On the uniqueness and regularity of the Primitive Equations imposing additional anisotropic regularity.

F. Guillén-González♣*†, M.A. Rodríguez-Bellido♠

♣ Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Aptdo. 1160, 41080 Sevilla, Spain. e-mail: guillen@numer.us.es
♠ Dpto. de Matemática Aplicada I, E. T. S. de Arquitectura, Universidad de Sevilla, Avda. Reina Mercedes, 2, 41012 Sevilla, Spain. e-mail: angeles@us.es

Abstract

In this note, we prove that given $\mathbf{u}$ a weak solution of the Primitive Equations, imposing an additional condition on the vertical derivative of the velocity $\mathbf{u}$ (concretely $\partial_z \mathbf{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$), then two different results hold; namely, uniqueness of weak solution (any weak solution associated to the same data that $\mathbf{u}$ must coincide with $\mathbf{u}$) and global in time strong regularity for $\mathbf{u}$ (without "smallness assumptions" on the data).

Both results are proved when either Dirichlet or Robin type conditions on the bottom are considered. In the last case, a domain with a strictly bounded from below depth has to be imposed, even for the uniqueness result.

Key words: Weak-strong uniqueness, Primitive Equations, anisotropic estimates, strong solution

1 Introduction

The Primitive Equations are related with a great variety of geophysical fluids [9, 10, 12]. This system can be deduced asymptotically from the Navier-Stokes equations with anisotropic (eddy) viscosity, when the aspect ratio (quotient between vertical and horizontal characteristic dimensions) tends to zero [1, 2, 3]. The 3D system can be written as follows: to find $\mathbf{u} : (0,T)\times\Omega \to \mathbb{R}^2$, the horizontal velocity field, and $p_s : (0,T)\times S \to \mathbb{R}$, a surface potential function (involving the pressure), verifying:

$$(PE)\begin{cases} \partial_t \mathbf{u} - \nu_H \Delta_H \mathbf{u} - \nu_z \partial^2_{zz}\mathbf{u} + \alpha \mathbf{u}^\perp + & \\ \qquad +(\mathbf{u}\cdot\nabla_H)\mathbf{u} + u_3 \partial_z \mathbf{u} + \nabla_H p_s = \mathbf{F} & \text{in } (0,T)\times\Omega, \\ \nabla_H \cdot \langle \mathbf{u}\rangle = 0 & \text{in } (0,T)\times S, \\ \mathbf{u}_{|t=0} = \mathbf{u}_0 & \text{in } \Omega, \\ \nu_z \partial_z \mathbf{u}_{|\Gamma_s} = \boldsymbol{\Upsilon}, \quad \mathbf{u}_{|\Gamma_l} = \mathbf{0} & \text{in } (0,T), \\ \text{either } \mathbf{u}_{|\Gamma_b} = \mathbf{0} \quad \text{or} \quad ((\nu_H \nabla_H \mathbf{u}, \nu_z \partial_z \mathbf{u})\cdot \mathbf{n} + \beta \mathbf{u})_{|\Gamma_b} = \mathbf{0} & \text{in } (0,T), \end{cases}$$

where u_3 is the vertical velocity, that becomes now a diagnostic variable, depending on the horizontal velocity $\mathbf{u}$ as follows:

$$u_3(\mathbf{x},z) = \int_z^0 \nabla_H \cdot \mathbf{u}(\mathbf{x},s)\,ds. \tag{1}$$

We consider the domain $\Omega = \{(x,y,z) = (\mathbf{x},z) \in \mathbb{R}^3 \,/\, \mathbf{x} \in S,\ -h(\mathbf{x}) < z < 0\}$, with $S \subset \mathbb{R}^2$ a bounded open set (the surface) and $h : S \to \mathbb{R}$ a non-negative continuous function (the depth).

*Corresponding author

†The first author has been partially financed by the project BFM2000-1317.

Its boundary is decomposed as $\partial\Omega = \Gamma_b \cup \Gamma_l \cup \Gamma_s$ where $\Gamma_s = \{(\mathbf{x}, 0) : \mathbf{x} \in S\}$ is the surface, $\Gamma_l = \{(\mathbf{x}, z) \in \mathbb{R}^3 : \mathbf{x} \in \partial S, -h(\mathbf{x}) < z < 0\}$ are the side-walls and $\Gamma_b = \{(\mathbf{x}, z) \in \mathbb{R}^3 : \mathbf{x} \in S, z = -h(\mathbf{x})\}$ is the bottom.

We have denoted by $\langle \mathbf{u} \rangle(t;\mathbf{x}) = \int_{-h(\mathbf{x})}^0 \mathbf{u}(t;\mathbf{x},z)dz$ the vertical integration of $\mathbf{u}$. The horizontal operators Δ_H and ∇_H represent $\partial^2_{xx} + \partial^2_{yy}$ and $(\partial_x, \partial_y)^t$ respectively. The constants $\nu_H, \nu_z > 0$ are the viscosity coefficients and $\mathbf{n}$ is the outward normal vector on the bottom. The external forces are data denoted by $\mathbf{F} : (0,T) \times \Omega \to \mathbb{R}^2$, and $\alpha \mathbf{u}^\perp = \alpha(-u_2, u_1)^t$ models the Coriolis forces, with $\alpha \in \mathbb{R}$ depending on the latitude. We consider either homogeneous Dirichlet or Robin type boundary conditions on the bottom (with $\beta : S \to \mathbb{R}$ a non-negative data function depending on the rugosity of the bottom) and Neumann boundary conditions on the surface, where $\mathbf{\Upsilon} : (0,T) \times S \to \mathbb{R}^2$ is a data function depending on the wind force. The Neumann conditions on the bottom are also considered taking $\beta = 0$.

Notice that Primitive Equations are variants of the Navier-Stokes equations. Now, the pressure field depends only on $\mathbf{x}$ (but not on z). However, the explicit form of u_3 given in (1) implies that the system is no longer parabolic respect to $(\mathbf{u}, u_3)$ and the regularity of u_3 and $\nabla_H \cdot \mathbf{u}$ are comparable, hence the nonlinear term corresponding to the vertical convection $u_3 \partial_z \mathbf{u}$ is less regular that in the Navier-Stokes case.

The existence of weak solution of (PE) were given in [10, 9]. The existence of local in time strong solution (or global for small enough data) is proved in [7] for the 2D case (where S is a real interval) and in [6] for the 3D case, using strong regularity results for the stationary linear case given in [14]. On the other hand, some results of weak/strong uniqueness were given in [7, 6], always imposing additional regularity hypothesis over the horizontal and vertical derivatives of $\mathbf{u}$.

In this work, we weaken these additional hypothesis found in [7, 6] supposing only additional regularity over the vertical derivative $\partial_z \mathbf{u}$ (avoiding the additional regularity over $\nabla_H \mathbf{u}$). Moreover, we will also prove that this same additional regularity implies global strong regularity when the data are more regular but without smallness assumptions.

We think that the anisotropy between horizontal and vertical scales could produce anisotropic regularity for the solution. Indeed, this occurs in the 2D case; existence (and uniqueness) of weak solution u for the 2D model such that $\partial_z u$ has also weak regularity, i.e. $\partial_z u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$, is proved in [4] for Robin boundary conditions on the bottom and in [5] for Dirichlet conditions. In this line, the existence of weak solution for (PE) with only weak regularity for $\partial_z \mathbf{u}$ (even local in time or global for small enough data) is an interesting open problem, that we are going to analyse in a future work.

2 The main results.

Basically, $\mathbf{u}$ is a **weak solution** for (PE) in $(0,T)$, if $\mathbf{u} \in L^2(0,T;H^1(\Omega))^2 \cap L^\infty(0,T;L^2(\Omega)^2)$ and verifies the restriction $\nabla_H \cdot \langle \mathbf{u} \rangle = 0$, the Dirichlet conditions in the trace sense and the momentum equations jointly with the Neumann and Robin conditions in a variational sense ([10, 6]). Moreover, if $\mathbf{u} \in L^\infty(0,T;H^1(\Omega)^2) \cap L^2(0,T;H^2(\Omega)^2)$ and $\partial_t \mathbf{u} \in L^2(0,T;L^2(\Omega)^2)$, $\mathbf{u}$ is a **strong solution** for (PE) in $(0,T)$.

Theorem 2.1 (Uniqueness of solution) *Let $\mathbf{u}$ be a weak solution of (PE) in $(0,T)$. If there exists a weak solution $\bar{\mathbf{u}}$ of (PE) in $(0,T)$ verifying the additional regularity:*

$$\partial_z \bar{\mathbf{u}} \in L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;H^1(\Omega)^2), \tag{2}$$

then both solution coincided in $[0,T)$. When Robin conditions are considered on the bottom, the assumption $h \geq h_{\min} > 0$ in S has to be imposed.

Remark 2.1 *Notice that we have reduced the hypotheses on $\bar{\mathbf{u}}$ imposed in [6] for getting uniqueness of weak/strong solution. Concretely, in [6] we considered*

$$\nabla_H \bar{\mathbf{u}} \in L^2(0,T;L^\infty_z L^2_{\mathbf{x}}) \quad \text{and} \quad \partial_z \bar{\mathbf{u}} \in L^\infty(0,T;L^2(\Omega)^2) \cap L^2(0,T;H^1(\Omega)^2)$$

(see the next section for the definition of the anisotropic space $L^\infty_z L^2_{\mathbf{x}}$). Therefore, we have removed the hypothesis for $\nabla_H \bar{\mathbf{u}}$.

Theorem 2.2 (Global strong regularity) *Let $S \subseteq \mathbb{R}^2$ with $\partial S \in C^3$ and $h \in C^3(\bar{S})$ with $h \geq h_{\min} > 0$ in $\bar{S}$. Suppose that $\mathbf{u}_0 \in H^1(\Omega)$ with $\nabla_H \cdot \langle \mathbf{u}_0 \rangle = \mathbf{0}$ (and $\mathbf{u}_{0|\Gamma_b} = 0$ in the case of Dirichlet conditions on the bottom), $\mathbf{F} \in L^2(0,T;L^2(\Omega)^2)$ and $\boldsymbol{\Upsilon} \in L^2(0,T;H_0^{1/2+\varepsilon}(\Gamma_s)^2) \cap L^\infty(0,T;H^{-1/2}(\Gamma_s)^2)$ for some $\varepsilon > 0$ such that $\partial_t \boldsymbol{\Upsilon} \in L^2(0,T;H^{-3/2}(\Gamma_s)^2)$ with $\boldsymbol{\Upsilon}(0) \in H^{-1/2}(\Gamma_s)^2$. If $\partial_z \mathbf{u}$ verifies the additional regularity of (2), then $\mathbf{u}$ is a strong solution of (PE) in $(0,T)$.*

3 Some auxiliary anisotropic estimates.

Let us to introduce the anisotropic $L^{p,q}$ spaces for any exponents $p, q \in [1, +\infty]$. It will said say that a function v belongs to $L^q_z L^p_{\mathbf{x}}(\Omega)$ if:

$$v(\cdot, z) \in L^p(S_z) \quad \text{and} \quad \|v(\cdot,z)\|_{L^p(S_z)} \in L^q(-h_{\max}, 0),$$

where $h_{\max} = \max_{\bar{S}} h$ and $S_z = \{\mathbf{x} \in S : (\mathbf{x}, z) \in \Omega\}$ for each $z \in (-h_{\max}, 0)$.

We will use the following three anisotropic results, the first one has already been considered and proved in [6], and the other ones are new in this work (see Appendix for the proofs).

Lemma 3.1 *a) Let $v \in H^1(\Omega)$. Then $v \in L^2_z L^4_{\mathbf{x}}(\Omega)$ and verifies:*

$$\|v\|_{L^2_z L^4_{\mathbf{x}}} \leq C \|v\|^{1/2}_{L^2(\Omega)} \|v\|^{1/2}_{H^1(\Omega)} \tag{3}$$

b) Let $\mathbf{v} \in L^2(\Omega)^2$ such that $\nabla_H \cdot \mathbf{v} \in L^2(\Omega)$, and v_3 defined as in (1). Then, $v_3 \in L^\infty_z L^2_{\mathbf{x}}(\Omega)$ and

$$\|v_3\|_{L^\infty_z L^2_{\mathbf{x}}} \leq h^{1/2}_{\max} \|\nabla_H \cdot \mathbf{v}\|_{L^2(\Omega)} \tag{4}$$

Lemma 3.2 *Let $v \in H^1(\Omega)$ such that $\partial_z v \in H^1(\Omega)$ and $v|_{\Gamma_b} = 0$. Then $v \in L^\infty_z L^4_{\mathbf{x}}(\Omega)$ and*

$$\|v\|_{L^\infty_z L^4_{\mathbf{x}}} \leq C \|v\|^{1/4}_{L^2(\Omega)} \|v\|^{1/4}_{H^1(\Omega)} \|\partial_z v\|^{1/4}_{L^2(\Omega)} \|\partial_z v\|^{1/4}_{H^1(\Omega)} \tag{5}$$

Lemma 3.3 *Assume $h \geq h_{\min} > 0$ in S.*
a) Let $v \in L^2(\Omega)$ such that $\partial_z v \in L^2(\Omega)$. Then $v \in L^\infty_z L^2_{\mathbf{x}}(\Omega)$ and

$$h_{\min} \|v\|^2_{L^\infty_z L^2_{\mathbf{x}}} \leq \|v\|^2_{L^2(\Omega)} + 2\|v\|_{L^2(\Omega)} \|\partial_z v\|_{L^2(\Omega)}. \tag{6}$$

b) Let $v \in H^1(\Omega)$ such that $\partial_z v \in H^1(\Omega)$. Then $v \in L^\infty_z L^4_{\mathbf{x}}(\Omega)$ and

$$h^{1/2}_{\min} \|v\|_{L^\infty_z L^4_{\mathbf{x}}} \leq C \|v\|^{1/4}_{L^2(\Omega)} \|v\|^{1/4}_{H^1(\Omega)} \left(\|v\|^{1/4}_{L^2(\Omega)} \|v\|^{1/4}_{H^1(\Omega)} + \|\partial_z v\|^{1/4}_{L^2(\Omega)} \|\partial_z v\|^{1/4}_{H^1(\Omega)} \right) \tag{7}$$

4 Proof of the main results in the Dirichlet case.

Proof of Theorem 2.1**:** We follow the direct method to prove uniqueness, used for instance in [11] for the 3D Navier-Stokes equations. Denoting $\mathbf{v} = \mathbf{u} - \bar{\mathbf{u}}$ and $v_3 = u_3 - \bar{u}_3$, one has [6] (see [13] for more details):

$$\begin{aligned} \frac{1}{2}\|\mathbf{v}(t)\|^2_{L^2(\Omega)} &+ \nu \int_0^t \|\mathbf{v}(s)\|^2_{H^1(\Omega)} ds \\ &\leq - \int_0^t \int_\Omega [\mathbf{v} \cdot \nabla_H \bar{\mathbf{u}} + v_3 \partial_z \bar{\mathbf{u}}] \cdot \mathbf{v} \, d\Omega \, ds := I_1 + I_2, \end{aligned} \tag{8}$$

where $\nu = \min\{\nu_H, \nu_z\}$. With respect to the proof of uniqueness done in [6], we will change the treatment of the term I_1. Now, integrating by parts and applying (3) and (5) one has (here, Dirichlet condition on Γ_b is used)

$$\begin{aligned} I_1 &= \int_0^t \int_\Omega [(\mathbf{v}\cdot\nabla_H)\mathbf{v}\cdot\bar{\mathbf{u}} + (\nabla_H\cdot\mathbf{v})\,\mathbf{v}\cdot\bar{\mathbf{u}}]\,d\Omega\,ds \\ &\le C\int_0^t\int_\Omega |\mathbf{v}|\,|\nabla_H\mathbf{v}|\,|\bar{\mathbf{u}}|\,d\Omega\,ds \le C\int_0^t \|\mathbf{v}\|_{L^2_zL^4_{\mathbf{x}}}\|\nabla_H\mathbf{v}\|_{L^2(\Omega)}\|\bar{\mathbf{u}}\|_{L^\infty_zL^4_{\mathbf{x}}}\,ds \\ &\le C\int_0^t \|\mathbf{v}\|^{1/2}_{L^2(\Omega)}\|\nabla_H\mathbf{v}\|^{3/2}_{L^2(\Omega)}\|\bar{\mathbf{u}}\|^{1/4}_{L^2(\Omega)}\|\bar{\mathbf{u}}\|^{1/4}_{H^1(\Omega)}\|\partial_z\bar{\mathbf{u}}\|^{1/4}_{L^2(\Omega)}\|\partial_z\bar{\mathbf{u}}\|^{1/4}_{H^1(\Omega)}\,ds. \end{aligned}$$

Using the Young inequality for the indexes $(4/3, 4)$, we have:

$$I_1 \le \varepsilon\int_0^t \|\mathbf{v}\|^2_{H^1(\Omega)}\,ds + C_\varepsilon\int_0^t \|\bar{\mathbf{u}}\|_{L^2(\Omega)}\|\bar{\mathbf{u}}\|_{H^1(\Omega)}\|\partial_z\bar{\mathbf{u}}\|_{L^2(\Omega)}\|\partial_z\bar{\mathbf{u}}\|_{H^1(\Omega)}\|\mathbf{v}\|^2_{L^2(\Omega)}\,ds.$$

We bound I_2 as in [6] (using (3) and (4)), obtaining

$$\begin{aligned} I_2 &= \int_0^t\int_\Omega v_3\,\partial_z\bar{\mathbf{u}}\cdot\mathbf{v}\,d\Omega\,ds \le \int_0^t \|v_3\|_{L^\infty_zL^2_{\mathbf{x}}}\|\partial_z\bar{\mathbf{u}}\|_{L^2_zL^4_{\mathbf{x}}}\|\mathbf{v}\|_{L^2_zL^4_{\mathbf{x}}}\,ds \\ &\le \varepsilon\int_0^t \|\mathbf{v}\|^2_{H^1(\Omega)}\,ds + C_\varepsilon\int_0^t \|\partial_z\bar{\mathbf{u}}\|^2_{L^2(\Omega)}\|\partial_z\bar{\mathbf{u}}\|^2_{H^1(\Omega)}\|\mathbf{v}\|^2_{L^2(\Omega)}\,ds \end{aligned}$$

Using the previous bounds in (8), one has:

$$\|\mathbf{v}(t)\|^2_{L^2(\Omega)} + \nu\int_0^t \|\mathbf{v}(s)\|^2_{H^1(\Omega)}\,ds \le C\int_0^t a(s)\,\|\mathbf{v}(s)\|^2_{L^2(\Omega)}\,ds \tag{9}$$

where $a = \|\bar{\mathbf{u}}\|_{L^2(\Omega)}\|\bar{\mathbf{u}}\|_{H^1(\Omega)}\|\partial_z\bar{\mathbf{u}}\|_{L^2(\Omega)}\|\partial_z\bar{\mathbf{u}}\|_{H^1(\Omega)} + \|\partial_z\bar{\mathbf{u}}\|^2_{L^2(\Omega)}\|\partial_z\bar{\mathbf{u}}\|^2_{H^1(\Omega)}$. Since $a \in L^1(0,T)$ (thanks to the regularity hypothesis for $\bar{\mathbf{u}}$ and $\partial_z\bar{\mathbf{u}}$), we are in the hypothesis of Gronwall Lemma, hence the uniqueness is deduced. ∎

Proof of Theorem 2.2**:** First, we lift the boundary data $\boldsymbol{\Upsilon}$ using an adequate (strong) solution $(\mathbf{e}, q_s)$ of a stationary hydrostatic Stokes system. Observe that hypothesis $\partial_t\boldsymbol{\Upsilon} \in L^2(0,T;H^{-3/2}(\Gamma_s)^2)$ implies that $\partial_t\mathbf{e} \in L^2(0,T;L^2(\Omega)^2)$ (see [8]). Then, we reason over the homogeneous variables $(\mathbf{v}, v_3, \pi_s) = (\mathbf{u}-\mathbf{e}, u_3-e_3, p_s-q_s)$, verifying:

$$\left\{\begin{array}{rcll} \partial_t\mathbf{v} - \nu_H\Delta_H\mathbf{v} - \nu_z\partial^2_{zz}\mathbf{v} + \mathbf{u}\cdot\nabla_H\mathbf{v} + v_3\partial_z\mathbf{u} + \nabla_H\pi_s &=& \mathbf{G} & \text{in } (0,T)\times\Omega, \\ \nabla_H\cdot\langle\mathbf{v}\rangle &=& \mathbf{0} & \text{in } (0,T)\times S, \\ \mathbf{v}|_{t=0} &=& \mathbf{v}_0 & \text{in } \Omega, \\ \nu_z\partial_z\mathbf{v}|_{\Gamma_s} = \mathbf{0}, \quad \mathbf{v}|_{\Gamma_b} = \mathbf{0}, \quad \mathbf{v}|_{\Gamma_l} &=& \mathbf{0} & \text{in } (0,T), \end{array}\right. \tag{10}$$

where $\mathbf{v}_0 = \mathbf{u}_0 - \mathbf{e}(0)$ and $\mathbf{G} = \mathbf{F} - \partial_t\mathbf{e} - \mathbf{u}\cdot\nabla_H\mathbf{e} + e_3\partial_z\mathbf{u}$. Thanks to the additional regularity of $\partial_z\mathbf{u}$ and the strong regularity of $\mathbf{e}$, one has that $\mathbf{G} \in L^2(0,T;L^2(\Omega)^2)$. Indeed, in the convective terms, we have products of $\mathbf{u}$ (and e_3) belonging to $L^4_tL^\infty_zL^4_{\mathbf{x}}$, by $\partial_z\mathbf{u}$ (and $\nabla_H\mathbf{e}$) belonging to $L^4_tL^2_zL^4_{\mathbf{x}}$ (accordingly Lemmas 3.1 and 3.2). Using the same argument than in [6], we apply a Galerkin method, using $A\mathbf{v}_m$ as test functions, where A is the hydrostatic Stokes operator and $\mathbf{v}_m$ its eigenfunctions. In order to bound the convection terms, we use the inequalities (3) and (5) as follows (for simplicity, we drop the m-indexes):

$$\begin{aligned} \int_\Omega \mathbf{u}\cdot\nabla_H\mathbf{v}\cdot A\mathbf{v}\,d\Omega &\le \|\mathbf{u}\|_{L^\infty_zL^4_{\mathbf{x}}}\|\nabla_H\mathbf{v}\|_{L^2_zL^4_{\mathbf{x}}}\|A\mathbf{v}\|_{L^2(\Omega)} \\ &\le C\|\mathbf{u}\|^{1/4}_{L^2(\Omega)}\|\mathbf{u}\|^{1/4}_{H^1(\Omega)}\|\partial_z\mathbf{u}\|^{1/4}_{L^2(\Omega)}\|\partial_z\mathbf{u}\|^{1/4}_{H^1(\Omega)}\|\nabla_H\mathbf{v}\|^{1/2}_{L^2(\Omega)}\|A\mathbf{v}\|^{3/2}_{L^2(\Omega)} \\ &\le \varepsilon\|A\mathbf{v}\|^2_{L^2(\Omega)} + a(t)\|\nabla_H\mathbf{v}\|^2_{L^2(\Omega)}, \end{aligned} \tag{11}$$

where $a(t) = C\|\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{u}(t)\|_{H^1(\Omega)}\|\partial_z\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))}\|\partial_z\mathbf{u}(t)\|_{H^1(\Omega)}$, and

$$\begin{aligned}\int_\Omega v_3\partial_z\mathbf{u}\cdot A\mathbf{v}\,d\Omega &\le \|v_3\|_{L_z^\infty L_{\mathbf{x}}^4}\|\partial_z\mathbf{u}\|_{L_z^2L_{\mathbf{x}}^4}\|A\mathbf{v}\|_{L^2(\Omega)}\\ &\le C\|\nabla_H\cdot\mathbf{v}\|_{L^2(\Omega)}^{1/2}\|\partial_z\mathbf{u}\|_{L^2(\Omega)}^{1/2}\|\partial_z\mathbf{u}\|_{H^1(\Omega)}^{1/2}\|A\mathbf{v}\|_{L^2(\Omega)}^{3/2} \le \varepsilon\|A\mathbf{v}\|_{L^2(\Omega)}^2 + b(t)\|\nabla_H\mathbf{v}\|_{L^2(\Omega)}^2,\end{aligned} \tag{12}$$

where $b(t) = C\|\partial_z\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega))}\|\partial_z\mathbf{u}(t)\|^2_{H^1(\Omega)}$. The additional regularity for $\partial_z\mathbf{u}$ guarantees that $a,\, b \in L^1(0,T)$. Then,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{v}\|^2_{L^2(\Omega)} + \|A\mathbf{v}\|^2_{L^2(\Omega)} \le (a(t)+b(t))\|\nabla\mathbf{v}\|^2_{L^2(\Omega)} + \|\mathbf{G}(t)\|^2_{L^2(\Omega)}.$$

Gronwall's Lemma allows us to conclude that $\mathbf{v} \in L^\infty(0,T;H^1(\Omega)^2)\cap L^2(0,T;H^2(\Omega)^2)$ and, thanks to the strong regularity of $\mathbf{e}$, one has the same regularity for $\mathbf{u}$. Regularity for $\partial_t\mathbf{u}$ is followed by a standard way. ■

5 Proofs for Robin conditions on the bottom

Proof of Theorem 2.1**:** In this case, one arrives at (8) with the supplementary non-negative term $\int_0^t\int_{\Gamma_b}\beta|\mathbf{v}|^2d\sigma\,ds$ in the left hand-side. Now, it is necessary to change the bound for the term I_1 in (8). Indeed, using directly (8) (without by parts integration), we get:

$$I_1 \le \int_0^t \|\mathbf{v}\|^2_{L_z^2L_{\mathbf{x}}^4}\|\nabla_H\bar{\mathbf{u}}\|_{L_z^\infty L_{\mathbf{x}}^2} \le \int_0^t \|\mathbf{v}\|_{L^2(\Omega)}\|\mathbf{v}\|_{H^1(\Omega)}\|\nabla_H\bar{\mathbf{u}}\|_{L_z^\infty L_{\mathbf{x}}^2} \tag{13}$$

In order to bound $\nabla_H\bar{\mathbf{u}}$, we cannot use the following inequality (proved in [6])

$$\|\nabla_H\bar{\mathbf{u}}\|_{L_z^\infty L_{\mathbf{x}}^2} \le C\|\nabla_H\bar{\mathbf{u}}\|^{1/2}_{L^2(\Omega)}\|\nabla_H\bar{\mathbf{u}}\|^{1/2}_{H^1(\Omega)}$$

because $\nabla_H\bar{\mathbf{u}} \notin H^1(\Omega)$. Instead of this, we will use Lemma 3.3 a). Indeed, applying (6) for $v = \nabla_H\bar{\mathbf{u}}$ in (13), we arrive at

$$I_1 \le \varepsilon\int_0^t \|\mathbf{v}\|^2_{H^1(\Omega)}\,ds + \frac{C_\varepsilon}{h_{\min}}\int_0^t \Big\{\|\nabla_H\bar{\mathbf{u}}\|^2_{L^2(\Omega)} + \|\nabla_H\bar{\mathbf{u}}\|_{L^2(\Omega)}\|\partial_z(\nabla_H\bar{\mathbf{u}})\|_{L^2(\Omega)}\Big\}\|\mathbf{v}\|^2_{L^2(\Omega)}\,ds$$

Adding this expression to the estimate for I_2, one can also prove uniqueness of solution. ■

Proof of Theorem 2.2**:** The main difference in the proof is the use of inequality (7) instead of (5) of Lemma 3.2 in order to bound the convective terms. ■

A Appendix

Proof of Lemma 3.2**.** We will use the following inequality, proved in [6]; for any $p, q \in [1,+\infty]$ with $q > p$,

$$\|v\|_{L_{\mathbf{x}}^qL_z^p} \le \|v\|_{L_z^pL_{\mathbf{x}}^q} \tag{14}$$

With the same arguments one can change the integration order, i.e.,

$$\|v\|_{L_z^qL_{\mathbf{x}}^p} \le C\|v\|_{L_{\mathbf{x}}^pL_z^q} \tag{15}$$

Let v in the hypothesis of Lemma 3.2. Since $v_{|\Gamma_b} = 0$, we have

$$v(\mathbf{x},z)^4 = \left(2\int_{-h(\mathbf{x})}^z v(\mathbf{x},s)\,\partial_z v(\mathbf{x},s)ds\right)^2 \le 4\|v(\mathbf{x},\cdot)\|^2_{L_z^2}\|\partial_z v(\mathbf{x},\cdot)\|^2_{L_z^2}$$

Consequently,

$$\|v(\mathbf{x},\cdot)\|_{L^\infty_z} \le \sqrt{2}\,\|v(\mathbf{x},\cdot)\|^{1/2}_{L^2_z}\|\partial_z v(\mathbf{x},\cdot)\|^{1/2}_{L^2_z}. \tag{16}$$

Taking $L^4_{\mathbf{x}}$-norm,

$$\|v\|_{L^4_{\mathbf{x}}L^\infty_z} \le \sqrt{2}\,\|v\|^{1/2}_{L^4_{\mathbf{x}}L^2_z}\|\partial_z v\|^{1/2}_{L^4_{\mathbf{x}}L^2_z}.$$

Now using (14) and (3), we obtain:

$$\|v\|_{L^4_{\mathbf{x}}L^\infty_z} \le \sqrt{2}\,\|v\|^{1/2}_{L^2_zL^4_{\mathbf{x}}}\|\partial_z v\|^{1/2}_{L^2_zL^4_{\mathbf{x}}} \le C\|u\|^{1/4}_{L^2(\Omega)}\|v\|^{1/4}_{H^1(\Omega)}\|\partial_z v\|^{1/4}_{L^2(\Omega)}\|\partial_z v\|^{1/4}_{H^1(\Omega)}.$$

To finish, it suffices to consider (15) in the left hand side, getting (5). ■

Proof of Lemma 3.3. For any function $g = g(z)$ defined in $z \in (-h(\mathbf{x}), 0)$ with $\mathbf{x} \in S$, we write $g^2(z) = g^2(z') + 2\int_{z'}^z g(s)\,\partial_z g(s)\,ds$. Integrating in $z' \in (-h(\mathbf{x}), 0)$, $h(\mathbf{x})\,g^2(z) \le \|g\|^2_{L^2_z} + 2\|g\|_{L^2_z}\|\partial_z g\|_{L^2_z}$, thus $h(\mathbf{x})^{1/2}\|g\|_{L^\infty_z} \le \|g\|_{L^2_z} + \sqrt{2}\|g\|^{1/2}_{L^2_z}\|\partial_z g\|^{1/2}_{L^2_z}$. Applying the previous inequality to $g = v(\mathbf{x},\cdot)$ and bounding from below $h(\mathbf{x}) \ge h_{\min}$, we get

$$h^{1/2}_{\min}\|v(\mathbf{x},\cdot)\|_{L^\infty_z} \le \|v(\mathbf{x},\cdot)\|_{L^2_z} + \sqrt{2}\|v(\mathbf{x},\cdot)\|^{1/2}_{L^2_z}\|\partial_z v(\mathbf{x},\cdot)\|^{1/2}_{L^2_z}. \tag{17}$$

In order to prove estimate (7), we follow the same argument that in the proof of Lemma 3.2, replacing (16) by (17) and adapting the calculus therein. Finally, (6) follows directly taking $L^2_{\mathbf{x}}$-norm in (17). ■

References

[1] P. Azérad & F. Guillén-González, *Mathematical justification of the hydrostatic approximation in the Primitive Equations of Geophysical fluid dynamics.* Siam J. Math. Anal. ,Vol. 33, No. 4, 847-859 (2001).

[2] O. Besson & M. R. Laydi, *Some Estimates for the Anisotropic Navier-Stokes Equations and for the Hydrostatic Approximation,* M^2AN-Mod. Math. Ana. Nume., Vol. 7, 855-865 (1992).

[3] D. Bresch, F. Guillén-González, N. Masmoudi & M. A. Rodríguez-Bellido, *Asymptotic derivation of a Navier condition for the Primitive Equations,* to appear in Asymptotic Analysis.

[4] D. Bresch, F. Guillén-González, N. Masmoudi & M. A. Rodríguez-Bellido, *On the uniqueness of weak solutions of the two-dimensional primitive equations.* Diff. and Integral Eqs. ,Volume 16, Number 1, 77-94 (2003).

[5] D. Bresch, A. Kazhikhov & J. Lemoine, *On the two-dimensional hydrostatic equations.* Submitted to Arch. Rat. Mech. Anal., (2001).

[6] F. Guillén-González, N. Masmoudi & M. A. Rodríguez-Bellido, *Anisotropic estimates and strong solutions of the Primitive Equations.* Diff. and Integral Eqs. ,Volume 14, Number 11, 1381-1408 (2001).

[7] F. Guillén-González & M.A. Rodríguez-Bellido. *On the strong solutions of the Primitive Equations in 2D domains,* Nonlinear Analysis **50**, 621-646 (2002).

[8] F. Guillén-González, M. A. Rodríguez-Bellido & M. A. Rojas-Medar, *Hydrostatic Stokes equations with non-smooth Neumann data.* Submitted.

[9] R. Lewandowski, *Analyse Mathématique et Océanographie,* Masson, (1997).

[10] J. L. Lions, R. Teman & S. Wang,*On the equations of the large scale Ocean.* Nonlinearity, 5, 1007-1053 (1992).

[11] P. L. Lions, *Mathematical Topics in Fluids Mechanics,* Vol.1, Incompressible Models, Univ. Paris-Dauphine & École Polytechnique, Oxford Univ. Press. Inc., NY, (1996).

[12] J. Pedlosky, *Geophysical fluid dynamics,* Springer-Verlag, (1987).

[13] M. A. Rodríguez Bellido, Ph. Thesis, University of Seville (2001).

[14] M. Ziane, *Regularity Results for Stokes Type Systems.* Applicable Analysis, Vol. 58, 263-292 (1995).