$ and $f = 0$ decrease exponentially in time; while the solution has a polinomial decay in time if a power law with $p \geq 2$ is considered. More specific studies at this scope can be found in [11], focused on the time asymptotic behaviour of the planets orbit through the Boussinesq approximation.

The purpose of this paper will be the study of the set of times where a global weak solution cannot have the regularity necessary to be a strong solution, which are called **singular times**. We obtain two main results. First, under hypothesis of existence of a strong solution which "blows up" at infinite time, we will get (in Section 3) the existence of arbitrarily small singular times. Second, in Section 4 we will estimate the measure of the singular times set, using the Hausdorff dimension (in particular, only considering the regularity of a weak solution, this set has always zero Lebesgue's measure).

The norms related to the spaces $L^p(\Omega)$ will be denote by $\|\cdot\|_p$, and the norms related to another space F will be denote as $\|\cdot\|_F$.

2 Existence and uniqueness of solution.

Classical results of existence and uniqueness of solution (in the Dirichlet case) were obtained in [5] and [6], using compactness and monotony arguments. After that, more specific results are collected in [8], mainly in the case of periodic boundary conditions. We are going to review it on this section.

We consider the following spaces of functions with free divergence and periodic boundary conditions:

$$H = \left\{ v \in L^2(\Omega)^d : \nabla \cdot v = 0, (v \cdot n)|_{\Gamma_j} = -(v \cdot n)|_{\Gamma_{j+d}}, \int_\Omega v dx = 0 \right\},$$

$$V_p = \left\{ v \in W^{1,p}(\Omega)^d : \nabla \cdot v = 0, v|_{\Gamma_j} = v|_{\Gamma_{j+d}}, \int_\Omega v dx = 0 \right\},$$

(the condition of zero average let have spaces where Poincaré and Korn inequalities are satisfied).

Definition 2.1 (Weak solution) *Given $u_0 \in H$, $f \in L^2((0,T) \times \Omega)^d$ and $T > 0$, we say that $u : (0,T) \times \Omega \longrightarrow \mathbb{R}^d$ is a weak solution of $(NS)^p_{per}$ in $(0,T)$ if $u \in L^\infty(0,T;H) \cap L^p(0,T;V_p)$, and satifies the following variational formulation: $\forall \varphi \in C^1([0,T];V_p)$ such that $\varphi(T) = 0$,*

$$\int_0^T \int_\Omega \left\{ -u_i \frac{\partial \varphi_i}{\partial t} - u_j u_i \frac{\partial \varphi_i}{\partial x_j} + \tau_{i,j}(e(u)) e_{i,j}(\varphi) - f_i \varphi_i \right\} dxdt = \int_\Omega u_{0_i} \varphi_i(0) dx,$$

and the energy inequality: a.e. $t \in (0,T)$,

$$\frac{1}{2} \int_\Omega |u|^2 dx + \int_0^t \int_\Omega \tau^p_{i,j}(e(u)) e_{i,j}(u) dxds \leq \frac{1}{2} \int_\Omega |u_0|^2 dx + \int_0^t \int_\Omega f \cdot u dxds. \tag{1}$$

If $u : (0,+\infty) \times \Omega \longrightarrow \mathbb{R}^d$ and verifies the previous conditions for all $T > 0$, it will be said that u is a weak solution of $(NS)^p_{per}$ in $(0,+\infty)$.

Remark: *In the case of newtonian viscosity* ($\mu_\infty > 0$)*, we will also have* $u \in L^2(0,T;V_2)$.

Definition 2.2 (Strong solution) *Given* $u_0 \in V_p \bigcap V_2$ *and* $u : (0,T)\times\Omega \longrightarrow \mathbb{R}^d$ *a weak solution of* $(NS)^p_{per}$ *in* $(0,T)$*, we say that* u *is a strong solution of* $(NS)^p_{per}$ *in* $(0,T)$ *if moreover:*

$$i)\, u \in L^\infty(0,T;V_2),\ ii)\, u \in L^\infty(0,T;V_p),\ \frac{\partial u}{\partial t} \in L^2(0,T;H),\ \text{and } iii)\, u \in L^2(0,T;H^2(\Omega)).$$

In order to obtain $i)$ it will be necessary $u_0 \in V_2$, and for $ii)$ we will use $u_0 \in V_p$.

2.1 Existence of global solution in time.

We are going to focus on the threedimensional case ($d = 3$).

Theorem 2.3 (Carreau's laws without newtonian viscosity)

a) Let $u_0 \in H$ *and* $f \in \begin{cases} L^{p'}((0,T)\times\Omega)^3 & \text{if } p < 2, \\ L^2((0,T)\times\Omega)^3 & \text{if } p > 2. \end{cases}$ *If* $p > 9/5$*, then there exists a weak solution of* $(NS)^p_{per}$ *in* $(0,T)$.

b) Let $u_0 \in V_p$ *and* $f \in L^2((0,T)\times\Omega)^3$*. If* $p \geq 11/5$*, then there exists a strong solution of* $(NS)^p_{per}$ *in* $(0,T)$.

Remark: *For a power law without newtonian viscosity (and* $1 < p < 2$*), the part a) of the above result is also true.*

Corollary 2.4 (Case with newtonian viscosity)

a) Let $u_0 \in H$ *and* $f \in L^2((0,T)\times\Omega)^3$*. If* $p > 1$*, then there exists a weak solution of* $(NS)^p_{per}$ *in* $(0,T)$.

b) Let $u_0 \in V_p$ *and* $f \in L^2((0,T)\times\Omega)^3$*. If* $p \geq 11/5$*, then there exists a strong solution of* $(NS)^p_{per}$ *in* $(0,T)$.

The proofs of these results are based on the construction of approximated solutions via a Galerkin method. After estimating these solutions on appropiate spaces, a limit process by compactness will give the desired solution (see [2], [9]).

In view of the previous results, it is reasonable to ask, when $p \in (9/5, 11/5)$, about the possible existence of "singular times" (where a weak solution blows up in a stronger norm although their weak regularity is preserved). That is not possible when $d = 2$, because for two dimensional domains, one has the existence of a global strong solution for all $p > 1$. This is the reason we are going to restrict ourselves to the threedimensional case.

Remark: *In the case of homogeneous Dirichlet boundary conditions, the existence of a global weak solution is only known for* $p \geq 2$ *and the existence of a global strong solution for* $p > 20/9$ *(see [10]).*

2.2 Continuous dependence and uniqueness of weak/strong solution.

In this subsection, we are going to assume the existence of two solutions of $(NS)^p_{per}$ in $(0,T)$, u and v, where u is a strong solution with data $u_0 \in V_p \bigcap V_2$ and $f \in L^2((0,T) \times \Omega)^3$, and v is a weak solution with data $v_0 \in H$ and $g \in L^2((0,T) \times \Omega)^3$. We will see sufficient conditions to obtain continuous dependence and uniqueness results.

Theorem 2.5 (Carreau's laws and $p \geq 2$) *Under the above conditions,*

$$\|u - v\|^2_{L^\infty(0,T;L^2)} + \|u - v\|^p_{L^p(0,T;V_p)} \leq C \left\{ \|u_0 - v_0\|_2^2 + \|f - g\|^2_{L^2(0,T;L^2)} \right\}.$$

for some $C = C(T, \|u\|_{L^\infty(V_2)}) > 0$*. In particular, if* $u_0 \equiv v_0$ *and* $f \equiv g$*, one has the uniqueness of weak solutions assuming the existence of a strong solution.*

Corollary 2.6 (Case with newtonian viscosity and $p > 1$)

There exists $C = C(T, \|u\|_{L^\infty(V_2)}) > 0$ *such that:*

$$\|u - v\|^2_{L^\infty(0,T;L^2)} + \|u - v\|^2_{L^2(0,T;V_2)} \leq C \left\{ \|u_0 - v_0\|_2^2 + \|f - g\|^2_{L^2(0,T;L^2)} \right\}.$$

In particular, one has the uniqueness of solution in the same way of Theorem 2.5.

Uniqueness questions can also be seen in [5], [6], [7] and [8]. Here, we are also interested in the continuous dependence because of it will be used below.

Proof of Theorem 2.5**:** Thanks to the regularity of u, we can take u as a test function in the weak formulation of v, then a.e. $t \in (0,T)$:

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega v \cdot u dx + \int_0^t \int_\Omega \tau^p_{i,j}(e(v)) e_{i,j}(u) dx ds = \int_\Omega v_0 \cdot u(0) dx \\ \displaystyle + \int_0^t \int_\Omega g \cdot u dx ds + \int_0^t \int_\Omega ((v \cdot \nabla) u + \partial_t u) \cdot v dx ds. \end{array} \right. \tag{2}$$

Also, we can multiply the differential problem in u by v and integrate on $\Omega \times (0,T)$,

$$\left\{ \begin{array}{l} \displaystyle\int_0^t \int_\Omega \left(\frac{\partial u}{\partial t} + (v \cdot \nabla) u \right) \cdot v dx ds + \int_0^t \int_\Omega \tau^p_{i,j}(e(u)) e_{i,j}(v) dx ds \\ \displaystyle\qquad = \int_0^t \int_\Omega f \cdot v dx ds + \int_0^t \int_\Omega [(v-u) \cdot \nabla u] \cdot v dx ds. \end{array} \right. \tag{3}$$

Finally, u verifies the energy equality:

$$\frac{1}{2} \int_\Omega |u|^2 dx + \int_0^t \int_\Omega \tau^p_{i,j}(e(u)) e_{i,j}(u) dx ds = \int_0^t \int_\Omega f \cdot u dx ds + \frac{1}{2} \int_\Omega |u_0|^2 dx. \tag{4}$$

Adding (2) and (3), the terms $\displaystyle\int_0^t \int_\Omega \{[(v \cdot \nabla) u] \cdot v + \partial_t u \cdot v\} dx ds$ are cancelled. Then, a.e. $t \in (0,T)$:

$$\begin{array}{l} \displaystyle\int_\Omega v \cdot u dx + \int_0^t \int_\Omega \left[\tau^p_{i,j}(e(v)) e_{i,j}(u) + \tau^p_{i,j}(e(u)) e_{i,j}(v)\right] dx ds \\ \displaystyle = \int_\Omega v_0 \cdot u_0 dx + \int_0^t \int_\Omega (g \cdot u + f \cdot v) dx ds + \int_0^t \int_\Omega [((v-u) \cdot \nabla) u] \cdot v dx ds. \end{array} \tag{5}$$

Now, adding the energy inequality for v and (4), and subtracting (5), we obtain for $w = u - v$:

$$\begin{array}{l} \displaystyle\frac{1}{2} \|w(t)\|_2^2 + \int_0^t \int_\Omega [\tau^p_{i,j}(e(u)) - \tau^p_{i,j}(e(v))] e_{i,j}(w) dx ds \\ \displaystyle\le \frac{1}{2} \|u_0 - v_0\|_2^2 - \int_0^t \int_\Omega w_j \frac{\partial u_i}{\partial x_j} w_i dx ds + \int_0^t \int_\Omega (f-g)_i w_i dx ds. \end{array} \tag{6}$$

Since τ^p is a Carreau's laws with $p \ge 2$, we have ([8]):

$$[\tau^p_{i,j}(e(u)) - \tau^p_{i,j}(e(v))] e_{i,j}(u-v) \ge C_3 \left\{ |e(u-v)|^2 + |e(u-v)|^p \right\} \tag{7}$$

Therefore, if we use the Korn inequality: $\forall p > 1$, $\exists K_p > 0$ such that $\displaystyle\int_\Omega |e(w)|^p dx \ge K_p^p \|\nabla w\|_p^p$, we are able to bound lowerly the left hand side of (6) by

$$\frac{1}{2} \|w(t)\|_2^2 + C_3 \int_0^t \left\{ K_2^2 \|\nabla w(s)\|_2^2 + K_p^p \|\nabla w(s)\|_p^p \right\} ds.$$

On the other hand, we bound the last terms of the right hand side of (6) by:

$$\left| -\int_\Omega w_j \frac{\partial u_i}{\partial x_j} w_i dx \right| \le \int_\Omega |w|^2 |\nabla u| dx \le \|\nabla u\|_2 \|w\|_4^2 \le C_4 \|\nabla u\|_2 \|w\|_2^{1/2} \|\nabla w\|_2^{3/2}, \tag{8}$$

(using in the last bound the interpolation inequality $\|w\|_4 \le \|w\|_2^{1/4} \|w\|_6^{3/4}$ joint to the Sobolev embedding of H^1 in L^6, with constant C_4), and

$$\left| \int_\Omega (f_i - g_i) w_i dx \right| \le \|f-g\|_2 \|w\|_2 \le C_4' \|f-g\|_2 \|\nabla w\|_2 \tag{9}$$

(in the last bound, the Poincaré inequality with constant C_4' has been used). Using now the Young inequality (with exponent (4, 4/3) and (2, 2) respectively) in the two previous inequalities, we obtain:

$$\left| \int_\Omega w_j \frac{\partial u_i}{\partial x_j} w_i dx \right| \le C \|\nabla u\|_2^4 \|w\|_2^2 + \frac{K_2^2 C_3}{4} \|\nabla w\|_2^2, \tag{10}$$

$$\left| \int_\Omega (f_i - g_i) w_i dx \right| \le C' \|f-g\|_2^2 + \frac{K_2^2 C_3}{4} \|\nabla w\|_2^2. \tag{11}$$

According to all the previous estimations, we obtain:

$$\begin{aligned} &\|w(t)\|_2^2 + K_2^2 C_3 \int_0^t \|\nabla w(s)\|_2^2 ds + 2 C_3 K_p^p \int_0^t \|\nabla w(s)\|_p^p ds \\ &\le \|u_0 - v_0\|_2^2 + C \int_0^t \|\nabla u(s)\|_2^4 \|w(s)\|_2^2 ds + C' \|f-g\|_{L^2(L^2)}^2. \end{aligned} \tag{12}$$

Since u is a strong solution of $(NS)_{per}^p$, $u \in L^\infty(0,T;V_2)$, then, in particular, $\|\nabla u\|_2^4 \in L^1(0,T)$, and, therefore, we may use Gronwall lemma to finish the proof of Theorem. ∎

Proof of Corollary 2.6**:** It is similar to Theorem 2.5. In this case, the difference is the lower bound for the tensor in (7), which is $\mu_\infty \|\nabla w_j(t)\|_2^2$ instead of $C_3 \left\{ K_2^2 \|e(w_j(t))\|_2^2 + K_p^p \|e(w_j(t))\|_p^p \right\}$. This expression arise from the newtonian part of the tensor, because of the purely non-Newtonian part only verifies:

$$\int_\Omega \left\{ \tau_{k,l}^p(e(z_j)) - \tau_{k,l}^p(e(v)) \right\} e_{k,l}(w_j) dx \ge 0. \tag{13}$$

Thus, one only obtains $\|\nabla w\|_2^2$ in (12), hence the continuous dependence in $L^2(V_2)$ is deduced (instead of $L^p(V_p)$). ∎

Remark: (Case without newtonian viscosity and p > 1) *If we assume that the strong solution* *u verifies the additional hypothesis* $\nabla u \in L^1(0,T;L^\infty(\Omega)^3)$*, changing the bound of estimate* (8) *by* $C_4\|\nabla u\|_{L^\infty(\Omega)}\|w\|_2^2$ *we can conclude uniqueness for both laws.*

Remark: (**Power law with newtonian viscosity or Carreau's laws**) *If* $p \geq 5/2$*, one has the uniqueness of weak solutions of* $(NS)^p_{per}$ *in* $(0,T)$ *(see [5], [6]).*

Remark: *All the results of this subsection can be easily extended to the case of Dirichlet boundary conditions.*

2.3 Existence of local strong solution.

Theorem 2.7 (Carreau's laws without newtonian viscosity) *Let* $u_0 \in V_p \cap V_2$ *and* $f \in \begin{cases} L^q(0,T;L^{p'}(\Omega)^3), & \text{with } q > p' \quad (\text{if} \quad p < 2), \\ L^q(0,T;L^2(\Omega)^3), & \text{with } q > 2 \quad (\text{if} \quad p \geq 2). \end{cases}$ *If* $p > 5/3$*, then there exists* $T^* \in (0,T]$ *and a strong solution of* $(NS)^p_{per}$ *in* $(0,T^*)$ *(when* $p < 2$ *the strong solution obtained satisfies* $u \in L^2(0,T;W^{2,p}(\Omega))$ *instead of* $u \in L^2(0,T;H^2(\Omega))$*).*

Corollary 2.8 (Case with newtonian viscosity) *Let* $u_0 \in V_p \cap V_2$ *and* $f \in L^q(0,T;L^2(\Omega))^3$ *with* $q > 2$*. If* $p > 1$*, then there exists* $T^* \in (0,T]$ *and a strong solution of* $(NS)^p_{per}$ *in* $(0,T^*)$*.*

Proof of Theorem 2.7**:** We are going to follow the argument of [8]. Moreover, here we generalize the hypothesis on the regularity of f imposed in [8], where $f \in \begin{cases} L^\infty(0,T;L^{p'}(\Omega)^3), & \text{if} \quad p < 2, \\ L^\infty(0,T;L^2(\Omega)^3), & \text{if} \quad p \geq 2, \end{cases}$ is assumed. For $p \geq 11/5$, the result is obvious taking $T^* = T$ (Theorem 2.3). Therefore, let us suppose $5/3 < p < 11/5$. We divide the proof in two steps:

Step 1: Any weak solution u of $(NS)^p_{per}$ in $(0,T)$, obtained as in Theorem 2.3, such that verifies the additional regularity $u \in L^\infty(0,T;V_2)$, is also a strong solution of $(NS)^p_{per}$ in $(0,T)$ (i.e., the regularity conditions of definition 2.2 are verified).

Step 2: There exists $T^* \in (0,T]$ and a weak solution u of $(NS)^p_{per}$ in $(0,T^*)$, obtained as in Theorem 2.3, such that $u \in L^\infty(0,T^*;V_2)$.

We are interested in separating the proof in these two steps in order to remark the main difference between a weak and a strong solution: the $L^\infty(0,T;V_2)$ regularity. This will be an essential fact to define the singular (or blows up) times of a weak solution.

We are going to develop these two steps:

Step 1: Since $u \in L^\infty(0,T;V_2)$, it is sufficient to proof the regularity conditions $ii)$ and $iii)$ of a strong solution. To get it in a rigorous form, one can use the sequence of approximated solutions furnished by Galerkin method (choosing Stokes eigenfunctions with periodic boundary conditions as basis functions) and estimating them in the spaces of definition of a strong solution. For sake of simplicity, in order to demonstrate how one can get these estimations, we argue in a formal way on the weak solution u given in the hypothesis. First of all, since u is a weak solution, then

$$\|u\|_{L^\infty(0,T;H)} < +\infty, \quad \|u\|_{L^p(0,T;V_p)} < +\infty, \tag{14}$$

and, moreover, we assume the hypothesis $\|u\|_{L^\infty(0,T;V_2)} < +\infty$.

Taking the laplacian of u as a test function (that is possible due to the periodic conditions), integrating by parts and applying $(H3)_2$, one obtains ([8]):

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + C_1 I_p(u) \leq \|\nabla u\|_3^3 + \int_\Omega f \cdot \Delta u \, dx \tag{15}$$

where $I_p(u) = \displaystyle\int_\Omega (1+|e(u)|)^{p-2}\,|\nabla(e(u))|^2 dx$. We bound, $\|\nabla u\|_3^3 \leq C\|\nabla u\|_2^\alpha \|\nabla u\|_p^\beta \|\nabla u\|_{3p}^\gamma$ for α, β, $\gamma > 0$ such that $\alpha+\beta+\gamma = 3$ and $\dfrac{\alpha}{2}+\dfrac{\beta}{p}+\dfrac{\gamma}{3p} = 1$. Using the following property of $I_p(u)$ (see Lemma 3.24 in [8]): $\forall p > 1$,

$$\|\nabla u\|_{3p} \leq C\, I_p(u)^{1/p} \tag{16}$$

and applying an appropiate Young inequality, one has:

$$\|\nabla u\|_3^3 \leq \varepsilon I_p(u) + C_\varepsilon \|\nabla u\|_2^{\alpha p/(p-\gamma)} \|\nabla u\|_p^{\beta p/(p-\gamma)} \tag{17}$$

Now, choosing $\beta p/(p-\gamma) = p$,

$$\|\nabla u\|_3^3 \leq \varepsilon I_p(u) + C_\varepsilon \left(\|\nabla u\|_2^2\right)^\lambda \|\nabla u\|_p^p \tag{18}$$

where $\lambda = 2(3-p)/(3p-5) > 0$ (here it is used that $p > 5/3$). On the other hand, taking into account the following property of $I_p(u)$ (see [8]):

$$\begin{cases} \|D^2u\|_p \leq C\, I_p(u)^{1/2}\,(1+\|\nabla u\|_p)^{(2-p)/2} & \text{(if } p<2\text{)}, \\ \|D^2u\|_2 \leq C\, I_p(u)^{1/2} & \text{(if } p\geq 2\text{)}, \end{cases} \tag{19}$$

we obtain:

$$\int_\Omega f\cdot\Delta u\,dx \leq \begin{cases} \|f\|_{p'}\|D^2u\|_p \leq \varepsilon I_p(u) + C_\varepsilon\|f\|_{p'}^2(1+\|\nabla u\|_p)^{2-p} & \text{(if } p<2\text{)}, \\ \|f\|_2\|D^2u\|_2 \leq \varepsilon I_p(u) + C_\varepsilon\|f\|_2^2 & \text{(if } p\geq 2\text{)}. \end{cases} \tag{20}$$

Finally, if we replace (18) and (20) (for ε arbitrarily small), in (15), we obtain (omitting the constants):

$$\frac{d}{dt}\|\nabla u\|_2^2 + I_p(u) \leq \|\nabla u\|_2^{2\lambda}\|\nabla u\|_p^p + \begin{cases} \|f\|_{p'}^2(1+\|\nabla u\|_p)^{2-p} & \text{(if } p<2\text{)}, \\ \|f\|_2^2 & \text{(if } p\geq 2\text{)}. \end{cases} \tag{21}$$

Then, integrating between 0 and T,

$$\begin{aligned} \int_0^T I_p(u)dt \quad &\leq \quad \|\nabla u_0\|_2^2 + \|u\|_{L^\infty(0,T;V_2)}^{2\lambda}\int_0^T \|\nabla u\|_p^p dt \\ &+ \quad \begin{cases} \displaystyle\int_0^T \|f\|_{p'}^2(1+\|\nabla u\|_p)^{2-p}dt & \text{(if } p<2\text{)}, \\ \displaystyle\int_0^T \|f\|_2^2 dt & \text{(if } p\geq 2\text{)}. \end{cases} \end{aligned} \tag{22}$$

Taking into account regularity of u_0 and f, (14) and $u \in L^\infty(0,T;V_2)$, the right hand side of (22) is bounded. Therefore,

$$\int_0^T I_p(u)dt < +\infty. \tag{23}$$

On the other hand, considering $\dfrac{\partial u}{\partial t}$ as a test function:

$$\left\|\frac{\partial u}{\partial t}\right\|_2^2 + \int_\Omega \tau_{i,j}^p(e(u))\frac{\partial}{\partial t}e_{i,j}(u)dx = -\int_\Omega u_j\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial t}dx + \int_\Omega f_i\frac{\partial u_i}{\partial t}dx.$$

Using $(H1)$ and estimating the tensor term, one has:

$$\frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|_2^2 + \frac{d}{dt}\int_\Omega \mathcal{U}^p(e(u))dx \leq \|f\|_2^2 + I(u,\nabla u), \tag{24}$$

where $I(u,\nabla u)=\displaystyle\int_\Omega |u|^2|\nabla u|^2 dx$. Let now us to bound $I(u,\nabla u)$:

Case 1: If $p\geq 2$. Using (16), one has:

$$I(u,\nabla u)\leq \|u\|^2_{6p/(3p-2)}\|\nabla u\|^2_{3p}\leq C\|u\|^2_{V_2}\|\nabla u\|^2_{3p}\leq C\|u\|^2_{L^\infty(0,T;V_2)}I_p(u)^{2/p}. \tag{25}$$

Case 2: If $p\in(5/3,2)$. Now, we bound by:

$$I(u,\nabla u)=\int_\Omega |u|^2|\nabla u|^{2-p}|\nabla u|^p dx\leq \|u\|_6^2\|\nabla u\|^{2-p}_{3(2-p)}\|\nabla u\|^p_{3p} \tag{26}$$

As $3(2-p)\leq 2$ if only if $p\geq 4/3$ (it is true due $p>5/3$), the previous bound becomes:

$$I(u,\nabla u)\leq C\|u\|^{4-p}_{L^\infty(0,T;V_2)}I_p(u). \tag{27}$$

In both cases, (23) imply that the second term of (24) belong to $L^1(0,T)$. On the other hand, from $u_0\in V_p$ and the property (see Lemma 1.35 in [8]):

$$|\mathcal{U}^p(\eta)|\leq C\,(1+|\eta|)^p\,,\quad \forall\eta\in\mathbb{R}^{d\times d},$$

one has that $\displaystyle\int_\Omega \mathcal{U}^p(e(u_0))dx<+\infty$. Therefore, integrating (24) respect to time,

$$\int_0^t\left\|\frac{\partial u}{\partial t}\right\|_2^2 ds+\int_\Omega \mathcal{U}^p(e(u(t)))dx\leq C_T,\quad \forall t\in(0,T].$$

Now, using the property (see lemma 1.35 in [8]): $\displaystyle\int_\Omega \mathcal{U}^p(e(u))dx\geq C\left\{\|e(u)\|_p^p-|\Omega|\right\}$, we can deduce that $\dfrac{\partial u}{\partial t}\in L^2(0,T;H)$ and $u\in L^\infty(0,T;V_p)$, i.e., the regularity *ii)* of a strong solution. Finally, from (19), (23) and $u\in L^\infty(0,T;V_p)$, we get $u\in\begin{cases}L^2(0,T;W^{2,p}) & \text{if } p<2,\\ L^2(0,T;H^2) & \text{if } p\geq 2,\end{cases}$ and we finish step 1.

Step 2: We start from (15). But now, we choose in (17) $\dfrac{\beta p}{p-\gamma}=\dfrac{p}{1+\varepsilon}$, for $\varepsilon>0$, which leads us to the following inequality (instead of (21)):

$$\frac{d}{dt}\|\nabla u\|_2^2+I_p(u)\leq C(\|\nabla u\|_2^2)^{\lambda_\varepsilon}\|\nabla u\|_p^{p/(1+\varepsilon)}+\begin{cases}\|f\|^2_{p'}(1+\|\nabla u\|_p)^{2-p} & (\text{if } p<2),\\ \|f\|_2^2 & (\text{if } p\geq 2)\end{cases} \tag{28}$$

where $\lambda_\varepsilon = \dfrac{2(3-p)}{3p-5} + \dfrac{(5p-9)\varepsilon}{(3p-5)(1+\varepsilon)}$. In our case, as $p < \dfrac{11}{5}$, then $\lambda_\varepsilon > 1$. Dividing (28) by $(1+\|\nabla u\|_2^2)^{\lambda_\varepsilon}$ and integrating in $(0,t)$, $t \in [0,T]$, we have:

$$\frac{1}{\lambda_\varepsilon - 1}\frac{1}{(1+\|\nabla u_0\|_2^2)^{\lambda_\varepsilon-1}} + \int_0^t \frac{I_p(u(s))}{(1+\|\nabla u(s)\|_2^2)^{\lambda_\varepsilon}}ds$$
$$\le \|u\|_{L^p(0,T;V_p)}^{p/(1+\varepsilon)}\, t^{\varepsilon/(1+\varepsilon)} + C(f)\, t^a + \frac{1}{\lambda_\varepsilon - 1}\frac{1}{(1+\|\nabla u(t)\|_2^2)^{\lambda_\varepsilon-1}},$$

where

$$a = \begin{cases} 2\,(1/p' - 1/q) & \text{if } p < 2, \\ 2\,(1/2 - 1/q) & \text{if } p \ge 2, \end{cases} \quad \text{and} \quad C(f) = \begin{cases} \|f\|^2_{L^q(0,T;L^{p'})} & \text{if } p < 2, \\ \|f\|^2_{L^q(0,T;L^2)} & \text{if } p \ge 2. \end{cases}$$

Thus, $\|\nabla u(t)\|_2^2 \le C, \forall t \in [0, T_*]$, for $T_* \in (0,T]$ small enough, such that:

$$\|u\|_{L^p(V_p)}^{p/(1+\varepsilon)}\, {T_*}^{\varepsilon/(1+\varepsilon)} + C(f)\, {T_*}^a < \frac{1}{(\lambda_\varepsilon - 1)}\frac{1}{(1+\|\nabla u_0\|_2^2)^{\lambda_\varepsilon-1}}. \tag{29}$$

The proof of the Theorem 2.7 is finished. ∎

Remark: *From (29), we have that T_* depends on $C(f)$ and $\|u_0\|_{V_2}$ in a decreasing way, because of $\|u\|_{L^p(V_p)}$ depends on $C(f)$ and $\|u_0\|_{V_2}$ in a increasing way. Moreover, it is possible to obtain $T^* = T$ if $C(f)$ and $\|u_0\|_{V_2}$ are small enough (this result has been considered in [12]).*

Outline of the proof of Corollary 2.8**:** In the power law case, the definition of I_p has a slightly different form:

$$I_p(u) = \int_\Omega |e(u)|^{p-2}|\nabla(e(u))|^2 dx.$$

This $I_p(u)$ verifies $(19)_1$ and (16), but not $(19)_2$. This difficulty can be circumvented thanks to the newtonian viscosity, since $\mu_\infty\|\Delta u\|_2^2$ must be added to the left hand side of (15). In this case, the bound for the term $\|\nabla u\|_3^3$ of (15) is $\|\nabla u\|_3^3 \le \varepsilon\|u\|_{H^2}^2 + C_\varepsilon\|\nabla u\|_2^6$, hence we get $u \in L^2(0,T;H^2)$. Finally, to obtain $u \in L^\infty(0,T;V_p)$ and $\dfrac{\partial u}{\partial t} \in L^2(0,T;H)$, we can use the bound $I(u,\nabla u) \le C\|u\|_3^3\|u\|_{H^2}^2 \in L^1(0,T)$.

3 Blow up at finite time if a solution blows up at infinite time.

We study two cases: a) $2 \leq p < 11/5$ and Carreau's laws, b) $1 < p < 11/5$ and newtonian viscosity (power law or Carreau's laws). As in the previous section, in both cases we have: existence of global weak solution, uniqueness of strong/weak solution and existence of local strong solution.

The main results of this section are the following:

Theorem 3.1 ($\mathbf{2 \leq p < 11/5}$ and Carreau's laws) *Assume $f(t) = f \in L^2(\Omega)^3, \forall t \geq 0$ (f is independent of t). Assume there exists a strong solution u of $(NS)^p_{per}$ in $(0, +\infty)$, such that:*

$$\lim_{t \to +\infty} \sup \|u(t)\|_{V_p} = +\infty$$

Then, for any $T_1 > 0$, there exists $v_0 \in V_p$ such that the local strong solution of $(NS)^p_{per}$ with initial data v_0 blows up in the $L^\infty(V_2)$-norm before T_1, i.e., u is not strong solution in $(0, T_1)$.

Corollary 3.2 ($\mathbf{1 < p < 11/5}$ and newtonian viscosity) *Assume $f(t) = f \in L^2(\Omega)^3, \forall t \geq 0$. Assume there exists a strong solution u of $(NS)^p_{per}$ in $(0, +\infty)$, such that:*

$$\lim_{t \to +\infty} \sup \|u(t)\|_{V_2} = +\infty$$

Then, for any $T_1 > 0$, there exists $v_0 \in V_2$ such that the local strong solution of $(NS)^p_{per}$ with initial data v_0 blows up in the $L^\infty(V_2)$-norm before $T_1 > 0$, i.e., v is not strong solution in $(0, T_1)$.

For the proof of these results, it will be necessary the following technical lemma:

Lemma 3.3 *Assume one of the two above cases. Let u be a weak solution of $(NS)^p_{per}$ in $(0, +\infty)$. If $f \in L^\infty(0, +\infty; L^2(\Omega)^3)$, then, for each $\tau > 0$, there exists a constant $C = C(f, u_0, \tau) > 0$ such that in all interval of length τ, $[t, t + \tau]$, there exists $t_0 \in [t, t + \tau]$ such that:*

$$\|u(t_0)\|^\sigma_{V_\sigma} \leq C(f, u_0, \tau), \tag{30}$$

where $\sigma = \max\{p, 2\}$.

Remark: *The previous bound depends on the size of the interval (τ), but it is independent of the time position (t).*

Proof of Lemma 3.3: We consider two cases:

Case i) : $p \geq 2$ and $\mu_\infty = 0$. From the energy inequality,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + C_5\|u(t)\|_{V_p}^p \leq \|f\|_{L^\infty(L^2)}\|u(t)\|_2 \tag{31}$$

Using the Sobolev's embedding $V_p \hookrightarrow L^2$, one has

$$\frac{d}{dt}\|u(t)\|_2^2 + K\left(\|u(t)\|_2^2\right)^{p/2} \leq C_6\|f\|_{L^\infty(L^2)}^{p'} \tag{32}$$

Let $M = M(f, u_0) > 0$ be a large enough number, such that $\|u_0\|_2^2 \leq M$ and $C_6\|f\|_{L^\infty(L^2)}^{p'} < KM^{p/2}$. Then, one has $\|u(t)\|_2^2 \leq M$, $\forall t \geq 0$. Indeed, if we suppose the opposite, let $t' > 0$ be the first time such that $\|u(t')\|_2^2 = M$ and $\|u(t)\|_2^2 > M$, $\forall t > t'$ (near t'). Then, from (32), $\frac{d}{dt}\|u(t')\|_2^2 < 0$, so this norm decreases in t', and thus $\|u(t)\|_2^2 \leq M$, $\forall t > t'$ (near t'). This is in contradiction with the definition of t'.

Now, integrating (31) in $[t, t+\tau]$, we get:

$$C_5\int_t^{t+\tau}\|u(s)\|_{V_p}^p ds \leq \frac{M}{2} + C_6\tau\|f\|_{L^\infty(L^2)}M^{1/2}$$

Defining $C = C(f, u_0, \tau)$ such that

$$\frac{1}{C_5}\left(\frac{M}{2} + \tau C_6\|f\|_{L^\infty(L^2)}M^{1/2}\right) = C(f, u_0, \tau)\frac{\tau}{2}$$

and denoting by λ the Lebesgue's measure on $\mathbb{R}$, from the previous inequality we have:

$$\lambda\left(\left\{s \in [t, t+\tau] \text{ such that } \|u(s)\|_{V_p}^p \geq \rho\right\}\right) \leq \frac{\tau}{2}c(f, u_0, \tau)\rho^{-1}.$$

Taking $\rho = C(f, u_0, \tau)$,

$$\lambda\left(\left\{s \in [t, t+\tau] \text{ such that } \|u(s)\|_{V_p}^p \geq C(f, u_0, \tau)\right\}\right) \leq \frac{\tau}{2} < \lambda([t, t+\tau])$$

and, therefore (30) holds.

Case ii) : $\mu_\infty > 0$. Now, using the V_2-norm to controll the right hand side of the energy inequality (instead of the V_p-norm), we obtain:

$$\frac{d}{dt}\|u(t)\|_2^2 + \mu_\infty K\|u(t)\|_2^2 + C_5\|\nabla u(t)\|_p^p \le C_6\|f\|_{L^\infty(L^2)}^2 \tag{33}$$

Thus, we can argue like in the case **i)**, defining this time $C = C(f, u_0, \tau)$ such that:

$$\int_t^{t+\tau} \|u(s)\|_{V_\sigma}^\sigma ds \le C(f, u_0, \tau)\frac{\tau}{2}. \qquad \blacksquare$$

Remark: *Lemma* 3.3 *is also true in more general cases. For example, for* $p \ge 6/5$ *and* $\mu_\infty = 0$, *if we always consider the* V_p*-norm.*

Proof of Theorem 3.1**:**

Let $T_1 > 0$ and $t_j \to +\infty$ such that $\lim_{j\to+\infty}\|u(t_j)\|_{V_p} = +\infty$. Accordingly Lemma 3.3 (now $\sigma = p$), we may find a time $a_j \in [t_j - T_1, t_j], \forall j \ge 1$ such that:

$$\|u(a_j)\|_{V_p}^p \le C(f, u_0, T_1) \tag{34}$$

where c is independent of j. ¿From the compact embedding of V_p in H and (34), there exists $v_0 \in V_p$ and a subsequence of $\{u(a_j)\}_{j\ge1}$ (that we will denote as the sequence) such that $u(a_j) \longrightarrow v_0$ weakly in V_p and strongly in H. Now, we consider the strong solutions of $(NS)_{per}^p$ (with second member f):

$$z_j(s) = u(a_j + s) \text{ : solution in } (0, +\infty), \text{ with initial data } u(a_j),$$

$$v(s) \text{ : solution in } (0, T^*), \text{ with initial data } v_0 \in V_p$$

($T^* = T^*\left(\|v_0\|_{V_2}, \|f\|_{L^2}\right) > 0$, see Theorem 2.7).

To finish the proof, we will see that v is not a strong solution in $[0, T_1]$. Arguing by contradiction, let us suppose that $T^* \ge T_1$. Since z_j and v are solutions of $(NS)_{per}^p$ associated to the same f, the sequence difference $w_j(t) = z_j(t) - v(t), \quad \forall t \in [0, T_1]$, satisfies:

$$\frac{1}{2}\frac{d}{dt}\|w_j(t)\|_2^2 + \int_\Omega \left[\tau_{k,l}(e(z_j)) - \tau_{k,l}(e(v))\right] e_{k,l}(w_j)dx = \int_\Omega (w_j \cdot \nabla v)\, w_j dx. \tag{35}$$

Following the continuous dependence argument of Theorem 2.5, we obtain:

$$\frac{d}{dt}\|w_j(t)\|_2^2 + C_3\left\{K_2^2\|\nabla w_j(t)\|_2^2 + K_p^p\|\nabla w_j(t)\|_p^p\right\} \le C\|\nabla v\|_2^4\|w_j\|_2^2. \tag{36}$$

Since $v \in L^\infty(0,T_1;V_2)$, we have $\int_0^t \|\nabla v(s)\|_2^4 < Ct \le CT_1 < +\infty$. Therefore, applying Gronwall's lemma to (36), one has for all $t \in [0,T_1]$,

$$\|w_j(t)\|_2^2 \le \|w_j(0)\|_2^2 \, exp\left(C\int_0^t \|\nabla v(s)\|_2^4 ds\right) \le \|w_j(0)\|_2^2\, e^{CT_1}. \tag{37}$$

In particular, since $w_j(0) \to 0$ in H then $\|w_j(t)\|_2^2 \longrightarrow 0$ as $j \longrightarrow +\infty$. Now, if we integrate (36) in $(0,T_1)$,

$$C_3\left\{K_2^2\int_0^{T_1}\|w_j(s)\|_{V_2}^2 ds + K_p^p\int_0^{T_1}\|w_j(s)\|_{V_p}^p ds\right\} \le C\int_0^{T_1}\|\nabla v(s)\|_2^4\|w_j(s)\|_2^2 ds + \|w_j(0)\|_2^2.$$

Thus, from the Lebesgue's Dominated Convergence theorem, we have $\int_0^{T_1}\|w_j(s)\|_{V_p}^p ds \longrightarrow 0$, as $j \to +\infty$, and, in particular, $\|w_j(t)\|_{V_p}^p \longrightarrow 0$ a.e. $t \in [0,T_1]$, as $j \to +\infty$. Therefore, if we define

$$J = \left\{t \in [0,T_1] : \lim_{j\to+\infty}\|w_j(t)\|_{V_p} = 0\right\},$$

then the set $[0,T_1]/J$ has Lebesgue's measure zero and moreover, for a fixed $t \in J$, there exists $j_0 = j_0(t)$ such that $\forall j \ge j_0$, $\|w_j(t)\|_{V_p} \le 1$. On the other hand, since $v \in L^\infty(0,T_1;V_p) \cap C_\omega([0,T_1];V_p)$ (i.e. $t \in [0,T_1] \to \langle v(j),h\rangle \in \mathbb{R}$ is continuous for all $h \in V_p'$, see [13]), we have $\|v(t)\|_{V_p} \le \|v\|_{L^\infty(0,T_1;V_p)} \equiv r$ $(\forall t \in [0,T_1])$. Thus,

$$\|z_j(t)\|_{V_2} \le C\|z_j(t)\|_{V_p} \le C\left\{\|w(t)\|_{V_p} + \|v(t)\|_{V_p}\right\} \le C\{1+r\}.$$

For $t=0$, we have $z_j(0) = u(a_j)$, and due to (34), we get $\|z_j(0)\|_{V_2} \le C$, $\forall j$. ¿From Theorem 2.7, there exists $T_2 = T_2(r,f)$, independent of j, such that: $\|z_j\|_{L^\infty(t,t+T_2;V_2)} \le C \quad \forall t \in J\cup\{0\}, \quad \forall j \ge j_0(t)$. Moreover, if we follow the proof of Theorem 2.7 (Step 1), we also obtain:

$$\|z_j\|_{L^\infty(t,t+T_2;V_p)} \le C \quad \forall t \in J\cup\{0\}, \quad \forall j \ge j_0(t). \tag{38}$$

Then, as we know that $z_j \in C_w([0,+\infty];V_p)$, one has $\|z_j(s)\|_{V_p} \leq C$, $\forall s \in [t,t+T_2]$. Hence, choosing a finite number of $t_i \in J \cup \{0\}$, $i = 1,2,...,m$, such that: $[0,T_1] \subset \bigcup_{i=1}^{m}[t_i, t_i+T_2]$, and considering $j_1 = \max_{i=1,...,m}\{j_0(t_i)\}$, one has that $\|z_j(t)\|_{V_p} \leq C$, $\forall t \in [0,T_1]$ and $\forall j \geq j_1$. But this is a contradiction because $\|z_j(t_j - a_j)\|_{V_p} = \|u(t_j)\|_{V_p} \longrightarrow +\infty$ as $t_j \to +\infty$ and $t_j - a_j \in [0,T_1]$. Hence, v is not a strong solution in $[0,T_1]$. ■

Proof of Corollary 3.2**:** The argument is similar to Theorem 3.1. The difference arise in the continuous dependence on $L^2(0,T;V_2)$ instead of $L^p(0,T;V_p)$ (see Corollary 2.6). Consequently, to finish we argue over (36) without the V_p norm. Notice that the choice of newtonian viscosity is essential to guarantee the continuous dependence of strong solution of $(NS)^p_{per}$ in the cases of power law ($p > 1$) and Carreau's laws ($p \in (1,2)$). ■

Remark: *The uniqueness result is used to identify the solution furnished by Theorem* 2.7 *(or Corollary* 2.8*) and the solution given in the hypothesis of the theorem* 3.1 *(or Corollary* 3.2*). In the cases without newtonian viscosity and* $p \in (5/3,2)$*, there exists at least a strong solution but uniqueness is an open problem. Therefore the previous argument can not be applied.*

4 Hausdorff dimension estimation of singular times.

Learning of the Subsection 2.3, we can define the set **S** of the singular times of a weak solution u of $(NS)^p_{per}$ as the times where the $L^\infty(V_2)$ norm of this solution blows up, i.e.:

$$S = \{b \in (0,T] : \lim_{t\uparrow b} sup \|u(t)\|_{V_2} = +\infty\}.$$

Clearly, S has Lebesgue's measure zero, due to the fact that $\displaystyle\int_0^T \|u(t)\|^2_{V_2} dt < +\infty$. Basically, in this section we will see that "S has a Hausdorff dimension smaller than $d = d(p)$, with $d(p) < 1$ and decreasing respect to p".

Definition 4.1 (Hausdorff dimension) *Let $X \subset M$ a compact subset of a metric space M. The d-dimensional Hausdorff measure of X is given by $\nu_H^d(X) = \lim_{r\to 0} \nu_{H,r}^d(X)$ where*

$$\nu_{H,r}^d(X) = \inf\{\sum_{i=1}^k r_i^d : X \subset \cup_{i=1}^k B_i, B_i \text{ open balls in } M \text{ of radius } r_i \le r\}.$$

Finally, the Hausdorff dimension of X is given by $d_H(X) = \inf\{d > 0 : \nu_H^d(X) = 0\}$.

We study the same cases of the previous section, because we will need uniqueness of weak/strong solution and existence of local strong and global weak solution in our reasoning. With this purpose in mind, it is necessary to assume the regularity for the data (f, u_0) used in Theorem 2.7 and Corollary 2.8 respectively.

Let u be a weak solution of $(NS)^p_{per}$ in $(0,T)$, associated to these data (f, u_0). The main results of this section are the following:

Theorem 4.2 ($\mathbf{2 \le p < 11/5}$ and power law with newtonian viscosity or Carreau's laws). *Assume $f \in L^q(0,T;L^2(\Omega)^3)$ ($2 < q \le +\infty$) and $u_0 \in V_p$. Then, there exists a compact set $E \subset [0,T]$, such that $S \subseteq E$ and $d_H(E) \le d(p,q)$, where*

$$d(p,q) = \begin{cases} \dfrac{q(7-3p)-4(p-2)}{2(q-5p+9)} & \text{if } q < \dfrac{34}{13} \text{ and } q \le \dfrac{2(7p-12)}{3p-4} \\ \dfrac{q(20-9p)}{2[(4-p)q+(12-7p)]} & \text{otherwise.} \end{cases}$$

Corollary 4.3 ($\mathbf{1 < p < 2}$ and newtonian viscosity). *Assume $f \in L^q(0,T;L^2(\Omega)^3)$ ($2 < q \le +\infty$) and $u_0 \in V_2$. Then, there exists a compact set $E \subset [0,T]$, such that $S \subseteq E$ and $d_H(E) \le d(q)$, where $d(q) = q/(2q-2)$.*

Remark: *Notice that $d(q)$, given in Corollary 4.3 is equal to $d(2,q)$ of Theorem 4.2.*

Proof of Theorem 4.2: We divide the proof in three steps:

Step 1) General method to estimate the Hausdorff dimension for singular times.

Step 2) Some estimates of $d_H(E)$.

Step 3) Comparison of these estimates.

Step 1. General method to estimate the Hausdorff dimension for singular times. In this paragraph, we generalize the study made in [3] (in the case of the Navier-Stokes problem). For each t_0 such that $\|u(t_0)\|_{V_2 \cap V_p} < +\infty$, the results of theorems of existence of local strong solution and uniqueness imply that $u|_{[t_0, t_0+T^*]}$ is a strong solution, i.e., $u \in L^\infty(t_0, t_0 + T^*; V_2)$, for any T^* depending on $\|u(t_0)\|_{V_2}$ and $\|f\|_{L^q(L^2)}$, for $q > 2$. On the other hand, we consider the maximal interval of time containing t_0 where u is a strong solution, $I \subset [0, T]$. More specifically:

a) $I \subset [0, T], t_0 \in I$

b) $\forall J \supset I$, with $J \neq I$ one has $u|_J \notin L^\infty(J; V_2)$

The existence of a maximal interval follows from the set Z of intervals $J \subset [0, T]$ such that $t_0 \in J$ and $u|_J \in L^\infty(J; V_2)$ is not empty and if J_1, $J_2 \in Z$ then $J_1 \cup J_2 \in Z$. Moreover, I is open on the right side if the upper bound of I is not T.

We can find, at most, a countable number of disjoint maximal intervals $\{I_j\}_{j=1}^\infty$ (by the uniqueness of solution). Moreover, Lebesgue's measure of $[0, T] \setminus \bigcup_{j=1}^\infty I_j$ is zero. Let $I = I_j$ one of them. Denoting by a_j, b_j their end points, we have that b_j is a singular time if only if $b_j \neq T$, hence necessary, $\limsup_{t \uparrow b_j} \|u(t)\|_{V_\sigma} = +\infty$ (with $\sigma = \max(p, 2)$). We define the compact set $E = [0, T] \setminus \bigcup_{j=1}^\infty \overset{\circ}{I}_j$, where I_j is the maximal interval of regularity constructed previously. To estimate the d-dimensional Hausdorff measure ($d \in (0, 1)$) of E, we first observe that in the definition of ν_H^d we can use closed intervals instead of open intervals if $M = \mathbb{R}$ (M is the metric space in the definition of ν_H^d). Let $m \in \mathbb{N}$ and $E_m = [0, T] \setminus \bigcup_{j=1}^m \overset{\circ}{I}_j$. Then, $E_m \supset E$ and $(E_m) \searrow E$. Clearly, E_m is the union of a finite number of closed intervals (which may be degenerated to a point); i.e., $E_m = \bigcup_{j=1}^{k_m} K_j^{(m)}$, where $K_j^{(m)}$ are closed intervals, not empty and disjoint (respect to j). By construction, $\overset{\circ}{I}_j \bigcap K_l^{(m)} = \emptyset$ for $j \leq m$ and if $I_j \cap K_l^{(m)} \neq \emptyset$, for any $j \geq m + 1$, then $I_j \subset K_l^{(m)}$ because of I_j is connected and the intervals $(K_l^{(m)})_l$ are disjoint. Thus, the sets

$$N_l^{(m)} = \{j \geq m + 1; I_j \cap K_l^{(m)} \neq \emptyset\} = \{j \geq m + 1; I_j \subset K_l^{(m)}\}$$

are disjoint (respect to l). Denotting by $|\cdot|$ the Lebesgue's measure, we will get $\sum_{l=1}^{k_m} |K_l^{(m)}| \leq \sum_{l=1}^{k_m} \sum_{j\in N_l^{(m)}} |I_j| \leq \sum_{j=m+1}^{\infty} |I_j| = \varepsilon_m$. Moreover $\varepsilon_m \to 0$, because of $\sum_{j\geq 1} |I_j| \leq T$, since the $\mathring{I}_j$ are disjoint. In order to obtain the d-dimensional Hausdorff measure, we calculate:

$$\sum_{l=1}^{k_m} |K_l^{(m)}|^d \leq \sum_{l=1}^{k_m} \sum_{j\in N_l^{(m)}} |I_j|^d \leq \sum_{j\geq m+1} |I_j|^d, \quad 0 < d < 1,$$

(where we have used the fact that $(x+y)^d \leq x^d + y^d, \forall x, y \geq 0$). As $\{K_l^{(m)}\}_{l=1}^{k_m}$ is a cover by closed sets of E_m (so also of E) with intervals of radius $\leq \varepsilon_m/2$, we get $\nu^d_{H,\varepsilon_m/2}(E) \leq \sum_{j=m+1}^{\infty} |I_j|^d = \delta_m$. Therefore, if we prove that $\sum_{j\geq 1} |I_j|^d < +\infty$, then $\delta_m \longrightarrow 0$, and thus $\nu^d_H(E) = 0$ and, in particular, $d_H(E) \leq d$.

Step 2. Some estimates of $d_H(E)$. We want to demonstrate that $\sum_{j\geq 1} |I_j|^d < +\infty$, for any d: $0 < d < 1$. At the same time, we will perform two type of estimates: **1)** using a combination of the $L^\infty(V_p)$ and $L^\infty(V_2)$ regularities, and **2)** using only the $L^\infty(V_2)$ regularity.

1) Taking $\dfrac{\partial u}{\partial t}$ as a test function and integrating in Ω, we arrived to (24). On the other hand, taking $-\Delta u$ as a test function and integrating in Ω, if we take into account $(19)_2$ in Carreau's laws case (or the term $\mu_\infty \|\Delta u\|_2^2$ in the power law with newtonian viscosity case), we can get (omiting constants):

$$\frac{d}{dt}\|\nabla u\|_2^2 + I_p(u) \leq \int_\Omega |u|^2 |\nabla u|^2 dx + \|f\|_2^2 \tag{39}$$

Adding (39) to (24), we arrive at the inequality (up to constants):

$$\frac{d}{dt}\left\{\|\nabla u\|_2^2 + \int_\Omega \mathcal{U}^p(e(u))dx\right\} + \left\|\frac{\partial u}{\partial t}\right\|_2^2 + I_p(u) \leq \|f\|_2^2 + I(u, \nabla u). \tag{40}$$

The main difficulty is to bound $I(u, \nabla u)$. For this, we argue as follows:

$$I(u, \nabla u) = \int_\Omega |u|^2 |\nabla u|^r |\nabla u|^{2-r} dx \leq \|u\|_{p^*}^2 \|\nabla u\|_p^{(5p-8)/2} \|\nabla u\|_{3p}^{(12-5p)/2}$$

where we have chosen $r = r(p) = (5p-8)/2$, and p^* denotes the Sobolev exponent of p. So, applying (16) and the Young inequality with exponents $2p/(7p-12), 2p/(12-5p)$, we have

$$I(u, \nabla u) \leq C\|\nabla u\|_p^{(5p-4)/2} I_p(u)^{(12-5p)/2p} \leq \varepsilon I_p(u) + C_\varepsilon \|\nabla u\|_p^{p(5p-4)/(7p-12)}$$

Finally, from Korn inequality and the property $C_1\|e(u)\|_p^p \leq 2p(p-1)\int_\Omega \mathcal{U}^p(e(u))dx$, $\forall p \geq 2$ (see [8]), we arrive at:

$$I(u,\nabla u) \leq \varepsilon I_p(u) + C'_\varepsilon \left(\int_\Omega \mathcal{U}^p(e(u)dx\right)^{(5p-4)/(7p-12)} \tag{41}$$

Then, defining $J(u) = \|\nabla u\|_2^2 + J_p(u)$, from (40) and (41) we can deduce:

$$\frac{d}{dt}\{1+J(u)\} + \left\|\frac{\partial u}{\partial t}\right\|_2^2 + I_p(u) \leq \|f\|_2^2 + \{1+J(u)\}^{\lambda_1(p)} \tag{42}$$

where $\lambda_1(p) = (5p-4)/(7p-12)$.

Now, we also consider two cases, depending on the regularity of f:

1.1) Case $\mathbf{f \in L^\infty(0,T;L^2(\Omega)^3)}$: Dividing (42) by $\{1+J(u)\}^{\lambda_1(p)}$:

$$-\frac{1}{\lambda_1-1}\frac{d}{dt}\left(\frac{1}{\{1+J(u(t))\}^{\lambda_1-1}}\right) + \frac{\|\partial u/\partial t\|_2^2 + I_p(u)}{\{1+J(u(t))\}^{\lambda_1}} \leq \|f\|_2^2 + 1$$

where the right hand side belongs to $L^\infty(0,T)$ and integrating between t_0 y t ($t > t_0$) (taking into account that $\lambda_1 - 1 > 0$)

$$\frac{1}{\{1+J(u(t_0))\}^{\lambda_1-1}} \leq \frac{1}{\{1+J(u(t))\}^{\lambda_1-1}} + C(t-t_0).$$

Therefore, arguing as in Theorem 2.7 we obtain the following condition is sufficient for the existence of a local strong solution in $[t_0,t]$:

$$C(t-t_0) < \frac{1}{\{1+J(u(t_0))\}^{\lambda_1-1}}.$$

Accordingly, if I_j is the interval of maximal solution containing t_0 and $b = \sup I_j$, one has:

$$C(b-t_0)^{-1/(\lambda_1-1)} \leq 1 + J(u(t_0)) \tag{43}$$

and taking $\int_{I_j} dt_0$, we get $|I_j|^{1-1/(\lambda_1-1)} \leq C\int_{I_j}\{1+J(u(t_0))\}dt_0$. Thus,

$$\sum_{j\geq 1}|I_j|^{1-1/(\lambda_1-1)} \leq C\sum_{j\geq 1}\int_{I_j}\{1+J(u(t_0))\}\,dt_0 \leq C\int_0^T\{1+J(u(t_0))\}\,dt_0 < +\infty,$$

where we have used the property $J_p(u) \le C\left\{\|e(u)\|_p^p + |\Omega|\right\}$, see [8], and $u \in L^p(0,T;V_p)$. Then, we are under the hypothesis of **Step 1** for $d = d_1(p,\infty) = 1 - \dfrac{1}{\lambda_1 - 1} = \dfrac{20-9p}{2(4-p)} \ge d_H(E)$. Notice that the function $d_1(p,\infty)$ is decreasing on p and $d_1(p,+\infty) \to 1/2$ as $p \to 2^+$ (that was the bound obtained in the newtonian case [3]).

1.2) Case $\mathbf{f} \in \mathbf{L^q(0,T;L^2(\Omega)^3)}$ with $\mathbf{q > 2}$: Now, dividing (42) by $\{1+J(u(t))\}^{\lambda_1 - 2/q}$, we arrive at the expression:

$$\frac{1}{1-\lambda_1+2/q}\frac{d}{dt}\left(\frac{1}{\{1+J(u)\}^{\lambda_1-1-2/q}}\right) + \frac{\|\partial u/\partial t\|_2^2 + I_p(u)}{\{1+J(u)\}^{\lambda_1-2/q}}$$
$$\le \|f\|_2^2 + \{1+J(u(t))\}^{2/q} \in L^{q/2}(0,T).$$

So, integrating in time between t_0 and t, one obtains the expression:

$$\frac{1}{\{1+J(u(t_0))\}^{\lambda_1-(q+2)/q}} \le \frac{1}{\{1+J(u(t))\}^{\lambda_1-(q+2)/q}} + C(t-t_0)^{(q-2)/q}.$$

Therefore, now the condition which is sufficient for the existence of a local strong solution in $[t_0,t]$ is:

$$C(t-t_0)^{(q-2)/q} < \frac{1}{\{1+J(u(t_0))\}^{\lambda_1-(q+2)/q}}.$$

Thus, if $b = \sup I_j$:

$$C(b-t_0)^{-\frac{q-2}{q}(\lambda_1-\frac{q+2}{q})^{-1}} \le 1 + J(u(t_0)), \tag{44}$$

which, arguing as before, implies that $\sum_{j\ge1}|I_j|^d < +\infty$ with

$$d = d_1(p,q) = 1 - \frac{q-2}{q}\left(\lambda_1 - \frac{q+2}{q}\right)^{-1} = \frac{q(20-9p)}{2[(4-p)q+(12-7p)]} \ge d_H(E).$$

The function $d_1(p,q)$ is decreasing on p (fixed q) and decreasing on q (fixed p) and $d_1(p,q) \to d_1(p,+\infty)$ as $q \to +\infty$.

2) Starting only from the inequality obtained taking $-\Delta u$ as a test function, see (15), we may obtain (up to constants):

$$\frac{d}{dt}\|\nabla u\|_2^2 + I_p(u) \le \|\nabla u\|_3^3 + \|f\|_2^2. \tag{45}$$

We distinguish again two cases, depending on the regularity of f.

2.1) Case $\mathbf{f \in L^\infty(0,T;L^2(\Omega)^3)}$: We bound $\|\nabla u\|_3^3$ by:

$$\|\nabla u\|_3^3 \le \|\nabla u\|_2^{6(p-1)/(3p-2)} \|\nabla u\|_{3p}^{3p/(3p-2)} \le \varepsilon I_p(u) + C_\varepsilon \|\nabla u\|_2^{6(p-1)/(3p-5)}$$

where we have applied Hölder and Young inequalities, and the property (16). Then (45) becomes (up to constants):

$$\frac{d}{dt}\|\nabla u\|_2^2 + I_p(u) \le \|f\|_2^2 + \|\nabla u\|_2^{2\lambda_2} \tag{46}$$

where $\lambda_2 = 3(p-1)/(3p-5)$. Dividing by $\left(1+\|\nabla u\|_2^2\right)^{\lambda_2}$ in (46):

$$-\frac{1}{\lambda_2 - 1}\frac{d}{dt}\left(\frac{1}{\{1+\|\nabla u\|_2^2\}^{\lambda_2-1}}\right) + \frac{I_p(u)}{(1+\|\nabla u\|_2^2)^{\lambda_2}} \le \|f\|_2^2 + 1, \tag{47}$$

where the right hand side belongs to $L^\infty(0,T)$. Arguing as in paragraph **1.1)**, we obtain $d_2(p,\infty) = \dfrac{7-3p}{2} \ge d_H(E)$. Again, the function $d_2(p,\infty)$ is decreasing on p and $d_2(p,+\infty) \to 1/2$ as $p \to 2^+$.

2.2) Case $\mathbf{f \in L^q(0,T;L^2(\Omega)^3), q > 2}$: We bound $\|\nabla u\|_3^3$ like in **Step** 1 of Theorem 2.7 (see (17)). In this case, we take $\beta p/(p-\gamma) = 2p/q$, arriving to the inequality:

$$\frac{d}{dt}\left(1+\|\nabla u\|_2^2\right) + I_p(u) \le \|f\|_2^2 + \|\nabla u\|_p^{2p/q}\left(\|\nabla u\|_2^2\right)^{\lambda_3} \tag{48}$$

where $\lambda_3 = \{3(p-1)q - 2(5p-9)\}/q(3p-5)$. Dividing now by $\left(1+\|\nabla u\|_2^2\right)^{\lambda_3}$:

$$-\frac{1}{\lambda_3 - 1}\frac{d}{dt}\left(\frac{1}{\{1+\|\nabla u\|_2^2\}^{\lambda_3-1}}\right) + \frac{I_p(u)}{(1+\|\nabla u\|_2^2)^{\lambda_3}} \le \|f\|_2^2 + \|\nabla u\|_p^{2p/q}$$

and the right hand side belongs to $L^{q/2}(0,T)$. Following a similar reasoning to the paragraph **1.2)**, we can conclude that

$$d = d_2(p,q) = 1 - \frac{q-2}{q}\frac{1}{\lambda_3 - 1} = \frac{q(7-3p) - 4(p-2)}{2(q-5p+9)} \ge d_H(E)$$

Again, $d_2(p,q)$ is a decreasing function on p (fixed q) and a decreasing function on q (fixed p) and $d_2(p,q) \to d_2(p,+\infty)$ as $q \to +\infty$.

Step 3. Comparison of these estimates. We are going to compare the bounds d_1 and d_2 obtained in **Step 2**.

When $f \in L^\infty(0,T;L^2(\Omega)^3)$, then $d_1(p,\infty) = (20-9p)/2(4-p)$ and $d_2(p,\infty) = (7-3p)/2$ and it is easy to see that the best estimate comes from $d_1(p,+\infty)$, i.e., $d_1(p,+\infty) \le d_2(p,+\infty)$, hence we choose $d(p,+\infty) = d_1(p,+\infty)$.

When $f \in L^q(0,T;L^2(\Omega)^3)$ with $q > 2$, then

$$d_1(p,q) = \frac{q(20-9p)}{2[(4-p)q + (12-7p)]} \quad \text{and} \quad d_2(p,q) = \frac{q(7-3p) - 4(p-2)}{2(q-5p+9)}$$

We have that $d_1(p,q) \le d_2(p,q)$ if only if $q \ge 2(7p-12)/(3p-4) = g(p)$. The function $g(p)$ is increasing on p and $g(p) \in (2, 34/13)$ if $p \in (2, 11/5)$. In particular, if $q \ge 34/13$, then always $d_1(p,q) \le d_2(p,q)$, hence we must choose $d(p,q) = d_1(p,q)$. Otherwise ($q < 34/13$), we have that $d_1(p,q) \le d_2(p,q)$ if only if $q \ge g(p)$, hence we must choose $d(p,q) = d_1(p,q)$ in this case and $d(p,q) = d_2(p,q)$ if $q \le g(p)$. In a intuitive way, as $2p/q$ is the power of $\|\nabla u\|_p$ in the reasoning to obtain $d_2(p,q)$ (see (48)), then p/q has to be large enough to make the L^p-regularity more important than the L^2-regularity, so in these cases $d_2(p,q)$ can improve the estimation of $d_1(p,q)$.

■

Proof of Corollary 4.3**:** Taking into account **Step 1** of the Theorem 4.2, we only consider the **Step 2: Calculus for $\mathbf{1 < p < 2}$ and newtonian viscosity.** In these cases, it is not worth to use a combination of the $L^\infty(V_p)$ and $L^2(V_2)$ regularities (as we have made in **Step 2** paragraph **1)** of Theorem 4.2), because now L^p is "less regular" than L^2. This is the reason why we only use $-\Delta u$ as a test function (not $\dfrac{\partial u}{\partial t}$), obtaining:

$$\frac{d}{dt}\|\nabla u\|_2^2 + \mu_\infty \|\Delta u\|_2^2 \le \|f\|_2^2 + \int_\Omega (u\cdot\nabla)\cdot u(-\Delta u)dx.$$

The difference with the case **Step 2** of Theorem 4.2 is the substitution of $I_p(u)$ for $\|\Delta u\|_2^2$. Now, if we use the L^2 regularity of the Stokes problem with periodic boundary conditions, i.e.:

$$\|u\|_{H^2}^2 \le C\|\Delta u\|_2^2, \quad \forall u \in H^2 \cap V_2,$$

then depending on the treatment of the term $\int_\Omega u\cdot\nabla u(-\Delta u)dx$, we are going to consider two cases:

$$\frac{d}{dt}\|\nabla u\|_2^2+\|u\|_{H^2}^2\le\|f\|_2^2+\begin{cases} I(u,\nabla u) & \textbf{Case 1)}\\ \|\nabla u\|_3^3 & \textbf{Case 2)}\end{cases} \tag{49}$$

1) Now, we bound $I(u,\nabla u)$ in a different way as in Theorem 4.2. Indeed,

$$I(u,\nabla u)=\int_\Omega |u|^2|\nabla u||\nabla u|dx\le C\|\nabla u\|_2^3\|u\|_{H^2}\le\varepsilon\|u\|_{H^2}^2+C_\varepsilon(\|\nabla u\|_2^2)^3. \tag{50}$$

Replacing (50) in (49) (for ε small enough), the inequality obtained is (up to constants):

$$\frac{d}{dt}\|\nabla u\|_2^2+\|u\|_{H^2}^2\le\|f\|_2^2+(\|\nabla u\|_2^2)^3. \tag{51}$$

Depending on the regularity of f, we distinguish again two subcases.

1.1) Case $\mathbf{f}\in\mathbf{L}^\infty(\mathbf{0},\mathbf{T};\mathbf{L}^2(\boldsymbol{\Omega})^3)$: Dividing by $(1+\|\nabla u\|_2^2)^3$ in (51):

$$-\frac{1}{2}\frac{d}{dt}\left\{\frac{1}{(1+\|\nabla u\|_2^2)^2}\right\}+\frac{I_p(u)}{(1+\|\nabla u\|_2^2)^3}\le\|f\|_2^2+1\in L^\infty(0,T). \tag{52}$$

Integrating in time between t_0 and t and arguing similary to the paragraph **1.1)** in **Step 2** of Theorem 4.2 (changing $I_p(u)$ by $\|u\|_{H^2}^2$), we arrive at $d=1/2\ge d_H(E)$.

1.2) Case $\mathbf{f}\in\mathbf{L}^\mathbf{q}(\mathbf{0},\mathbf{T};\mathbf{L}^2(\boldsymbol{\Omega})^3),\mathbf{q}>\mathbf{2}$: Now dividing (51) by $(1+\|\nabla u\|_2^2)^{3-2/q}$ and using the analogous reasoning of **1.2)** in Theorem 4.2 (changing $I_p(u)$ by $\|u\|_{H^2}^2$), we arrive at $d=d(q)=\frac{q}{2q-2}\ge d_H(E)$. The function $d(q)$ is decreasing on q.

2) Here, we bound $\|\nabla u\|_3^3\le\|\nabla u\|_2^{3/2}\|\nabla u\|_6^{3/2}\le\varepsilon\|u\|_{H^2}^2+C_\varepsilon\|\nabla u\|_2^6$, hence we obtain again (51). Therefore, the estimations are the same of **1)**, and the proof is finished. ■

Remark: *Notice that, in the previous arguments, the contribution of the newtonian viscosity is essential, not only to demonstrate the existence and uniqueness of strong solution, but also to estimate the Hausdorff dimension of the singular times.*

Remark: *In general, it is more convenient to bound $\|\nabla u\|_3^3$ than $I(u,\nabla u)$; for instance, the bound of $\|\nabla u\|_3^3$ gives us the existence of global strong solutions for $p\ge 11/5$, and if we bound $I(u,\nabla u)$*

one has the result for $p \geq 20/9$*, see* [*4*]*. However, the estimations obtained in Corollary* 4.3 *are the same because we can control both terms by* $\|u\|^2_{H^2}$*.*

Conclusion: The smoothness of a weak solution increase when p increase. The results obtained in this Section quantify this property as a disminution on the size of the set of singular times.

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