Periodic solutions in unbounded domains for the Boussinesq system [*]

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Abstract

Assuming that the external forces of the system are small enough, the reference temperature being a periodic function, we study the existence, the uniqueness and the regularity of time-periodic solutions for the Boussinesq equations in several classes of unbounded domains of $\mathbb{R}^n$. Our analysis is based on the framework of weak-L^p spaces.

Keywords: Boussinesq equations, Strong Periodic Solutions, Unbounded domains.
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1 Introduction

Let Ω be as either the whole space $\mathbb{R}^n, n \geq 3$, either the half space $\mathbb{R}^n_+, n \geq 3$, either a bounded domain in $\mathbb{R}^n, n \geq 3$, or an exterior domain in $\mathbb{R}^n, n \geq 4$, with boundary $\partial\Omega$ of class $C^{2+\mu}(\mu > 0)$.

We consider the following nonstationary Boussinesq equations in Ω :

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p \;=\; \beta \, \theta \, \mathbf{g} + \boldsymbol{\Psi}, \quad \mathbf{x} \in \Omega, \; t \in \mathbb{R}, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} \;=\; 0, \quad \mathbf{x} \in \Omega, \; t \in \mathbb{R}, \tag{1.2}$$

$$\frac{\partial \theta}{\partial t} - \chi \Delta \theta + (\mathbf{u} \cdot \nabla)\theta \;=\; f, \quad \mathbf{x} \in \Omega, \; t \in \mathbb{R}, \tag{1.3}$$

$$\mathbf{u} \;=\; \mathbf{0} \text{ on } \partial\Omega, \tag{1.4}$$

$$\theta \;=\; 0 \text{ on } \partial\Omega, \tag{1.5}$$

where $\mathbf{g}$ represents the gravitational field at $\mathbf{x}$, f is the reference temperature, $\boldsymbol{\Psi}$ is an external force and ρ, ν, β, χ are positive physical constants which denote, respectively, the density, the kinematic viscosity, the coefficient of volume expansion and the thermal conductance. The unknowns are $\mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^n$, $p(\mathbf{x}, t) \in \mathbb{R}$ and $\theta(\mathbf{x}, t) \in \mathbb{R}$ representing respectively, the velocity field, the pressure and the temperature of the fluid at point $(x, t) \in \Omega \times \mathbb{R}$. Boussinesq equations describe the evolution of the temperature and velocity field of a viscous incompressible Newtonian fluid. For an extensive discussion on the physical origin of the equations (1.1)-(1.3), see [4].

We are interested in the study of the time-periodic solutions for the system (1.1)-(1.5) when the reference temperature is a periodic function with the same period. Without loss of generality, we have taken the constants ρ, ν, β, χ equal to one. To avoid some technical complexities in the study of (1.1)-(1.5), throughout this paper we assume $\boldsymbol{\Psi} = \mathbf{0}$. Several works have been made in the mathematical analysis of system (1.1)-(1.5); see, for instance, [3], [5], [7], [13], [6] and papers cited therein. The time-periodic solutions for the Boussinesq equations in bounded domains was considered in [13]. The analysis was made via the Galerkin's method. Indeed, in [13] it was considered a class of nonlinear evolution equations in a separable Hilbert space generalizing several models of hydrodynamics. However, the study of periodic solutions for system (1.1)-(1.5) has not been investigated in unbounded domains. Hence, the purpose of the present paper is to prove the existence and uniqueness of strong periodic solutions for problem (1.1)-(1.5) in the framework of Semigroups Theory on the Lorentz spaces, more explicitly, on the theory of weak-$\mathbf{L}^p$ spaces. We construct the time-periodic solutions using $\mathbf{L}^{p,q} - \mathbf{L}^{r,s}$ estimates for the semigroups generated by the Stokes and Laplace operators. If Ω is an exterior domain, we need to assume $n \geq 4$ in order to obtain the gradient bounds for the semigroups generated by the Stokes and the Laplace operators in $\mathbf{L}^{(p,\infty)}$ (see Lemma 3.2).

This work is motivated by the existence results of periodic solutions for the Navier-Stokes equations. In unbounded domains, this subject has been investigated in [11], [14], [15], [18] and [19]. In particular, in [14] was proved the existence of a unique time-periodic solution on the whole space $\mathbb{R}^3$ for small external force. The problem in the half-space $\mathbb{R}^3_+$ was considered in [15]. In [11], making use of $\mathbf{L}^p - \mathbf{L}^r$ estimates for the semigroup generated by the Stokes operator, time-periodic solutions were constructed for small time-periodic forces. The stability

of these solutions was considered in [18]. However, the existence of strong time-periodic solutions in general unbounded domains is still an open problem. More complete references, including results for bounded domains, can be found in [11], [14], [15].

This paper is organized as follows. In Section §2, we give some preliminaries about Lorentz spaces and state our main results. Section §3 is devoted to prove the existence and the uniqueness of strong periodic solutions.

2   Preliminaries and Results

Before stating our results we introduce some functional spaces. $\mathbf{C}_{0,\sigma}^{\infty}(\Omega)$ denotes the set of all C^{∞}−real functions $\boldsymbol{\varphi} = (\varphi_1, ..., \varphi_n)$ with compact support in Ω, such that $div\ \boldsymbol{\varphi} = 0$. The closure of $\mathbf{C}_{0,\sigma}^{\infty}$ with respect to norm $\mathbf{L}^r, 1 < r < \infty$, is denoted by $\mathbf{L}_{\sigma}^r(\Omega)$. Let us recall the Helmholtz decomposition: $\mathbf{L}^r(\Omega) = \mathbf{L}_{\sigma}^r(\Omega) \oplus \mathbf{G}^r(\Omega), \ 1 < r < \infty$, where $\mathbf{G}^r(\Omega) = \{\nabla p \in \mathbf{L}^r(\Omega) : p \in L_{loc}^r(\overline{\Omega})\}$ (c.f. [8]). P_r denotes the projection operator from $\mathbf{L}^r(\Omega)$ onto $\mathbf{L}_{\sigma}^r(\Omega)$. The Stokes operator $\mathbf{A}_r = -P_r\Delta$ with domain $D(\mathbf{A}_r) = \{\mathbf{u} \in \mathbf{H}^{2,r}(\Omega) : \mathbf{u}|_{\partial\Omega} = \mathbf{0}\} \cap \mathbf{L}_{\sigma}^r$. It is well known that $-\mathbf{A}_r$ generates a uniformly bounded analytic semigroup $\{e^{-t\mathbf{A}_r}\}_{t\geq 0}$ of class C_0 in $\mathbf{L}_{\sigma}^r$ (c.f. [9]).

We denote by B_q the Laplace operator in $L^q(\Omega), 1 < q < \infty$, with homogeneous Dirichlet boundary conditions: $B_q = -\Delta$ with domain $D(B_q) = W^{2,q}(\Omega) \bigcap W_0^{1,q}(\Omega)$. We recall that the operator $-B_q$ generates a uniformly bounded analytic semigroup $\{e^{-tB_q}\}_{t\geq 0}$ in $L^q(\Omega)$ of class C_0 .

Now we introduce some preliminaries about the Lorentz spaces. For details see [1]. Let $1 < p \leq \infty$ and $1 \leq q \leq \infty$. A Lebesgue measurable function f defined on a domain $\Omega \subset \mathbb{R}^n$ belongs to Lorentz space $L^{(p,q)}(\Omega)$ if the quantity

$$\|f\|_{(p,q)} = \begin{cases} \left(\frac{q}{p}\int_0^{\infty}\left[t^{\frac{1}{p}}f^{**}(t)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} & , \text{ if } \quad 1 < p < \infty, 1 \leq q < \infty \\ \sup_{t>0} t^{\frac{1}{p}}f^{**}(t) & , \text{ if } \quad 1 < p \leq \infty, q = \infty \end{cases},$$

is finite, where

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\ ds, \quad f^*(t) = \inf\{s > 0 : m\{\mathbf{x} \in \Omega : |f(\mathbf{x})| > s\} \leq t\}, t > 0,$$

with m denoting the Lebesgue measure on $\mathbb{R}^n$. The spaces $L^{(p,q)}$ with the norm $\|f\|_{(p,q)}$ are Banach spaces. Note that $L^p(\Omega) = L^{(p,p)}(\Omega)$. When $q = \infty$, $L^{(p,\infty)}(\Omega)$ are called the Marcinkiewicz spaces or weak-L^p spaces. Moreover, $L^{(p,q_1)}(\Omega) \subset L^p(\Omega) \subset L^{(p,q_2)}(\Omega) \subset L^{(p,\infty)}(\Omega)$ for $0 < q_1 \leq p \leq q_2 \leq \infty$. We recall that the space $C_0^{\infty}(\Omega)$ is not dense in $L^{(p,\infty)}(\Omega)$. Borchers and Miyakawa [2] established the following Helmholtz decomposition of the Lorentz spaces extending the operator P_r to a bounded operator on $\mathbf{L}^{(r,d)}(\Omega)$, which we denote by $P_{r,d}$. Setting $\mathbf{L}_{\sigma}^{(r,d)}(\Omega) = Range(P_{r,d})$ and $\mathbf{G}^{(r,d)}(\Omega) = Kernel(P_{r,d})$, then $\mathbf{L}^{(r,d)}(\Omega) = \mathbf{L}_{\sigma}^{(r,d)}(\Omega) \oplus \mathbf{G}^{(r,d)}(\Omega)$,

with $\mathbf{L}_\sigma^{(r,d)}(\Omega) = \{\mathbf{u} \in \mathbf{L}^{(r,d)}(\Omega) : \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0\}$ and $\mathbf{G}^{(r,d)}(\Omega) = \{\nabla v \in \mathbf{L}^{(r,d)}(\Omega) :$
$v \in L_{loc}^{(r,d)}(\bar{\Omega})\}$. For simplicity, we shall abbreviate the projection operator and the Stokes and Laplace operators on Lorentz spaces by P, $\mathbf{A}$, B, respectively. In view of [2], the operators $-\mathbf{A}, -B$ generate bounded analytic semigroups on $\mathbf{L}_\sigma^{(p,q)}(\Omega)$ and $L^{(p,q)}(\Omega)$, respectively. However, we recall that if $q = \infty$, this semigroups are not strongly continuous at $t = 0$.

Applying the operator P on the equations (1.1)-(1.2), from (1.1)-(1.5) we obtain the following problem of parabolic type:

$$\mathbf{u}_t + \mathbf{A}\mathbf{u} + P[(\mathbf{u} \cdot \nabla)\mathbf{u}] = P(\theta\,\mathbf{g}), \quad t \in \mathbb{R} \tag{2.6}$$

$$\theta_t + B\theta + (\mathbf{u} \cdot \nabla)\theta = f, \quad t \in \mathbb{R}. \tag{2.7}$$

The system (2.6)-(2.7), with periodic in time conditions, has associated the following system of integral equations

$$\mathbf{u}(t) = -\int_{-\infty}^t e^{-(t-s)\mathbf{A}} P[(\mathbf{u} \cdot \nabla)\mathbf{u}]\,ds + \int_{-\infty}^t e^{-(t-s)\mathbf{A}} P(\theta\mathbf{g})\,ds \tag{2.8}$$

$$\theta(t) = -\int_{-\infty}^t e^{-(t-s)B}(\mathbf{u} \cdot \nabla)\theta\,ds + \int_{-\infty}^t e^{-(t-s)B} f\,ds. \tag{2.9}$$

Throughout this paper we assume the following assumptions on the external force f and the field $\mathbf{g}$:

Assumption 1.

(CASE 1). If Ω is either the whole space $\mathbb{R}^n$, a bounded domain in $\mathbb{R}^n$, with boundary of class $C^{2+\mu}(\mu > 0)$, or the half space $\mathbb{R}_+^n$, $n \geq 3$, we consider $r, \tilde{r}, q, \tilde{q}$ verifying $2 < r, \tilde{r} < n$, $\frac{n}{2} < q, \tilde{q} < n$, $\frac{1}{r} - \frac{1}{\tilde{r}} < \min\{\frac{2}{n} - \frac{1}{q}, \frac{2}{n} - \frac{1}{\tilde{q}}\}$.

(CASE 2). If Ω is an exterior domain in $\mathbb{R}^n$, $n \geq 4$, with boundary of class $C^{2+\mu}(\mu > 0)$, we consider $r, \tilde{r}, q, \tilde{q}$ such that $\frac{2n}{(n-1)} \leq r, \tilde{r} < n$, $\frac{n}{2} < q, \tilde{q} < n$, $\frac{1}{r} - \frac{1}{\tilde{r}} < \min\{\frac{2}{n} - \frac{1}{q}, \frac{2}{n} - \frac{1}{\tilde{q}}\}$.
For each $r, \tilde{r}$ and $q, \tilde{q}$ we assume that f satisfies

$$f \in BC(\mathbb{R}; L^{(\tilde{p},\infty)}(\Omega) \cap L^{(\tilde{l},\infty)}(\Omega)), \tag{2.10}$$

for $1 < \tilde{p}, \tilde{l} < \infty$ with $\frac{1}{\tilde{r}} + \frac{2}{n} < \frac{1}{\tilde{p}}$, $\frac{1}{\tilde{q}} < \frac{1}{\tilde{l}} < \frac{1}{\tilde{q}} + \frac{1}{n}$ provided $n \geq 4$ in both CASES (1,2). (Note that as $n < 2\tilde{q}$, $\tilde{r} < n$, then the inequality $1/\tilde{q} < 1/\tilde{l} < 1/\tilde{q} + 1/n$ implies that $1/\tilde{l} < 2/n + 1/\tilde{r}$). If $n = 3$, in the CASE 1, we assume that f satisfies

$$\begin{cases} f \in BC(\mathbb{R}, L^{(\tilde{l},\infty)}(\Omega)) \text{ such that} \\ f(s) = B_{\tilde{p},\infty}^{\delta} h(s) \text{ for some } h \in BC(\mathbb{R}, D(B_{\tilde{p},\infty}^{\delta})), \end{cases} \tag{2.11}$$

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for $1 < \tilde{p} < min\{\tilde{r}, \tilde{q}\}$, and $\delta > 0$ satisfying $\frac{3}{2\tilde{p}} + \delta > 1 + max\{1 + \frac{3}{2\tilde{r}}, \frac{1}{2} + \frac{3}{2\tilde{q}}\}$ and $1/\tilde{q} < 1/\tilde{l} < 1/\tilde{q} + 1/3$, where $B_{\tilde{p},\infty}^{\delta}$ denotes the power δ of the operator B on $L^{\tilde{p},\infty}$. With respect to the field $\mathbf{g}$ we make the following assumptions:

$$\mathbf{g} \in \mathbf{L}^{(a,\infty)}(\Omega) \cap \mathbf{L}^{(b,\infty)}(\Omega),$$

where a and b are such that: $\frac{1}{a} > \frac{2}{n} + \frac{1}{r} - \frac{1}{\tilde{r}}, \quad \frac{1}{b} < \frac{1}{n} + \frac{1}{q} - \frac{1}{\tilde{r}}, \quad (b > 1, a > 1).$

Remark 2.1 *Condition (2.11) can be replaced by $f(s) = \nabla \cdot \mathbf{G}(s)$, $\mathbf{G}(s) = (G_1, ..., G_n) \in BC(\mathbb{R}; \mathbf{L}^{(\tilde{p},\infty)}(\Omega))$ with $\nabla \mathbf{G}(t) \in BC(\mathbb{R}; \mathbf{L}^{(\tilde{p},\infty)}(\Omega))^n$ for $1 < \tilde{p} < \infty$ with $1/\tilde{r} + 1/3 < 1/\tilde{p}$. This implies that $f(s) = \Delta h(s)$ for some $h \in BC(\mathbb{R}; D(B_{\tilde{p},\infty}))$.*

Our main results are stated as follows:

Theorem 2.2 *Let f be a periodic function with period $\tau > 0$ (i.e, for all $t \in \mathbb{R}$, $f(t) = f(t+\tau)$) satisfying Assumption 1. Then, if the quantities*

$$\sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{p},\infty)} + \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{l},\infty)}, \qquad n \geq 4, \text{ in the CASES 1 and 2,}$$

$$\sup_{s\in\mathbb{R}} \|h(s)\|_{(\tilde{p},\infty)} + \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{l},\infty)}, \qquad n = 3, \text{ in the CASE 1,}$$

$$\|\mathbf{g}\|_{(b,\infty)} + \|\mathbf{g}\|_{(a,\infty)}, \qquad \text{in the CASES 1 and 2,}$$

are small enough, then there exists a periodic solution $(\mathbf{u}, \theta)$ of (2.8),(2.9), with the same period τ of the external force, in the class $\mathbf{u} \in BC(\mathbb{R}; \mathbf{L}_{\sigma}^{(r,\infty)}(\Omega))$, $\theta \in BC(\mathbb{R}; L^{(\tilde{r},\infty)}(\Omega))$, with $\nabla \mathbf{u} \in BC(\mathbb{R}; \mathbf{L}^{(q,\infty)}(\Omega))^n$, $\nabla \theta \in BC(\mathbb{R}; L^{(\tilde{q},\infty)}(\Omega))^n$. Moreover, within this class, if

$$\sup_{s\in\mathbb{R}} \|\mathbf{u}(s)\|_{(r,\infty)} + \sup_{s\in\mathbb{R}} \|\nabla\mathbf{u}(s)\|_{(q,\infty)}, \quad \sup_{s\in\mathbb{R}} \|\theta(s)\|_{(\tilde{r},\infty)} + \sup_{s\in\mathbb{R}} \|\nabla\theta(s)\|_{(\tilde{q},\infty)}$$

are small enough, then the solution is unique.

Theorem 2.3 *Under the assumptions of Theorem 2.2, if f is Hölder continuous with values in $L^{(r^*,\infty)}(\Omega)$ and $\mathbf{g} \in \mathbf{L}^{(r^*,\infty)}(\Omega)$, then, for all $n < r^* < q^* = nq/(n-q)$, the periodic solution given by Theorem 2.2 satisfies*

1. $\mathbf{u} \in BC(\mathbb{R}; \mathbf{L}_{\sigma}^{(n,\infty)}(\Omega)) \cap C^1(\mathbb{R}; \mathbf{L}_{\sigma}^{r^*}(\Omega))$, $\theta \in BC(\mathbb{R}; L^{(n,\infty)}(\Omega)) \cap C^1(\mathbb{R}; L^{r^*}(\Omega))$.

2. $\mathbf{A}_n\mathbf{u} \in C(\mathbb{R}; \mathbf{L}_{\sigma}^{r^*}(\Omega))$, $B_n\theta \in C(\mathbb{R}; L^{r^*}(\Omega))$.

3. *For all $t \in \mathbb{R}$, (2.6) and (2.7) are satisfied in $\mathbf{L}_{\sigma}^{r^*}(\Omega)$ and $L^{r^*}(\Omega)$, respectively.*

3   Existence, Uniqueness and Regularity of Periodic Solutions

In this section we prove Theorem 2.2 and Theorem 2.3. Let us first recall the Hölder's inequality and some $L^{(r,\infty)} - L^{(p,\infty)}$ estimates for the semigroups $\{e^{-t\mathbf{A}}\}_{t\geq 0}, \{e^{-tB}\}_{t\geq 0}$.

Proposition 3.1 (Generalized Hölder's inequality [16]) *Let* $1 < p_1,\ p_2,\ r\ <\ \infty,\ f\ \in$ $L^{(p_1,q_1)}(\Omega)$ *and* $g \in L^{(p_2,q_2)}(\Omega)$ *where* $\frac{1}{p_1} + \frac{1}{p_2} < 1$, *then* fg *belongs to* $L^{(r,s)}(\Omega)$ *where* $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$, *and* $s \geq 1$ *is any number such that* $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}$. *Moreover,*

$$\|fg\|_{(r,s)} \leq C(r)\|f\|_{(p_1,q_1)}\|g\|_{(p_2,q_2)} \ . \tag{3.12}$$

Lemma 3.2 *([2], [19]).*

1. *Let* Ω *be either the whole space* $\mathbb{R}^n$, *a bounded domain in* $\mathbb{R}^3$ *with boundary* $\partial\Omega$ *of class* $C^{2+\mu}(\mu > 0)$, *or the half space* $\mathbb{R}^n_+$, $n \geq 3$. *Then*

$$\|\nabla^j e^{-tA}\mathbf{a}\|_{(r,1)} \ \leq \ ct^{-n/2(1/p-1/r)-j/2}\|\mathbf{a}\|_{(p,1)},\ 1 < p \leq r < \infty,$$

   for all $\mathbf{a} \in \mathbf{L}^{(p,1)}_\sigma(\Omega), j = 0, 1$ *and all* $t > 0$, *where* $c = c(n,p,r)$.

2. *Let* Ω *be an exterior domain in* $\mathbb{R}^n$, $n \geq 4$ *with boundary* $\partial\Omega$, *of class* $C^{2+\mu}(\mu > 0)$. *Then*

$$\begin{aligned}\|e^{-tA}\mathbf{a}\|_{(r,1)} \ &\leq \ ct^{-n/2(1/p-1/r)}\|\mathbf{a}\|_{(p,1)},\ 1 < p \leq r < \infty,\\ \|\nabla e^{-tA}\mathbf{a}\|_{(r,1)} \ &\leq \ ct^{-n/2(1/p-1/r)-1/2}\|\mathbf{a}\|_{(p,1)},\ 1 < p \leq r \leq n,\end{aligned}$$

   for all $\mathbf{a} \in \mathbf{L}^{(p,1)}_\sigma(\Omega)$ *and all* $t > 0$, *where* $c = c(n,p,r)$.

Remark 3.3 *Estimates in Lemma 3.2 for* $e^{-tA}\mathbf{a}$ *and* $\nabla e^{-tA}\mathbf{a}$ *in the norm* $L^{(r,\infty)}$ *with respect to the data in* $L^{(p,\infty)}$ *are true, because they are obtained by duality. Similar estimates hold for the semigroup* $\{e^{-tB}\}_{t\geq 0}$. *The estimates above hold in the particular case of* L^p *spaces (c.f. [11]).*

Lemma 3.4 *([2]). Let* Ω *be as the CASE 1 and CASE 2 and suppose that* $1 < q < n$, $1 \leq d \leq \infty$ *and* $q^* = nq/(n-q)$. *If* $\phi \in L^{(p,\infty)}(\Omega)$ *for some* $p < \infty$ *and* $\nabla\phi \in L^{(q,d)}(\Omega)^n$, *then* $\phi \in L^{(q^*,d)}(\Omega)$ *and the estimate* $\|\phi\|_{(q^*,d)} \leq C\|\nabla\phi\|_{(q,d)}$ *holds with* $C > 0$ *independent of* ϕ.

We denote by X the space of scalar functions $\{u \in BC(\mathbb{R}; L^{(\tilde{r},\infty)}) : \nabla u \in BC(\mathbb{R}; L^{(\tilde{q},\infty)})^n\}$ with the norm $\|\cdot\|_X$ defined as

$$\|u\|_X \equiv \sup_{s\in\mathbb{R}} \|u(s)\|_{(\tilde{r},\infty)} + \sup_{s\in\mathbb{R}} \|\nabla u(s)\|_{(\tilde{q},\infty)}.$$

We also defined by $\mathbf{Y}$ the space of vector functions $\{\mathbf{u} \in BC(\mathbb{R}; \mathbf{L}_\sigma^{(r,\infty)}(\Omega)) : \nabla\mathbf{u} \in BC(\mathbb{R}; \mathbf{L}^{(q,\infty)}(\Omega))^n\}$ with the norm $\|\cdot\|_Y$ defined as

$$\|\mathbf{u}\|_Y \equiv \sup_{s\in\mathbb{R}} \|\mathbf{u}(s)\|_{(r,\infty)} + \sup_{s\in\mathbb{R}} \|\nabla\mathbf{u}(s)\|_{(q,\infty)}.$$

X and $\mathbf{Y}$ are Banach spaces. We define the following operators $\mathbf{F}_1$ and G on $\mathbf{Y} \times \mathbf{Y}$ and $\mathbf{Y} \times X$, respectively, by

$$\mathbf{F}_1(\mathbf{u},\mathbf{v})(t) \;=\; -\int_{-\infty}^t e^{-(t-s)\mathbf{A}} P[(\mathbf{u}\cdot\nabla)\mathbf{v}](s)\,ds, \tag{3.13}$$

$$G(\mathbf{u},\theta)(t) \;=\; -\int_{-\infty}^t e^{-(t-s)B}(\mathbf{u}\cdot\nabla)\theta(s)\,ds. \tag{3.14}$$

3.1 Proof of Theorem 2.2

We construct a periodic solution of Problem (2.8)-(2.9) according to the following scheme:

$$\mathbf{u}_{m+1}(t) \;=\; \mathbf{F}(\mathbf{u}_m,\theta_m)(t), \quad \theta_{m+1}(t) \;=\; \theta_0(t) + G(\mathbf{u}_m,\theta_m)(t), \tag{3.15}$$

where

$$\mathbf{u}_0(t) \;=\; \int_{-\infty}^t e^{-(t-s)\mathbf{A}} P(\theta_0\mathbf{g})\,ds, \;\; \theta_0(t) = \int_{-\infty}^t e^{-(t-s)B} f\,ds, \tag{3.16}$$

$$\mathbf{F}(\mathbf{u}_m,\theta_m)(t) \;=\; \mathbf{F}_1(\mathbf{u}_m,\mathbf{u}_m)(t) + \int_{-\infty}^t e^{-(t-s)\mathbf{A}}\{P(\mathbf{g}\theta_m)\}\,ds, \tag{3.17}$$

$$G(\mathbf{u}_m,\theta_m)(t) \;=\; -\int_{-\infty}^t e^{-(t-s)B}(\mathbf{u}_m\cdot\nabla)\theta_m(s)\,ds. \tag{3.18}$$

Remark 3.5 *In (1.1), when $\boldsymbol{\Psi}$ is not zero, in the scheme above we consider*
$\mathbf{u}_0(t) = \int_{-\infty}^t e^{-(t-s)\mathbf{A}} P(\boldsymbol{\Psi})(s)\,ds$ *and* $\mathbf{u}_{m+1} = \mathbf{u}_0(t) + \mathbf{F}(\mathbf{u}_m,\theta_m)(t).$

Let us first obtain some estimates for approximations above. We shall need the following lemmas.

Lemma 3.6 *Let $r, \tilde{r}, q$ and $\tilde{q}$ be as Theorem 2.2. Then, we have*

$$\sup_{s\in\mathbb{R}} \|\mathbf{F}_1(\mathbf{u},\mathbf{v})\|_{(r,\infty)} \;\leq\; c_1 \sup_{s\in\mathbb{R}} \|\mathbf{u}(s)\|_{(r,\infty)}\Big(\sup_{s\in\mathbb{R}} \|\mathbf{v}(s)\|_{(r,\infty)} + \sup_{s\in\mathbb{R}} \|\nabla\mathbf{v}(s)\|_{(q,\infty)}\Big),$$

$$\sup_{s\in\mathbb{R}} \|\nabla\mathbf{F}_1(\mathbf{u},\mathbf{v})\|_{(q,\infty)} \;\leq\; c_1 \sup_{s\in\mathbb{R}} \|\nabla\mathbf{v}(s)\|_{(q,\infty)}\Big(\sup_{s\in\mathbb{R}} \|\mathbf{u}(s)\|_{(r,\infty)} + \sup_{s\in\mathbb{R}} \|\nabla\mathbf{u}(s)\|_{(q,\infty)}\Big),$$

$$\sup_{s\in\mathbb{R}} \|G(\mathbf{u},\theta)\|_{(\tilde{r},\infty)} \;\leq\; c_2 \sup_{s\in\mathbb{R}} \|\mathbf{u}(s)\|_{(r,\infty)}\Big(\sup_{s\in\mathbb{R}} \|\theta(s)\|_{(\tilde{r},\infty)} + \sup_{s\in\mathbb{R}} \|\nabla\theta(s)\|_{(\tilde{q},\infty)}\Big),$$

$$\sup_{s\in\mathbb{R}} \|\nabla G(\mathbf{u},\theta)\|_{(\tilde{q},\infty)} \;\leq\; c_2 \sup_{s\in\mathbb{R}} \|\nabla\theta(s)\|_{(\tilde{q},\infty)}\Big(\sup_{s\in\mathbb{R}} \|\mathbf{u}(s)\|_{(r,\infty)} + \sup_{s\in\mathbb{R}} \|\nabla\mathbf{u}(s)\|_{(q,\infty)}\Big),$$

for every $\mathbf{u},\mathbf{v} \in \mathbf{Y}$, $\theta \in X$, where $c_1 = c_1(n,r,q), c_2 = c_2(n,r,q,\tilde{r},\tilde{q}).$

Proof. The proof is an application of Lemma 3.2. In fact,

$$G(\mathbf{u}, \theta)(t) = -\int_{-\infty}^{t-1} e^{-(t-s)B}(\mathbf{u} \cdot \nabla)\theta(s)ds - \int_{t-1}^{t} e^{-(t-s)B}(\mathbf{u} \cdot \nabla)\theta(s)\,ds = G_1(t) + G_2(t).$$

Then for all $\psi \in C_0^\infty$ and for all $t \in \mathbb{R}$, we have

$$\begin{aligned}
|(G_1(t), \psi)| &\leq \int_{-\infty}^{t-1} \|\nabla e^{-(t-s)B}\psi\|_{((r\tilde{r}/(r+\tilde{r}))',1)} \|\theta\mathbf{u}\|_{(r\tilde{r}/(r+\tilde{r}),\infty)} ds \\
&\leq c \sup_{s\in\mathbb{R}} \|\theta(s)\|_{(\tilde{r},\infty)} \sup_{s\in\mathbb{R}} \|\mathbf{u}(s)\|_{(r,\infty)} \int_{-\infty}^{t-1} (t-s)^{-n/2r-1/2} \|\psi\|_{(\tilde{r}',1)}.
\end{aligned}$$

Hence, by duality, for all $t \in \mathbb{R}$, $\|G_1(t)\|_{(\tilde{r},\infty)} \leq c \sup_{s\in\mathbb{R}} \|\theta(s)\|_{(\tilde{r},\infty)} \sup_{s\in\mathbb{R}} \|\mathbf{u}(s)\|_{(r,\infty)}$.

$$\begin{aligned}
\|G_2(t)\|_{(\tilde{r},\infty)} &\leq \int_{t-1}^{t} (t-s)^{-n/2(1/r+1/\tilde{q}-1/\tilde{r})} \|\mathbf{u}(s)\|_{(r,\infty)} \|\nabla\theta(s)\|_{(\tilde{q},\infty)} ds \\
&\leq c \sup_{s\in\mathbb{R}} \|\mathbf{u}(s)\|_{(r,\infty)} \sup_{s\in\mathbb{R}} \|\nabla\theta(s)\|_{(\tilde{q},\infty)}.
\end{aligned}$$

Now, using Lemma 3.2 and Lemma 3.4 (for $d = \infty$), we get

$$\begin{aligned}
\|\nabla G(\mathbf{u}, \theta)\|_{(\tilde{q},\infty)} &\leq \int_{-\infty}^{t-1} \|\nabla e^{-(t-s)B}(\mathbf{u} \cdot \nabla)\theta(s)\|_{(\tilde{q},\infty)}\,ds \\
&+ \int_{t-1}^{t} \|\nabla e^{-(t-s)B}[(\mathbf{u} \cdot \nabla)\theta](s)\|_{(\tilde{q},\infty)}\,ds \\
&\leq c \sup_{s\in\mathbb{R}} \|\mathbf{u}(s)\|_{(r,\infty)} \sup_{s\in\mathbb{R}} \|\nabla\theta(s)\|_{(\tilde{q},\infty)} \int_{-\infty}^{t-1} (t-s)^{-n/2r-1/2}\,ds \\
&+ c \sup_{s\in\mathbb{R}} \|\mathbf{u}(s)\|_{(q^*,\infty)} \sup_{s\in\mathbb{R}} \|\nabla\theta(s)\|_{(\tilde{q},\infty)} \int_{t-1}^{t} (t-s)^{-n/2q}\,ds \\
&\leq c \left(\sup_{s\in\mathbb{R}} \|\mathbf{u}(s)\|_{(r,\infty)} + \sup_{s\in\mathbb{R}} \|\nabla\mathbf{u}(s)\|_{(q,\infty)}\right) \sup_{s\in\mathbb{R}} \|\nabla\theta(s)\|_{(\tilde{q},\infty)},
\end{aligned}$$

for all $t \in \mathbb{R}$ and $c = c(n, r, q)$. This complete the proof of the last two estimates of lemma. The first two estimates are obtained similarly. ∎

Lemma 3.7 *Let θ_0 be defined as in (3.16). Then $\theta_0 \in X$.*

Proof. If f satisfies (2.10), then using Lemma 3.2 we obtain

$$\begin{aligned}
\|\theta_0(t)\|_{(\tilde{r},\infty)} &\leq c \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{p},\infty)} \int_{-\infty}^{t-1} (t-s)^{-n/2(1/\tilde{p}-1/\tilde{r})}\,ds \\
&+ c \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{l},\infty)} \int_{t-1}^{t} (t-s)^{-n/2(1/\tilde{l}-1/\tilde{r})}\,ds.
\end{aligned}$$

This is valid for all $t \in \mathbb{R}$. The constant $c = c(n, \tilde{r}, \tilde{p}, \tilde{l})$. From (2.10), that is, $1/\tilde{r}+2/n < 1/\tilde{p}$ and $1/\tilde{l} < 2/n + 1/\tilde{r}$, we conclude that each integral above is finite and consequently, $\|\theta_0(t)\|_{(\tilde{r},\infty)} \leq$

$c_1 \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{p},\infty)} + c_2 \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{l},\infty)}$, where $c_1 = c(n,\tilde{p},\tilde{r})$ and $c_2 = c(n,\tilde{l},\tilde{r})$. A similar analysis proves that

$$
\begin{aligned}
\|\nabla\theta_0(t)\|_{(\tilde{q},\infty)} \;\leq\; & \widetilde{c}_1 \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{p},\infty)} \int_{-\infty}^{t-1} (t-s)^{-n/2(1/\tilde{p}-1/\tilde{q})-1/2}\,ds \\
& + \; \widetilde{c}_2 \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{l},\infty)} \int_{t-1}^{t} (t-s)^{-n/2(1/\tilde{l}-1/\tilde{q})-1/2}\,ds,
\end{aligned}
$$

for all $t \in \mathbb{R}$ and $\widetilde{c}_1 = c(n,\tilde{p},\tilde{q})$ and $\widetilde{c}_2 = c(n,\tilde{l},\tilde{q})$. As $1/\tilde{p} > 1/\tilde{r} + 2/n > 1/n + 1/\tilde{q}$ and $1/\tilde{l} < 1/\tilde{q} + 1/n$, the two integrals above converge.

Now, if $n = 3$, the previous analysis is wrong because it will be necessary $3/2(1/\tilde{p}-1/\tilde{r}) > 1$, with $\tilde{p} > 1$ and this is not possible. Consequently, we assume a new condition; in fact, if f satisfies (2.11), using the following estimate (which is a consequence of the analytic properties of the semigroup):

$$
\|B^\delta e^{-tB}\mathbf{a}\|_{(\tilde{p},\infty)} \leq C\,t^{-\delta}\|\mathbf{a}\|_{(\tilde{p},\infty)}, \ \forall\,\mathbf{a}\in L^{(\tilde{p},\infty)}, \ t>0, \ c=c(\tilde{p},\delta), \ \delta\geq 0,
$$

and using Lemma 3.2, we obtain

$$
\begin{aligned}
\|\theta_0(t)\|_{(\tilde{r},\infty)} &\leq \int_{-\infty}^{t-1} \|e^{-(t-s)B}B^\delta h(s)\|_{(\tilde{r},\infty)}\,ds + \int_{t-1}^{t} \|e^{-(t-s)B}f(s)\|_{(\tilde{r},\infty)}\,ds \\
&\leq c\int_{-\infty}^{t-1} (t-s)^{-3/2(1/\tilde{p}-1/\tilde{r})}\|B^\delta e^{-(t-s)B/2}h(s)\|_{(\tilde{p},\infty)}ds + c\int_{t-1}^{t} (t-s)^{-3/2(1/\tilde{l}-1/\tilde{r})}\|f(s)\|_{(\tilde{l},\infty)}ds \\
&\leq c\int_{-\infty}^{t-1} (t-s)^{-3/2(1/\tilde{p}-1/\tilde{r})-\delta}\|h(s)\|_{(\tilde{p},\infty)}\,ds + \sup_{s\in\mathbb{R}}\|f(s)\|_{(\tilde{l},\infty)}\int_{t-1}^{t} (t-s)^{-3/2(1/\tilde{l}-1/\tilde{r})}\,ds \\
&\leq c\left(\sup_{s\in\mathbb{R}} \|h(s)\|_{(\tilde{p},\infty)} + \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{l},\infty)}\right),
\end{aligned}
$$

for all $t \in \mathbb{R}$ with $c = c(n,\tilde{r},\tilde{p},\tilde{l},\delta)$. A similar estimate can be obtained for $\|\nabla\theta_0\|_{(\tilde{q},\infty)}$, $(n=3)$. This proves the lemma. ∎

Now we will estimate the terms $\mathbf{F}(\mathbf{u}_m,\theta_m)$ and $G(\mathbf{u}_m,\theta_m)$. We start with the following lemma

Lemma 3.8 *The terms* $\|\mathbf{F}(\mathbf{u}_m,\theta_m)\|_Y$, $\|G(\mathbf{u}_m,\theta_m)\|_X$ *given by (3.17),(3.18) satisfy*

$$
\|\mathbf{F}(\mathbf{u}_m,\theta_m)\|_Y \;\leq\; 2c_1\|\mathbf{u}_m\|_Y^2 + c_3\|\theta_m\|_X, \tag{3.19}
$$

$$
\|G(\mathbf{u}_m,\theta_m)\|_X \;\leq\; 2c_2\|\mathbf{u}_m\|_Y\|\theta_m\|_X, \tag{3.20}
$$

where c_1, c_2 *are as in Lemma 3.6 and* c_3 *depends on* $\mathbf{g}$ *but is independent of* m.

Proof. We will prove that

$$
\left\|\int_{-\infty}^{t} e^{-(t-s)\mathbf{A}} P(\mathbf{g}\theta_m)(s)\,ds\right\|_Y \leq c_3\|\theta_m\|_X. \tag{3.21}
$$

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In fact,

$$\left\| \int_{-\infty}^{t} e^{-(t-s)\mathbf{A}} P(\mathbf{g}\theta_m)(s) \, ds \right\|_{(r,\infty)} \leq \int_{-\infty}^{t-1} \| e^{-(t-s)\mathbf{A}} P(\mathbf{g}\theta_m)(s) \|_{(r,\infty)} \, ds$$
$$+ \int_{t-1}^{t} \| e^{-(t-s)\mathbf{A}} P(\mathbf{g}\theta_m)(s) \|_{(r,\infty)} \, ds.$$

Note that

$$\int_{-\infty}^{t-1} \| e^{-(t-s)\mathbf{A}} P(\mathbf{g}\theta_m)(s) \|_{(r,\infty)} \, ds \leq c \, \|\mathbf{g}\|_{(a,\infty)} \sup_{s \in \mathbb{R}} \|\theta_m(s)\|_{(\tilde{r},\infty)} \int_{-\infty}^{t-1} (t-s)^{\gamma} \, ds,$$

where $\gamma = -n/2(1/a + 1/\tilde{r} - 1/r)$. As $1/a > 2/n - 1/\tilde{r} + 1/r$, the last integral converges. Moreover, for $\xi = -n/2(1/b + 1/\tilde{r} - 1/r)$ we have

$$\int_{t-1}^{t} \| e^{-(t-s)\mathbf{A}} P(\mathbf{g}\theta_m)(s) \|_{(r,\infty)} \, ds \leq c \, \|\mathbf{g}\|_{(b,\infty)} \sup_{s \in \mathbb{R}} \|\theta_m(s)\|_{(\tilde{r},\infty)} \int_{t-1}^{t} (t-s)^{\xi} \, ds,$$

By Assumption 1, $\mathbf{g} \in \mathbf{L}^{(b,\infty)}(\Omega)$ with $b > 1$ and $1/b < 1/n + 1/q - 1/\tilde{r}$. As $r < n$, $n/2 < q$, we have that $1/r + 1/n > 2/n > 1/q$, and therefore $1/b < 1/n + 1/q - 1/\tilde{r} < 1/n + 1/n = 2/n$ which implies that $1/b < 2/n + 1/r - 1/\tilde{r}$ and thus, the last integral converges.

On the other hand

$$\left\| \nabla \int_{-\infty}^{t} e^{-(t-s)\mathbf{A}} P(\mathbf{g}\theta_m)(s) \right\|_{(q,\infty)} ds \leq \int_{-\infty}^{t-1} \| \nabla e^{-(t-s)\mathbf{A}} P(\mathbf{g}\theta_m)(s) \|_{(q,\infty)} \, ds$$
$$+ \int_{t-1}^{t} \| \nabla e^{-(t-s)\mathbf{A}} P(\mathbf{g}\theta_m)(s) \|_{(q,\infty)} \, ds,$$

where we estimate the first integral as

$$\int_{-\infty}^{t-1} \| \nabla e^{-(t-s)\mathbf{A}} P(\mathbf{g}\theta_m)(s) \|_{(q,\infty)} \, ds \leq c \, \|\mathbf{g}\|_{(a,\infty)} \sup_{s \in \mathbb{R}} \|\theta_m(s)\|_{(\tilde{r},\infty)} \int_{-\infty}^{t-1} (t-s)^{\zeta} \, ds,$$

for $\zeta = -n/2(1/a + 1/\tilde{r} - 1/q) - 1/2$. As $n < q$ and $1/\tilde{r} - 1/r > 2/n - 1/a$, we conclude that $1/a + 1/\tilde{r} - 1/q > 2/n + 1/r - 1/q > 1/n + 1/r > 1/n$. Hence the last integral converges. Analogously, we can show that

$$\int_{t-1}^{t} \| \nabla e^{-(t-s)\mathbf{A}} P(\mathbf{g}\theta_m)(s) \|_{(q,\infty)} \, ds \leq c \, \|\mathbf{g}\|_{(b,\infty)} \sup_{s \in \mathbb{R}} \|\theta_m(s)\|_{(\tilde{r},\infty)} \int_{t-1}^{t} (t-s)^{\xi} \, ds,$$

with $\xi = -n/2 \, (1/b + 1/\tilde{r} - 1/q) - 1/2$ and thus the integral converges. Hence, we prove inequality (3.21) with $c_3 = c \left(\|\mathbf{g}\|_{(b,\infty)} + \|\mathbf{g}\|_{(a,\infty)} \right)$ and $c = c(n, a, b, r, q, \tilde{r}, \tilde{q})$ independent of m. Therefore, from Lemma 3.6 and estimate (3.21), we obtain (3.19). Inequality (3.20) is obtained applying directly the last two inequalities of Lemma 3.6. ∎

After these Lemmas we back to the proof of Theorem 2.2.

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Existence.

From scheme (3.15), Lemma 3.7 and Lemma 3.8, we obtain:

$$\|\mathbf{u}_{m+1}\|_Y \leq 2c_1\|\mathbf{u}_m\|_Y^2 + c_3\|\theta_m\|_X, \tag{3.22}$$

$$\|\theta_{m+1}\|_X \leq \|\theta_0\|_X + 2c_2\|\mathbf{u}_m\|_Y\|\theta_m\|_X. \tag{3.23}$$

Let $a_m = \max\{\|\mathbf{u}_m\|_Y, \|\theta_m\|_X\}$, $m = 1, 2, \ldots$ $a_0 = \|\theta_0\|_X$. Therefore, from (3.22) and (3.23) it follows that $a_{m+1} \leq a_0 + \tilde{c}a_m^2 + c_3 a_m$, $\tilde{c} = max(2c_1, 2c_2)$. Hence, if

$$c_3 < 1, \quad 4a_0\tilde{c} < (1 - c_3)^2, \tag{3.24}$$

then, the sequence $\{a_m\}_{m=0}^{\infty}$ is bounded and

$$a_m \leq \frac{(1 - c_3) - \sqrt{(1 - c_3)^2 - 4\tilde{c}a_0}}{2\tilde{c}} \equiv k, \ \forall m = 0, 1, \ldots \Rightarrow a_m \leq k < \frac{1}{2\tilde{c}}. \tag{3.25}$$

From now on, we assume (3.24) (Note that this condition implies a small condition for f). Making $\mathbf{w}_m = \mathbf{u}_m - \mathbf{u}_{m-1}$ ($\mathbf{u}_{-1} \equiv 0$), $\Theta_m = \theta_m - \theta_{m-1}$, ($\theta_{-1} \equiv 0$), we have

$$\mathbf{w}_{m+1}(t) = \mathbf{F}_1(\mathbf{w}_m, \mathbf{u}_m)(t) + \mathbf{F}_1(\mathbf{u}_{m-1}, \mathbf{w}_m)(t) + \int_{-\infty}^{t} e^{-(t-s)\mathbf{A}} P(\mathbf{g}\Theta_m)(s)\, ds,$$

$$\Theta_{m+1}(t) = G(\mathbf{w}_m, \theta_m)(t) + G(\mathbf{u}_{m-1}, \Theta_m)(t).$$

This equality implies that

$$\begin{aligned}\|\mathbf{w}_{m+1}\|_Y &\leq 2c_1\left(\|\mathbf{w}_m\|_Y\|\mathbf{u}_m\|_Y + \|\mathbf{u}_{m-1}\|_Y\|\mathbf{w}_m\|_Y\right) + c_3\|\Theta_m\|_X \\ &\leq 2\tilde{c}k\left(\|\mathbf{w}_m\|_Y + \|\Theta_m\|_X\right),\end{aligned} \tag{3.26}$$

provided $c_3 \leq \tilde{c}k$ (this condition and (3.24) imply a small condition for the field $\mathbf{g}$ in the norms $\|\cdot\|_{(a,\infty)}$ and $\|\cdot\|_{(b,\infty)}$). Moreover,

$$\|\Theta_{m+1}\|_X \leq 2c_2\|\mathbf{w}_m\|_Y\|\theta_m\|_X + 2c_2\|\mathbf{u}_{m-1}\|_Y\|\Theta_m\|_X \leq 2c_2 k\left(\|\mathbf{w}_m\|_Y + \|\Theta_m\|_X\right). \tag{3.27}$$

From (3.26)-(3.27), we obtain

$$\begin{aligned}\max\{\|\mathbf{w}_{m+1}\|_Y, \|\Theta_{m+1}\|_X\} &\leq 2\tilde{c}k \max\left\{\|\mathbf{w}_m\|_Y, \|\Theta\|_X\right\} \leq \ldots \\ &\leq (2\tilde{c}k)^{m+1}a_0, \ \forall m = 0, 1, \ldots\end{aligned} \tag{3.28}$$

Note that $\mathbf{u}_m(t) = \sum_{j=0}^{m} \mathbf{w}_j(t)$, $\theta_m(t) = \sum_{j=0}^{m} \Theta_j(t)$. Since $2\tilde{c}k < 1$ (by (3.25)), from (3.28) we can conclude that there exist functions $\mathbf{u} \in \mathbf{Y}$, $\theta \in X$ such that when $m \longrightarrow \infty$, $\mathbf{u}_m \longrightarrow \mathbf{u}$ in $\mathbf{Y}$, $\theta_m \longrightarrow \theta$ in X. Note that

$$\begin{aligned}\|\mathbf{F}_1(\mathbf{u}_m, \mathbf{u}_m)(t) - \mathbf{F}_1(\mathbf{u}, \mathbf{u})(t)\|_Y &\leq \|\mathbf{F}_1(\mathbf{u}_m - \mathbf{u}, \mathbf{u}_m)(t)\|_Y + \|\mathbf{F}_1(\mathbf{u}, \mathbf{u}_m - \mathbf{u})(t)\|_Y \\ &\leq 2c_1\|\mathbf{u}_m - \mathbf{u}\|_Y\|\mathbf{u}_m\|_Y + 2c_1\|\mathbf{u}\|_Y\|\mathbf{u}_m - \mathbf{u}\|_Y < \|\mathbf{u}_m - \mathbf{u}\|_Y, \ \forall m.\end{aligned}$$

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Then, in **Y**

$$\mathbf{F}_1(\mathbf{u}_m, \mathbf{u}_m)(t) \longrightarrow \mathbf{F}_1(\mathbf{u}, \mathbf{u})(t). \tag{3.29}$$

Analogously,

$$
\begin{aligned}
\|G(\mathbf{u}_m, \theta_m)(t) - G(\mathbf{u}, \theta)(t)\|_X &\leq 2c_2 \|\mathbf{u}_m - \mathbf{u}\|_Y \|\theta_m\|_X + 2c_2 \|\mathbf{u}\|_Y \|\theta_m - \theta\|_X \\
&< \|\mathbf{u}_m - \mathbf{u}\|_Y + \|\theta_m - \theta\|_X, \ \forall m.
\end{aligned}
$$

Then, in X

$$G(\mathbf{u}_m, \theta_m)(t) \longrightarrow G(\mathbf{u}, \theta)(t). \tag{3.30}$$

Finally, when $m \longrightarrow \infty$,

$$\left\| - \int_{-\infty}^{t} e^{-(t-s)\mathbf{A}} P(\mathbf{g}(\theta_m - \theta))(s) \, ds \right\|_Y \leq c_3 \|\theta_m - \theta\|_X \longrightarrow 0. \tag{3.31}$$

From (3.29), (3.30) and (3.31) we conclude that $(\mathbf{u}, \theta)$ is a solution of the system of integral equations (2.8)-(2.9).

Periodicity.

Being f a periodic function with period $\tau > 0$, the functions $\mathbf{u}_m$ and θ_m are also periodic with the same period τ for all $m = 0, 1, 2, ...$ Consequently, the limit $(\mathbf{u}, \theta)$ is periodic with period τ.

Uniqueness.

Suppose that $(\mathbf{u}_1, \theta_1)$ is another solution of (2.8)-(2.9), such that $\|\mathbf{u}_1\|_Y \leq k$, $\|\theta_1\|_X \leq k$, being k the constant of (3.25). Working as the proof of existence we get

$$
\begin{aligned}
\|\theta - \theta_1\|_X &\leq 2c_2 \, k \, \|\mathbf{u} - \mathbf{u}_1\|_Y + 2c_2 \, k \, \|\theta - \theta_1\|_X, \\
\|\mathbf{u} - \mathbf{u}_1\|_Y &\leq c_3 \, \|\theta - \theta_1\|_X + 2c_1 \, k \, \|\mathbf{u} - \mathbf{u}_1\|_Y.
\end{aligned}
$$

Hence, if $M \equiv \max\{\|\mathbf{u} - \mathbf{u}_1\|_Y, \|\theta - \theta_1\|_X\}$, we have $M \leq 2\,\widetilde{c}\,k\,M$, because $c_3 \leq \widetilde{c}k$, which implies that $\theta = \theta_1$ and $\mathbf{u} = \mathbf{u}_1$. Hence, the proof of Theorem 2.2 is finished.

3.2 Strong Solution. Proof of Theorem 2.3.

In this subsection we prove that if f and $\mathbf{g}$ satisfy adequate regularity conditions, then the periodic solution $(\mathbf{u}, \theta)$ constructed in Theorem 2.2 is also a solution of the differential system (1.1)-(1.5). For the proof of Theorem 2.3, we need a result about local existence of strong solutions for the initial boundary value problem associated to (1.1)-(1.5) that will be presented as Theorem 3.10. This result follows the arguments of Kato [10]. Let us first give the definition of strong solution for the initial value problem (1.1)-(1.5).

Definition 3.9 Let $\mathbf{a} \in \mathbf{L}_\sigma^{(n,\infty)}(\Omega)$, $b \in L^{(n,\infty)}(\Omega)$, $n < r^* < \infty$, f being Hölder continuous with values in $L^{(r^*,\infty)}(\Omega)$ and $\mathbf{g} \in \mathbf{L}^{(r^*,\infty)}(\Omega)$. A pair $(\mathbf{v}, w)$ defined on $(t_0, t_1) \times \Omega$ is called a **strong solution of (1.1)-(1.5) of class** $S_{r^*}(t_0, t_1)$, with initial value $(\mathbf{a}, b)$ if

1. $\mathbf{v} \in BC_w([t_0, t_1); \mathbf{L}_\sigma^{(n,\infty)}(\Omega)) \cap C^1((t_0, t_1); \mathbf{L}_\sigma^{r^*}(\Omega))$,

   $w \in BC_w([t_0, t_1); L^{(n,\infty)}(\Omega)) \cap C^1((t_0, t_1); L^{r^*}(\Omega))$,

2. $\mathbf{A}\mathbf{v} \in C((t_0, t_1); \mathbf{L}_\sigma^{r^*}(\Omega))$, $Bw \in C((t_0, t_1); L^{r^*}(\Omega))$, $t_0 < t < t_1$,

3. $\mathbf{v}_t + \mathbf{A}\mathbf{v} + P[(\mathbf{v} \cdot \nabla)\mathbf{v}] = P(w\mathbf{g})$ in $\mathbf{L}_\sigma^{r^*}(\Omega)$, for $x \in \Omega$, $t_0 < t < t_1$,

4. $w_t + Bw + (\mathbf{v} \cdot \nabla w) = f$ in $L^{r^*}(\Omega)$, for $x \in \Omega$, $t_0 < t < t_1$,

where BC_w denotes the class of bounded and weakly-$*$ continuous functions, together with $\lim_{t \to t_0^+}(\mathbf{v}(t), \boldsymbol{\phi}) = (\mathbf{a}, \boldsymbol{\phi})$, $\lim_{t \to t_0^+}(w(t), \varphi) = (b, \varphi)$, for all $\boldsymbol{\phi} \in \mathbf{L}_\sigma^{(n/(n-1),1)}(\Omega)$, $\varphi \in L^{(n/(n-1),1)}(\Omega)$.

Our result about the local existence of strong solutions now reads

Theorem 3.10 (i) (Existence). Let $n/2 < q < n$ and $1 < l < \infty$ be such that $1/q < 1/l < 1/q + 1/n$. Supposed that $\mathbf{a} \in \mathbf{L}_\sigma^{(n,\infty)}(\Omega) \cap \mathbf{L}_\sigma^{(q^*,\infty)}(\Omega)$, $b \in L^{(n,\infty)}(\Omega) \cap L^{(q^*,\infty)}(\Omega)$, where $q^* = nq/(n-q)$, $f \in BC(\mathbb{R}; L^{(l,\infty)}(\Omega))$ being Hölder continuous with values in $L^{(r^*,\infty)}(\Omega)$, $\mathbf{g} \in \mathbf{L}^{(b,\infty)}(\Omega) \cap \mathbf{L}^{(r^*,\infty)}(\Omega)$ with $b > n/2$ and $n < r^* < q^*$. Then, there exists $T \in (0, 1]$ such that for all $t_0 \in \mathbb{R}$ there exists a strong solution of class $S_{r^*}(t_0, t_0 + T)$ of problem (1.1)-(1.5) at $(t_0, t_0 + T)$ with initial value $\mathbf{v}(t_0) = \mathbf{a}, w(t_0) = b$. Moreover, the solution satisfies $\mathbf{v} \in BC((t_0, t_0 + T); \mathbf{L}_\sigma^{(q^*,\infty)}(\Omega))$, $w \in BC((t_0, t_0 + T); L^{(q^*,\infty)}(\Omega))$, with

$$\sup_{t_0 < t < t_0 + T} \|\mathbf{v}(t)\|_{(q^*,\infty)} \le C_1, \quad \sup_{t_0 < t < t_0 + T} \|w(t)\|_{(q^*,\infty)} \le C_2, \tag{3.32}$$

where C_1, C_2 are independent of t_0. Here T is estimated as

$$T \equiv \left[\frac{\tilde{k}}{c_1 \max\{\|\mathbf{a}\|_{(n/\alpha,\infty)}, \|b\|_{(n/\alpha,\infty)} + \|f\|_{BC(\mathbb{R};L^{(l,\infty)})}\}} \right]^{\frac{2}{1-\alpha}} \tag{3.33}$$

with $\tilde{k}/c_1 = \tilde{k}/c_1(n, q, l)$, $\alpha = n/q^*$.

(ii) (Uniqueness). There exists a constant $\gamma = \gamma(n, r^*)$ such that any solution $(\mathbf{v}, w)$ in the above class, satisfying

$$\limsup_{t \to t_0^+} t^{n/2(1/n - 1/r^*)} \|\mathbf{v}(t)\|_{r^*} \le \gamma, \quad \limsup_{t \to t_0^+} t^{n/2(1/n - 1/r^*)} \|w(t)\|_{r^*} \le \gamma,$$

is unique.

Proposition 3.11 *Let $n/2 < q < n$ and $1 < l < \infty$ be such that $1/q < 1/l < 1/q + 1/n$. Supposed that $\mathbf{a} \in \mathbf{L}_\sigma^{(n,\infty)}(\Omega) \cap \mathbf{L}_\sigma^{(q^*,\infty)}(\Omega)$, $b \in L^{(n,\infty)}(\Omega) \cap L^{(q^*,\infty)}(\Omega)$, where $q^* = nq/(n-q)$, $f \in BC(\mathbb{R}; L^{(l,\infty)}(\Omega))$ with value in $L^{(n,\infty)}(\Omega)$, $\mathbf{g} \in \mathbf{L}^{(b,\infty)}(\Omega) \cap \mathbf{L}^{(n,\infty)}(\Omega)$ with $b > n/2$. Then, there exists $T \in (0,1]$ and functions $\mathbf{v}$, w in the class*

$$(\mathbf{v}, w) \in BC_w([t_0, t_0 + T); \mathbf{L}_\sigma^{(n,\infty)}(\Omega) \times L^{(n,\infty)}(\Omega)),$$

with

$$
\begin{cases}
(\mathbf{v}, w) & \in \quad BC((t_0, t_0 + T); \mathbf{L}^{(q^*,\infty)}(\Omega) \times L^{(q^*,\infty)}(\Omega)), \\
(t - t_0)^{1/2}\nabla(\mathbf{v}, w) & \in \quad BC_w((t_0, t_0 + T); \mathbf{L}^{(n,\infty)}(\Omega) \times L^{(n,\infty)}(\Omega))^n,
\end{cases}
$$

such that for all $t_0 \in \mathbb{R}$ the following equalities are verified in $\mathbf{L}_\sigma^{(n,\infty)}(\Omega) \cap \mathbf{L}_\sigma^{r^*}(\Omega)$ and $L^{(n,\infty)}(\Omega) \cap L^{r^*}(\Omega)$, $n < r^* < q^*$, respectively

$$\mathbf{v}(t) \;=\; e^{-(t-t_0)\mathbf{A}}\mathbf{a} + \int_{t_0}^t e^{-(t-s)\mathbf{A}}P(w\mathbf{g}) - \int_{t_0}^t e^{-(t-s)\mathbf{A}}P[(\mathbf{v} \cdot \nabla)\mathbf{v}]\,ds, \qquad (3.34)$$

$$w(t) \;=\; e^{-(t-t_0)B}b + \int_{t_0}^t e^{-(t-s)B}f(s)\,ds - \int_{t_0}^t e^{-(t-s)B}(\mathbf{v} \cdot \nabla w)(s)\,ds. \qquad (3.35)$$

Moreover, the functions $(\mathbf{v}, w)$ satisfy

$$t^{1/4}(\mathbf{v}, w) \in BC((t_0, t_0 + T); \mathbf{L}_\sigma^{2n}(\Omega) \times L^{2n}(\Omega)).$$

Here T is estimated as (3.33), where $\tilde{k}/c_1 = \tilde{k}/c_1(n, q, l)$, $\alpha = n/q^*$.

Proof of Proposition 3.11: Let us construct the solutions of integral equations (3.34)-(3.35) according to the following scheme:

$$\mathbf{v}_{m+1}(t) = \mathbf{v}_0(t) + \int_{t_0}^t e^{-(t-s)\mathbf{A}}P(w_m\,\mathbf{g})ds - \int_{t_0}^t e^{-(t-s)\mathbf{A}}P[(\mathbf{v}_m \cdot \nabla)\mathbf{v}_m]\,ds, \qquad (3.36)$$

$$w_{m+1}(t) = w_0(t) + \int_{t_0}^t e^{-(t-s)B}f\,ds - \int_{t_0}^t e^{-(t-s)B}(\mathbf{v}_m \cdot \nabla)w_m\,ds, \qquad (3.37)$$

where $\mathbf{v}_0(t) = e^{-(t-t_0)\mathbf{A}}\mathbf{a}$, $w_0(t) = e^{-(t-t_0)B}b$.

Since this Lemma only deals with local existence of solutions, we may assume that $0 < T \le 1$. We observe that $\alpha = n/q^*$, $q^* = nq/(n-q)$, and as $< 2q$, then $0 < \alpha < 1$. We will need the following lemmas

Lemma 3.12 *The sequences (3.36), (3.37) satisfy the following estimates*

$$\sup_{t_0 < t < t_0 + T} (t - t_0)^{(1-\alpha)/2}\|\mathbf{v}_m(t)\|_{(n/\alpha,\infty)} \;\le\; K_{m,1}, \qquad (3.38)$$

$$\sup_{t_0 < t < t_0 + T} (t - t_0)^{(1-\alpha)/2}\|w_m(t)\|_{(n/\alpha,\infty)} \;\le\; K_{m,2}, \quad m = 0, 1, ... \qquad (3.39)$$

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for some positive constants $K_{m,1}, K_{m,2}$ which are independent of t_0. Moreover, there exists $(\mathbf{v}, w)$ with

$$(t - t_0)^{(1-\alpha)/2}(\mathbf{v}(\cdot), w(\cdot)) \in BC((t_0, t_0 + T); \mathbf{L}_\sigma^{(n/\alpha,\infty)}(\Omega) \times L^{(n/\alpha,\infty)}(\Omega)), \tag{3.40}$$

such that

$$\lim_{m\to\infty} \sup_{t_0 < t < t_0 + T} (t - t_0)^{(1-\alpha)/2}\|\mathbf{v}_m(t) - \mathbf{v}(t)\|_{(n/\alpha,\infty)} = 0, \tag{3.41}$$

$$\lim_{m\to\infty} \sup_{t_0 < t < t_0 + T} (t - t_0)^{(1-\alpha)/2}\|w_m(t) - w(t)\|_{(n/\alpha,\infty)} = 0. \tag{3.42}$$

Proof. The proof is done by induction. In fact, $\|\mathbf{v}_0(t)\|_{(n/\alpha,\infty)} \le c\|\mathbf{a}\|_{(n/\alpha,\infty)}$ and $\|w_0(t)\|_{(n/\alpha,\infty)} \le c\|b\|_{(n/\alpha,\infty)}$, for $t_0 < t < t_0 + T$, where c is independent of t_0. Consequently,

$$\sup_{t_0 < t < t_0 + T} (t - t_0)^{(1-\alpha)/2}\|\mathbf{v}_0(t)\|_{(n/\alpha,\infty)} \le cT^{(1-\alpha)/2}\|\mathbf{a}\|_{(n/\alpha,\infty)} \equiv K_{0,1} \tag{3.43}$$

$$\sup_{t_0 < t < t_0 + T} (t - t_0)^{(1-\alpha)/2}\|w_0(t)\|_{(n/\alpha,\infty)} \le cT^{(1-\alpha)/2}\|b\|_{(n/\alpha,\infty)} \equiv K_{0,2}. \tag{3.44}$$

Assume (3.38)-(3.39) are true. We will prove (3.38)-(3.39) for the case $m + 1$. Note that for all $\phi \in \mathbf{C}_{0,\sigma}^\infty$ and all $t_0 < t < t_0 + T$, Lemma 3.2 implies

$$\left|\left(-\int_{t_0}^t e^{-(t-s)\mathbf{A}}P[(\mathbf{v}_m \cdot \nabla)\mathbf{v}_m]\,ds, \phi\right)\right| \le \int_{t_0}^t \|\mathbf{v}_m(s)\|_{(n/\alpha,\infty)}^2\|\nabla e^{-(t-s)\mathbf{A}}\phi\|_{(n/(n-2\alpha),1)}$$

$$\le c\int_{t_0}^t (t - s)^{-\alpha/2 - 1/2}\|\mathbf{v}_m(s)\|_{(n/\alpha,\infty)}^2\,ds \cdot \|\phi\|_{(n/(n-\alpha),1)}$$

$$\le c\,B((1-\alpha)/2, \alpha)K_{m,1}^2(t - t_0)^{-(1-\alpha)/2}\|\phi\|_{(n/(n-\alpha),1)},$$

where $B(\cdot, \cdot)$ denotes the Beta function and $c = c(n, q)$ is independent of t_0. By duality we have

$$\left\|\int_{t_0}^t e^{-(t-s)\mathbf{A}}P[(\mathbf{v}_m \cdot \nabla)\mathbf{v}_m](s)\,ds\right\|_{(n/\alpha,\infty)} \le C_{1,1}K_{m,1}^2(t - t_0)^{-(1-\alpha)/2},$$

for all t, $t_0 < t < t_0 + T$, with $C_{1,1} = C_{1,1}(n, q)$, and moreover

$$\left\|\int_{t_0}^t e^{-(t-s)\mathbf{A}}P(\mathbf{g}\,w_m)(s)\,ds\right\|_{(n/\alpha,\infty)} \le c\int_{t_0}^t (t - s)^{-n/2b}\|\mathbf{g}\|_{(b,\infty)}\|w_m(s)\|_{(n/\alpha,\infty)}\,ds$$

$$\le c\,(t - t_0)^{-(1-\alpha)/2}\,K_{m,2}\,\|\mathbf{g}\|_{(b,\infty)}\,(t - t_0)^{1-n/2b}\,B\left(-\frac{n}{2b} + 1, \frac{(1+\alpha)}{2}\right)$$

$$\le (t - t_0)^{-(1-\alpha)/2}C_{2,1}K_{m,2}\,T^{1-n/2b},$$

being $C_{2,1} = c\,\|\mathbf{g}\|_{(b,\infty)}\,B\left(-\frac{n}{2b} + 1, \frac{(1+\alpha)}{2}\right)$. Hence

$$\sup_{t_0 < t < t_0 + T} (t - t_0)^{(1-\alpha)/2}\|\mathbf{v}_{m+1}\|_{(n/\alpha,\infty)} \le K_{0,1} + C_{1,1}K_{m,1}^2 + C_{2,1}\,T^{1-n/2b}\,K_{m,2}.$$

Moreover, for all $\phi \in \mathbf{C}_0^\infty$ and all $t_0 < t < t_0 + T$, we have

$$\left|\left(-\int_{t_0}^t e^{-(t-s)B}(\mathbf{v}_m \cdot \nabla w_m)(s)ds, \phi\right)\right| = \left|\int_{t_0}^t (w_m \otimes \mathbf{v}_m(s), \nabla e^{-(t-s)B}\phi)ds\right|$$

$$\le \int_{t_0}^t \|\mathbf{v}_m(s)\|_{(n/\alpha,\infty)}\|w_m(s)\|_{(n/\alpha,\infty)}\|\nabla e^{-(t-s)B}\phi\|_{(n/(n-2\alpha),1)}ds$$

$$\le c\int_{t_0}^t (t - s)^{-\alpha/2 - 1/2}\|\mathbf{v}_m(s)\|_{(n/\alpha,\infty)}\|w_m(s)\|_{(n/\alpha,\infty)} \cdot \|\phi\|_{(n/(n-\alpha),1)}ds$$

$$\le cB((1-\alpha)/2, \alpha)K_{m,1}K_{m,2}(t - t_0)^{-(1-\alpha)/2}\|\phi\|_{(n/(n-\alpha),1)}.$$

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By duality, for $t_0 < t < t_0 + T$, we get

$$\left\| \int_{t_0}^t e^{-(t-s)B}(\mathbf{v}_m \cdot \nabla w_m)(s)\, ds \right\|_{(n/\alpha,\infty)} \le C_{1,2} K_{m,2} K_{m,1} (t - t_0)^{-(1-\alpha)/2}.$$

Now, using Lemma 3.4 we obtain

$$
\begin{aligned}
\int_{t_0}^t \|e^{-(t-s)B} f(s)\|_{(n/\alpha,\infty)} &\le c \int_{t_0}^t \|\nabla e^{-(t-s)B} f(s)\|_{(q,\infty)} \\
&\le c\|f\|_{BC(\mathbb{R};L^{(l,\infty)})}(t - t_0)^{-(1-\alpha)/2+3/2-n/2l},
\end{aligned}
$$

for all $t_0 < t < t_0 + T$ with $c = c(n, q, l)$. Since $1/l < 1/q + 1/n$, we have $(1-\alpha)/2 < 3/2 - n/2l$ and hence the above estimate yields

$$(t - t_0)^{(1-\alpha)/2}\left\| \int_{t_0}^t e^{-(t-s)B} f(s) \right\|_{(n/\alpha,\infty)} \le c\|f\|_{BC(\mathbb{R},L^{(l,\infty)})} T^{(1-\alpha)/2}.$$

Consequently,

$$
\left\{
\begin{aligned}
&\sup_{t_0 < t < t_0+T} (t - t_0)^{(1-\alpha)/2}\|w_{m+1}(t)\|_{(n/\alpha,\infty)} \\
&\qquad \le K_{0,2} + c\|f\|_{BC(\mathbb{R};L^{(l,\infty)})} T^{(1-\alpha)/2} + C_{1,2} K_{m,1} K_{m,2}.
\end{aligned}
\right.
$$

Then, we can take $K_{m+1,1}, K_{m+1,2}$ being respectively,

$$K_{m+1,1} = K_{0,1} + C_{1,1} K_{m,1}^2 + C_{2,1} K_{m,2}, \tag{3.45}$$

$$K_{m+1,2} = K_{0,2} + c\|f\|_{BC(\mathbb{R};L^{(l,\infty)})} T^{(1-\alpha)/2} + C_{1,2} K_{m,1} K_{m,2}. \tag{3.46}$$

Setting $K_m = \max(K_{m,1}, K_{m,2})$, $m = 1, 2, ...$, from (3.43), (3.44), (3.45) and (3.46) we have

$$K_{m+1} \le K_0 + \widetilde{C} K_m^2 + C_{2,1} K_m,$$

$$K_0 = c_1 T^{(1-\alpha)/2} \max\{\|\mathbf{a}\|_{(n/\alpha,\infty)}, \|b\|_{(n/\alpha,\infty)} + \|f\|_{BC(\mathbb{R};L^{(l,\infty)})}\} \tag{3.47}$$

and $\widetilde{C} = \max\{C_{1,1}, C_{1,2}\}$. If we consider

$$C_{2,1} < 1, \ K_0 < \frac{(1 - C_{2,1})^2}{4\widetilde{C}}, \tag{3.48}$$

we have that

$$K_m < \frac{(1 - C_{2,1}) - \sqrt{(1 - C_{2,1})^2 - 4\widetilde{C} K_0}}{2\widetilde{C}} \equiv k < \frac{1}{2\widetilde{C}}, \ \forall m = 0, 1, 2, ... \tag{3.49}$$

Assuming (3.48) and working as Subsection 3.2, due to the uniform estimate with respect to m, we can conclude the existence of a couple $(\mathbf{v}, w)$ such that (3.40) holds and satisfying $(3.41) - (3.42)$. Thus, we finish the proof of Lemma 3.12. $\blacksquare$

Lemma 3.13 *If K_0 defined by (3.47) is small enough, then the limit $(\mathbf{v}, w)$ given by Lemma 3.12 satisfies the following estimate*

$$(t - t_0)^{1/2} \nabla(\mathbf{v}(\cdot), w(\cdot)) \in BC((t_0, t_0 + T); \mathbf{L}^{(n,\infty)}(\Omega) \times L^{(n,\infty)}(\Omega))^n, \tag{3.50}$$

with

$$\lim_{m \to \infty} \sup_{t_0 < t < t_0 + T} (t - t_0)^{1/2} \|\nabla \mathbf{v}_m(t) - \nabla \mathbf{v}(t)\|_{(n,\infty)} = 0,$$
$$\lim_{m \to \infty} \sup_{t_0 < t < t_0 + T} (t - t_0)^{1/2} \|\nabla w_m(t) - \nabla w(t)\|_{(n,\infty)} = 0.$$

Proof. The proof is done by induction. In fact, we will prove that

$$\sup_{t_0 < t < t_0 + T} (t - t_0)^{1/2} \|\nabla \mathbf{v}_m(t)\|_{(n,\infty)} \le J_{m,1}, \tag{3.51}$$

$$\sup_{t_0 < t < t_0 + T} (t - t_0)^{1/2} \|\nabla w_m(t)\|_{(n,\infty)} \le J_{m,2}, \tag{3.52}$$

for some constants $J_{m,1}, J_{m,2}$ which are independent of t_0, $m = 0, 1, ...$
Note that by Lemma 3.2

$$\|\nabla \mathbf{v}_0(t)\|_{(n,\infty)} \le C(t - t_0)^{-1/2} \|\mathbf{a}\|_{(n,\infty)}, \quad \|\nabla w_0(t)\|_{(n,\infty)} \le C(t - t_0)^{-1/2} \|b\|_{(n,\infty)},$$

where $C = C(n)$ is independent of t_0. Hence we can take $J_{0,1}$ and $J_{0,2}$, being respectively, $C\|\mathbf{a}\|_{(n,\infty)}$, $C\|b\|_{(n,\infty)}$.
Supposed inequalities (3.51)-(3.52) are true. Then

$$\left\| \nabla \int_{t_0}^t e^{-(t-s)\mathbf{A}} P[(\mathbf{v}_m \cdot \nabla)\mathbf{v}_m] \right\|_{(n,\infty)} \le \int_{t_0}^t (t - s)^{-n/2(\alpha/n) - 1/2} \|\mathbf{v}_m\|_{(n/\alpha,\infty)} \|\nabla \mathbf{v}_m\|_{(n,\infty)}$$

$$\le cK_{m,1} J_{m,1} \int_{t_0}^t (t - s)^{-\alpha/2 - 1/2} (s - t_0)^{\alpha/2 - 1} ds$$

$$\le cK_{m,1} J_{m,1} (t - t_0)^{-1/2} B((1 - \alpha)/2, \alpha/2) \le C_{3,1} k J_{m,1} (t - t_0)^{-1/2},$$

for all $t_0 < t < t_0 + T$, where $C_{3,1} = C_{3,1}(n, q)$ is independent of t_0. Moreover

$$\left\| \nabla \int_{t_0}^t e^{-(t-s)\mathbf{A}} P(\mathbf{g} w_m) ds \right\|_{(n,\infty)} \le \int_{t_0}^t (t - s)^{-n/2(\alpha/n) - 1/2} \|\mathbf{g}\|_{(n,\infty)} \|w_m\|_{(n/\alpha,\infty)} ds$$

$$\le c\|\mathbf{g}\|_{(n,\infty)} K_{m,2} \int_{t_0}^t (t - s)^{-(\alpha+1)/2} (s - t_0)^{-(1-\alpha)/2} ds$$

$$\le cB((1 - \alpha)/2, (1 + \alpha)/2) k \|\mathbf{g}\|_{(n,\infty)} \le C_{4,1} k \|\mathbf{g}\|_{(n,\infty)}.$$

Therefore,

$$\sup_{t_0 < t < t_0 + T} (t - t_0)^{1/2} \|\nabla \mathbf{v}_{m+1}\|_{(n,\infty)} \le J_{0,1} + C_{3,1} k J_{m,1} + C_{4,1} k \|\mathbf{g}\|_{(n,\infty)} T^{1/2}.$$

Now, for any t, $t_0 < t < t_0 + T$,

$$\left\| \nabla \int_{t_0}^t e^{-(t-s)B} (\mathbf{v}_m \cdot \nabla w_m) ds \right\|_{(n,\infty)} \le \int_{t_0}^t (t - s)^{-n/2(\alpha/n) - 1/2} \|\mathbf{v}_m\|_{(n/\alpha,\infty)} \|\nabla w_m\|_{(n,\infty)}$$

$$\le cK_{m,1} J_{m,2} \int_{t_0}^t (t - s)^{-\alpha/2 - 1/2} (s - t_0)^{\alpha/2 - 1} ds \le C_{2,2} k J_{m,2} (t - t_0)^{-1/2},$$

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where $C_{2,2}$ is independent of t_0. As

$$\left\|\nabla \int_{t_0}^t e^{-(t-s)B} f(s)ds\right\|_{(n,\infty)} \leq c(t-t_0)^{-1/2}\|f\|_{BC(\mathbb{R};L^{(n,\infty)})},$$

we conclude that

$$\sup_{t_0<t<t_0+T}(t-t_0)^{1/2}\|w_{m+1}(t)\|_{(n,\infty)} \leq J_{0,2} + C_{2,2}kJ_{m,2} + c\|f\|_{BC(\mathbb{R};L^{(n,\infty)})}.$$

Then we can take $J_{m+1,1}$ and $J_{m+1,2}$ being respectively,

$$\begin{aligned}
J_{m+1,1} &= J_{0,1} + C_{3,1}kJ_{m,1} + C_{4,1}k\|\mathbf{g}\|_{(n,\infty)} T^{1/2}, \\
J_{m+1,2} &= J_{0,2} + C_{2,2}kJ_{m,2} + c\|f\|_{BC(\mathbb{R};L^{(n,\infty)})}.
\end{aligned}$$

Let $J_m = \max\{J_{m,1}, J_{m,2}\}$, $m = 1, 2, \ldots$ and

$$J_0 = \text{Max}\{J_{0,1} + C_{4,1}k\|g\|_{(n,\infty)} T^{1/2}, \ J_{0,2} + c\|f\|_{BC(\mathbb{R};L^{(n,\infty)})}\},$$

then

$$J_{m+1} \leq J_0 + k\widehat{C}J_m,$$

where $\widehat{C} = \max\{C_{3,1}, C_{2,2}\}$. Consequently, if

$$k < 1/\widehat{C} \tag{3.53}$$

we have a uniform estimate for the sequence $\{J_m\}$ given by $J_m \leq \frac{J_0}{1-\widehat{C}k} \equiv J$, $m = 0, 1, \ldots$

Assuming (3.53), we can see that the limits $\mathbf{v}, w$ satisfy (3.50) and the proof of Lemma 3.13 is finished. ∎

Lemma 3.14 *The limit $(\mathbf{v}, w)$ given by Lemma 3.12 and Lemma 3.13 verifies*

$$(t-t_0)^{1/4}(\mathbf{v}(\cdot), w(\cdot) \in BC((t_0, t_0+T); \mathbf{L}_\sigma^{2n}(\Omega) \times L^{2n}(\Omega)),$$

with

$$\lim_{m\to\infty} \sup_{t_0<t<t_0+T}(t-t_0)^{1/4}\|\mathbf{v}_m(t) - \mathbf{v}(t)\|_{2n} = 0,$$
$$\lim_{m\to\infty} \sup_{t_0<t<t_0+T}(t-t_0)^{1/4}\|w_m(t) - w(t)\|_{2n} = 0.$$

Proof. As the previous lemmas, the proof is done by induction. In fact, we will prove that there exist some constants $N_{m,1}, N_{m,2}$, which are independent of t_0, such that

$$\|\mathbf{v}_m(t)\|_{2n} \leq N_{m,1}(t-t_0)^{-1/4}, \qquad \|w_m(t)\|_{2n} \leq N_{m,2}(t-t_0)^{-1/4}. \tag{3.54}$$

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Since $L^{(p_0,\infty)} \cap L^{(p_1,\infty)} \subset L^p$ and $\|f\|_p \leq C(p_0, p_1, \lambda)\|f\|_{(p_0,\infty)}^{1-\lambda}\|f\|_{(p_1,\infty)}^{\lambda}$ provided that $p_0 \neq p_1, 0 < \lambda < 1$ and $1/p = (1 - \lambda)/p_0 + \lambda/p_1$, we have

$$\|\mathbf{v}_0(t)\|_{2n} \leq C(t - t_0)^{-1/4}\|\mathbf{a}\|_{(n,\infty)}, \qquad \|w_0(t)\|_{2n} \leq C(t - t_0)^{-1/4}\|b\|_{(n,\infty)},$$

where $C = C(n)$ is independent of t_0. Hence, we define $N_{0,1}$ and $N_{0,2}$ as $C\|\mathbf{a}\|_{(n,\infty)}$ and $C\|b\|_{(n,\infty)}$, respectively.

Assuming true (3.54) for a given m, we can prove that (3.54) holds for the case $m + 1$. In fact, note that for all $\phi \in \mathbf{C}_{0,\sigma}^\infty$, $\varphi \in C_0^\infty$, we get

$$\left|\left(-\int_{t_0}^t e^{-(t-s)\mathbf{A}}P[(\mathbf{v}_m \cdot \nabla)\mathbf{v}_m](s)\,ds, \phi\right)\right| \leq \int_{t_0}^t \|\mathbf{v}_m \otimes \mathbf{v}_m\|_n \|\nabla e^{-(t-s)\mathbf{A}}\phi\|_{n'}\,ds$$
$$\leq C\int_{t_0}^t \|\mathbf{v}_m\|_{2n}^2(t - s)^{-3/4}\|\phi\|_{(2n)'}\,ds \leq c\,(t - t_0)^{-1/4}\,B(1/4, 1/2)\,N_{m,1}^2\,\|\phi\|_{(2n)'}$$

$$\left|\left(-\int_{t_0}^t e^{-(t-s)B}(\mathbf{v}_m \cdot \nabla w_m)(s)\,ds, \varphi\right)\right| \leq \int_{t_0}^t \|w_m \cdot \mathbf{v}_m\|_n \|\nabla e^{-(t-s)B}\varphi\|_{n'}\,ds$$
$$\leq C\int_{t_0}^t \|\mathbf{v}_m\|_{2n}\|w_m\|_{2n}(t - s)^{-3/4}\|\varphi\|_{(2n)'}\,ds \leq c\,(t - t_0)^{-1/4}\,B(1/4, 1/2)\,N_{m,1}\,N_{m,2}\,\|\varphi\|_{(2n)'}.$$

Hence by duality

$$\left\|\int_{t_0}^t e^{-(t-s)\mathbf{A}}P(\mathbf{v}_m \cdot \nabla \mathbf{v}_m)(s)ds\right\|_{2n} \leq C_{1,1}\,N_{m,1}^2\,(t - t_0)^{-1/4},$$
$$\left\|\int_{t_0}^t e^{-(t-s)B}(\mathbf{v}_m \cdot \nabla w_m)(s)ds\right\|_{2n} \leq C_{1,2}N_{m,1}\,N_{m,2}\,(t - t_0)^{-1/4}.$$

We also note that

$$\left\|\int_{t_0}^t e^{-(t-s)\mathbf{A}}P(\mathbf{g}w_m)(s)\,ds\right\|_{2n} \leq \int_{t_0}^t (t - s)^{-1/2}\|\mathbf{g}\|_{(n,\infty)}\|w_m(s)\|_{(2n)}\,ds$$
$$\leq c\,\|\mathbf{g}\|_{(n,\infty)}\,B(1/2, 3/4)\,(t - t_0)^{1/4} \leq C_{2,1}\,N_{m,2}\,(t - t_0)^{1/4}$$

and

$$\left\|\int_{t_0}^t e^{-(t-s)B}f(s)ds\right\|_{2n} \leq \int_{t_0}^t (t - s)^{-1/4}\|f(s)\|_{(n,\infty)}\,ds \leq c\|f\|_{BC(\mathbb{R};L^{(n,\infty)})}\,(t - t_0)^{3/4}.$$

The inequalities above imply that

$$\sup_{t_0 < t < t_0 + T}(t - t_0)^{1/4}\|\mathbf{v}_{m+1}\|_{2n} \leq N_{0,1} + C_{1,1}N_{m,1}^2 + C_{2,1}N_{m,2}\,T^{1/2}$$
$$\sup_{t_0 < t < t_0 + T}(t - t_0)^{1/4}\|w_{m+1}\|_{2n} \leq N_{0,2} + C_{1,2}N_{m,1}N_{m,2} + c\|f\|_{BC(\mathbb{R};L^{(n,\infty)})}\,T.$$

As before, setting $N_m = \max(N_{m,1}, N_{m,2}), m = 1, 2, \ldots$ and $N_0 = \max(N_{0,2}+c\|f\|_{BC(\mathbb{R};L^{(n,\infty)})}\,T, N_{0,1})$, we obtain $N_{m+1} \leq N_0 + \widetilde{C}N_m^2 + \widetilde{C}_{2,1}N_m$, with $\widetilde{C}_{2,1} = C_{2,1}\,T^{1/2}$ where $\widetilde{C} = \max(C_{1,1}, C_{1,2})$. If we consider

$$C_{2,1} < 1, \quad N_0 < \frac{(1 - C_{2,1})^2}{4\widetilde{C}}, \tag{3.55}$$

we have that the sequence $\{N_m\}_{m=0}^{m=\infty}$ is bounded with

$$N_m \leq \frac{(1 - C_{2,1}) - \sqrt{(1 - C_{2,1})^2 - 4N_0 \widetilde{C}}}{2\widetilde{C}}, \quad m = 0, 1, \ldots$$

Assuming (3.55) and working as Lemma 3.12 and Lemma 3.13, we conclude the proof. ∎

Lemma 3.15 *The limit* $(\mathbf{v}, w)$ *given by Lemma 3.12 and Lemma 3.13 verifies*

$$(\mathbf{v}, w) \in BC((t_0, t_0 + T); \mathbf{L}_\sigma^{(n,\infty)}(\Omega) \cap \mathbf{L}_\sigma^{(q^*,\infty)}(\Omega) \times L^{(n,\infty)}(\Omega) \cap L^{(q^*,\infty)}(\Omega)), \qquad (3.56)$$

with

$$\sup_{t_0 < t < t_0 + T} \|\mathbf{v}_m(t)\|_{(n/s,\infty)} \leq M_{1,s,m}, \quad s = \alpha, \quad s = 1, \qquad (3.57)$$

$$\sup_{t_0 < t < t_0 + T} \|w_m(t)\|_{(n/s,\infty)} \leq M_{2,s,m}, \quad s = \alpha, \quad s = 1, \qquad (3.58)$$

and $M_{1,s,m}, M_{2,s,m}$ *independent of* t_0.

Proof. Computations similar to Lemma 3.12, Lemma 3.13, yield to

$$M_{1,\alpha,0} = C\|\mathbf{a}\|_{(n/\alpha,\infty)}, \quad M_{1,1,0} = C\|\mathbf{a}\|_{(n,\infty)},$$
$$M_{2,\alpha,0} = C\|b\|_{(n/\alpha,\infty)}, \quad M_{2,1,0} = C\|b\|_{(n,\infty)},$$

where $C = C(n, s)$ is independent of t_0. Suppose by induction that (3.57), (3.58) are true. Note that

$$\left| \left(-\int_{t_0}^t e^{-(t-s)\mathbf{A}} P(\mathbf{v}_m \cdot \nabla \mathbf{v}_m)(s)\, ds, \phi \right) \right| \leq \left| \int_{t_0}^t (\mathbf{v}_m \otimes \mathbf{v}_m(s), \nabla e^{-(t-s)\mathbf{A}} \phi)\, ds \right|$$

$$\leq \int_{t_0}^t \|\mathbf{v}_m\|_{(n/\alpha,\infty)} \|\mathbf{v}_m\|_{(n/s,\infty)} \|\nabla e^{-(t-s)\mathbf{A}} \phi\|_{(n/(n-\alpha-s),1)}\, ds$$

$$\leq C K_{m,1} M_{1,s,m} \int_{t_0}^t (t-s)^{-\alpha/2-1/2} (s-t_0)^{-(1-\alpha)/2} \cdot \|\phi\|_{(n/(n-s),1)}\, ds$$

$$\leq C k M_{1,s,m} B((1-\alpha)/2, (1+\alpha)/2) \|\phi\|_{(n/(n-s),1)},$$

for all $\phi \in \mathbf{C}_{0,\sigma}^\infty$ and all $t_0 < t < t_0 + T$ and $C = C(n, q, s)$ independent of t_0. Consequently, by duality, for $s = 1, \alpha$, we have

$$\sup_{t_0 < t < t_0 + T} \left\| \int_{t_0}^t e^{-(t-s)\mathbf{A}} P(\mathbf{v}_m \cdot \nabla \mathbf{v}_m)(s)\, ds \right\|_{(n/s,\infty)} \leq C_{5,1}\, k\, M_{1,s,m}, \qquad (3.59)$$

where $C_{5,1}$ independent of t_0. Note that

$$\left\| \int_{t_0}^t e^{-(t-s)\mathbf{A}} P(\mathbf{g} w_m)(s)\, ds \right\|_{(n/s,\infty)} \leq c\, \|\mathbf{g}\|_{(b,\infty)} \int_{t_0}^t (t-s)^{-n/2b} \|w_m(s)\|_{(n/s,\infty)}\, ds$$

$$\leq c\, \|\mathbf{g}\|_{(b,\infty)} (t-t_0)^{1-n/2b} \leq C_{6,1}\, M_{2,s,m} (t-t_0)^{1-n/2b},$$

$$\left\| \int_{t_0}^{t} e^{-(t-s)B} f(s)\, ds \right\|_{(n/\alpha,\infty)} \le c\, \|f\|_{BC(\mathbb{R};L^{(l,\infty)})}\, T^a,$$

with $a = \frac{\alpha}{2} - \frac{n}{2l} + 1 > 0$. Moreover, for all $\varphi \in C_0^\infty$ and all $t_0 < t < t_0 + T$,

$$
\begin{aligned}
\left| \left(-\int_{t_0}^{t} e^{-(t-s)B} (\mathbf{v}_m \cdot \nabla w_m)(s)\, ds, \varphi \right) \right| &\le \left| \int_{t_0}^{t} (w_m \cdot \mathbf{v}_m(s), \nabla e^{-(t-s)B})\varphi)\, ds \right| \\
&\le \int_{t_0}^{t} \|\mathbf{v}_m\|_{(n/\alpha,\infty)} \|w_m\|_{(n/s,\infty)} \|\nabla e^{-(t-s)B}\varphi\|_{(n/(n-\alpha-s),1)}\, ds \\
&\le c\, K_{m,1}\, M_{2,s,m} \left(\int_{t_0}^{t} (t-s)^{-\alpha/2-1/2}(s-t_0)^{-(1-\alpha)/2}\, ds \right) \|\varphi\|_{(n/(n-s),1)} \\
&\le c\, k\, M_{2,s,m}\, B((1-\alpha)/2, (1+\alpha)/2)\, \|\varphi\|_{(n/(n-s),1)}.
\end{aligned}
$$

Thus, for $s = 1, \alpha$,

$$\sup_{t_0 < t < t_0+T} \left\| \int_{t_0}^{t} e^{-(t-s)B} (\mathbf{v}_m \cdot \nabla w_m)(s)\, ds \right\|_{(n/s,\infty)} \le C_{4,2}\, k\, M_{2,s,m}, \tag{3.60}$$

where $C_{4,2}$ is independent of t_0. Hence, from (3.59)-(3.60) we can take

$$
\begin{aligned}
M_{1,s,m+1} &= M_{1,s,0} + C_{5,1}\, k\, M_{1,s,m} + C_{6,1}\, M_{2,s,m}, \tag{3.61} \\
M_{2,s,m+1} &= M_{2,s,0} + c\, \|f\|_{BC(\mathbb{R};L^{(l,\infty)})}\, T^a + C_{4,2}\, k\, M_{2,s,m}. \tag{3.62}
\end{aligned}
$$

Setting

$$
\begin{aligned}
M_{s,m} &= \max\{M_{1,s,m}, M_{2,s,m}\}, \\
M_{s,0} &= \max\{M_{1,s,0}, M_{2,s,0} + c\, \|f\|_{BC(\mathbb{R};L^{(l,\infty)})}\, T^a\}, \\
\check{C} &= \max\{C_{5,1}, C_{4,2}\},
\end{aligned}
$$

from (3.61),(3.62) we obtain $M_{s,m+1} \le M_{s,0} + k\, \check{C}\, M_{s,m} + C_{6,1}\, M_{s,m}$, for $m = 0, 1, ..., s = 1, \alpha$. Then, if

$$k\check{C} + C_{6,1} < 1, \tag{3.63}$$

we have $M_{s,m} \le \frac{M_{s,0}}{1 - k\check{C} - C_{6,1}}$, $m = 0, 1, ..., s = 1, \alpha$, which yields to (3.56) with (3.57)-(3.58). ∎

Now, we continue the proof of Proposition 3.11. We can see that under conditions (3.48), (3.53) and (3.63), the limit $(\mathbf{v}, w)$ belongs to the class required in Proposition 3.11. Moreover, then following convergences hold in $\mathbf{L}_\sigma^{(n,\infty)}(\Omega)$, $\mathbf{L}_\sigma^{(n,\infty)}(\Omega)$ and $\mathbf{L}^{(n,\infty)}(\Omega)$, respectively

$$\int_{t_0}^{t} e^{-(t-s)\mathbf{A}} P[(\mathbf{v}_m \cdot \nabla)\mathbf{v}_m](s)\, ds \longrightarrow \int_{t_0}^{t} e^{-(t-s)\mathbf{A}} P[(\mathbf{v} \cdot \nabla)\mathbf{v}](s)\, ds,$$

$$\int_{t_0}^{t} e^{-(t-s)\mathbf{A}} P(w_m\, \mathbf{g})(s)\, ds \longrightarrow \int_{t_0}^{t} e^{-(t-s)\mathbf{A}} P(w\, \mathbf{g})(s)\, ds,$$

$$\int_{t_0}^{t} e^{-(t-s)B} P[(\mathbf{v}_m \cdot \nabla)w_m](s)\, ds \longrightarrow \int_{t_0}^{t} e^{-(t-s)\mathbf{A}} P[(\mathbf{v} \cdot \nabla)w](s)\, ds, \tag{3.64}$$

uniformly in $t \in (t_0, t_0 + t)$ as $m \to \infty$. In fact, note that by Lemma 3.2, Lemma 3.12 and Lemma 3.13, we have

$$
\left\| \int_{t_0}^{t} e^{-(t-s)\mathbf{A}} P[(\mathbf{v}_m \cdot \nabla)\mathbf{v}_m](s)\, ds - \int_{t_0}^{t} e^{-(t-s)\mathbf{A}} P[(\mathbf{v} \cdot \nabla)\mathbf{v}](s)\, ds \right\|_{(n,\infty)}
$$
$$
\leq \int_{t_0}^{t} (t-s)^{-\alpha/2} \|\mathbf{v}_m(s) - \mathbf{v}(s)\|_{(n/\alpha,\infty)} \|\nabla\mathbf{v}_m(s)\|_{(n,\infty)}\, ds
$$
$$
+ \int_{t_0}^{t} (t-s)^{-\alpha/2} \|\mathbf{v}(s)\|_{(n/\alpha,\infty)} \|\nabla(\mathbf{v}_m(s) - \mathbf{v}(s))\|_{(n,\infty)}\, ds
$$
$$
\leq c\, B(1 - \alpha/2, 1/2)\, J \sup_{t_0 < s < t_0 + T} (s - t_0)^{(1-\alpha)/2} \|\mathbf{v}_m(s) - \mathbf{v}(s)\|_{(n/\alpha,\infty)}
$$
$$
+ B(1 - \alpha/2, (\alpha+1)/2)\, k \sup_{t_0 < s < t_0 + T} (s - t_0)^{1/2} \|\nabla\mathbf{v}_m(s) - \nabla\mathbf{v}(s)\|_{(n,\infty)}),
$$

which converges to 0. On the other hand

$$
\left\| \int_{t_0}^{t} e^{-(t-s)\mathbf{A}} P(w_m g)(s)\, ds - \int_{t_0}^{t} e^{-(t-s)\mathbf{A}} P(w\mathbf{g})(s)\, ds \right\|_{(n,\infty)}
$$
$$
\leq \int_{t_0}^{t} \|e^{-(t-s)\mathbf{A}} P((w_m - w)\mathbf{g})(s)\|_{(n,\infty)}\, ds
$$
$$
\leq \int_{t_0}^{t} (t-s)^{-n/2b+(1-\alpha)/2} \|\mathbf{g}\|_{(b,\infty)} \|w_m(s) - w(s)\|_{(n/\alpha,\infty)}
$$
$$
\leq c\, T^{1-n/2b} \sup_{t_0 < s < t_0 + T} (s - t_0)^{(1-\alpha)/2} \|w_m(s) - w(s)\|_{(n/\alpha,\infty)} \longrightarrow 0.
$$

Analogously we obtain (3.64). Now we will prove the weak continuity on the initial dada. Firstly, we note that for any $\phi \in \mathbf{L}_\sigma^{(n',1)}(\Omega)$ and $\varphi \in L^{(n',1)}(\Omega)$ we have

$$
\begin{aligned}
|(e^{-(t-t_0)\mathbf{A}}\mathbf{a} - \mathbf{a}, \phi)| &= |(\mathbf{a}, e^{-(t-t_0)\mathbf{A}}\phi - \phi)| \\
&\leq \|\mathbf{a}\|_{(n,\infty)} \|e^{-(t-t_0)\mathbf{A}}\phi - \phi\|_{(n',1)} \to 0, \quad t \to t_0^+. \\
|(e^{-(t-t_0)B}b - b, \varphi)| &= |(b, e^{-(t-t_0)\mathbf{A}}\varphi - \varphi)| \\
&\leq \|b\|_{(n,\infty)} \|e^{-(t-t_0)B}\varphi - \varphi\|_{(n',1)} \to 0, \quad t \to t_0^+.
\end{aligned}
$$

As $\|\mathbf{v}\|_{(q^*,\infty)}, \|w\|_{(q^*,\infty)} \leq c$ and $(t - t_0)^{1/4}\|\mathbf{v}\|_{2n}, (t - t_0)^{1/4}\|w\|_{2n} \leq c$, we have

$$
\lim_{t \to t_0} \left(\int_{t_0}^{t} e^{-(t-s)\mathbf{A}} P(w\,\mathbf{g})(s)\, ds - \int_{t_0}^{t} e^{-(t-s)\mathbf{A}} P[(\mathbf{v} \cdot \nabla)\mathbf{v}](s)\, ds, \phi \right) = 0,
$$
$$
\lim_{t \to t_0} \left(\int_{t_0}^{t} e^{-(t-s)B} f(s)\, ds - \int_{t_0}^{t} e^{-(t-s)B}(\mathbf{v} \cdot \nabla w)(s)\, ds, \varphi \right) = 0.
$$

Indeed, note that when $t \to t_0^+$,

$$
\int_{t_0}^{t} \left(e^{-(t-s)B}[(\mathbf{v} \cdot \nabla)w], \varphi \right) \leq \int_{t_0}^{t} \|\mathbf{v}\|_{(q^*,\infty)} \|w\|_{2n} \|\nabla e^{-(t-s)B}\varphi\|_{(1-1/q^*-1/2n,1)}
$$
$$
\leq c \int_{t_0}^{t} (s - t_0)^{-1/4} (t - s)^{-n/2q+1/4} \|\varphi\|_{(n',1)}\, ds
$$
$$
\leq c\,(t - t_0)^{1-n/2q}\, B(3/4, -n/2q + 5/4)\, \|\varphi\|_{(n',1)} \to 0,
$$

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$$\int_{t_0}^t \left(e^{-(t-s)\mathbf{A}} P[(\mathbf{v} \cdot \nabla)\mathbf{v}], \boldsymbol{\phi} \right) \le \int_{t_0}^t \|\mathbf{v}\|_{(q^*,\infty)} \|\mathbf{v}\|_{2n} \|\nabla e^{-(t-s)\mathbf{A}} \boldsymbol{\phi}\|_{(1-1/q^*-1/2n,1)}$$

$$\le c \int_{t_0}^t (s-t_0)^{-1/4} (t-s)^{-n/2q+1/4} \|\boldsymbol{\phi}\|_{(n',1)} \, ds$$

$$\le c \, (t-t_0)^{1-n/2q} B(3/4, -n/2q + 5/4) \|\boldsymbol{\phi}\|_{(n',1)} \to 0,$$

$$\int_{t_0}^t \left(e^{-(t-s)\mathbf{A}} P(w \, \mathbf{g}), \boldsymbol{\phi} \right) ds \le \int_{t_0}^t \|e^{-(t-s)\mathbf{A}} P(w \, \mathbf{g})\|_{(n,\infty)} \|\boldsymbol{\phi}\|_{(n',1)} \, ds$$

$$\le c \, \|w\|_{q^*} \|\mathbf{g}\|_{(n,\infty)} \|\boldsymbol{\phi}\|_{(n',1)} \int_{t_0}^t (s-t_0)^{-n/2q^*} \, ds$$

$$\le c \, (t-t_0)^{1-n/2q^*} \|w\|_{q^*} \|\mathbf{g}\|_{(n,\infty)} \|\boldsymbol{\phi}\|_{(n',1)} \to 0,$$

and

$$\int_{t_0}^t \left(e^{-(t-s)B} f(s), \varphi \right) ds \le \int_{t_0}^t \|e^{-(t-s)B} f(s)\|_{(n,\infty)} \|\varphi\|_{(n',1)} \, ds$$

$$\le c \int_{t_0}^t \|f\|_{(n,\infty)} \, ds \, \|\varphi\|_{(n',1)} \le c \, (t-t_0) \to 0.$$

Collecting all the previous convergences and letting $m \to \infty$ in (3.36)-(3.37), we see that $(\mathbf{v}, w)$ is a solution of (3.34)-(3.35). Finally, we will estimate the time-interval T of existence in terms of the prescribed data. As k is determined by (3.49), there exists a constant $\tilde{k}$ independent of t_0 such that if $K_0 \le \tilde{k}$, then conditions (3.48),(3.53), (3.63) are satisfied. Now, from (3.47) we see that T may be chosen as (3.33).

Remark 3.16 *The solution $(\mathbf{v}, w)$ of integral equations (3.34)-(3.35) satisfies $(\mathbf{v}, w) \in BC(t_0, t_0 + T; \mathbf{L}_\sigma^p(\Omega) \times L^p(\Omega))$, for all $p \in (n, q^*)$, with $\|\mathbf{v}\|_p \le C\|\mathbf{v}\|_{(n,\infty)}^{1-\lambda} \|\mathbf{u}\|_{(q^*,\infty)}^\lambda$, $\|w\|_p \le C\|w\|_{(n,\infty)}^{1-\lambda} \|w\|_{(q^*,\infty)}^\lambda$, where λ is such that $1/p = (1-\lambda)/n + \lambda/q^*$.*

Note that being $\mathbf{a}$ and b elements of $\mathbf{L}_\sigma^{(n,\infty)}(\Omega) \cap \mathbf{L}_\sigma^{(q^*,\infty)}(\Omega)$ and $L^{(n,\infty)}(\Omega) \cap L^{(q^*,\infty)}(\Omega)$, respectively, we have that $\mathbf{a}$ and b belong to space $\mathbf{L}_\sigma^{r^*}(\Omega)$ and $L^{r^*}(\Omega)$, respectively. Consequently, the norms $\|e^{-(t-t_0)\mathbf{A}}\mathbf{a}\|_{r^*} \le C\|\mathbf{a}\|_{r^*}$ and $\|e^{-(t-t_0)B}b\|_{r^*} \le C\|b\|_{r^*}$ are finite. Moreover, it is not difficult to see that

$$\left| \left(\int_{t_0}^t e^{-(t-s)\mathbf{A}} P[(\mathbf{v} \cdot \nabla)\mathbf{v}], \boldsymbol{\phi} \right) \right| \le C \left(\int_{t_0}^t \|\mathbf{v}\|_{r^*}^2 (t-s)^{-\frac{n}{2}\left(\frac{1}{(r^*)'} - \frac{1}{(r^*/2)'}\right) - \frac{1}{2}} \right) \|\boldsymbol{\phi}\|_{(r^*)'}$$

$$\le C \, \|\mathbf{v}\|_{r^*}^2 \left(\int_{t_0}^t (t-s)^{-\frac{n}{2r^*} - \frac{1}{2}} \, ds \right) \|\boldsymbol{\phi}\|_{(r^*)'} \le C \, \|\boldsymbol{\phi}\|_{(r^*)'},$$

for all $\boldsymbol{\phi} \in \mathbf{C}_{0,\sigma}^\infty$. By duality, we have that $\int_{t_0}^t e^{-(t-s)\mathbf{A}} P[(\mathbf{v} \cdot \nabla)\mathbf{v}](s) \, ds \in \mathbf{L}_\sigma^{r^*}(\Omega)$.

Analogously, we can see that

$$\int_{t_0}^t e^{-(t-s)B}(\mathbf{v} \cdot \nabla w)(s) \, ds + \int_{t_0}^t e^{-(t-s)\mathbf{A}} P(w \, \mathbf{g})(s) \, ds \in \mathbf{L}^{r^*}(\Omega), \int_{t_0}^t e^{-(t-s)B} f(s) \, ds \in L^{r^*}(\Omega).$$

Therefore we conclude that equalities (3.34)-(3.35) are satisfied in $\mathbf{L}^{r^*}_\sigma(\Omega)$ and $L^{r^*}(\Omega)$, respectively. The proof of Proposition 3.11 is finished. □

Proof of Theorem 3.10. By the hypothesis of Theorem 3.10, we can apply Proposition 3.11 and hence, there exist $T \in (0,1]$ and functions $\mathbf{v}$ and w satisfying:

$$\begin{cases} \mathbf{v} \in BC_w([t_0, t_0 + T); \mathbf{L}^{(n,\infty)}_\sigma(\Omega)) \cap BC([t_0, t_0 + T); \mathbf{L}^{(q^*,\infty)}_\sigma(\Omega)), \\ w \in BC_w([t_0, t_0 + T); L^{(n,\infty)}(\Omega)) \cap BC([t_0, t_0 + T); L^{(q^*,\infty)}(\Omega)), \\ (t - t_0)^{1/2} \nabla \mathbf{v} \in BC((t_0, t_0 + T); \mathbf{L}^{(n,\infty)}(\Omega))^n, \\ (t - t_0)^{1/2} \nabla w \in BC((t_0, t_0 + T); L^{(n,\infty)}(\Omega))^n, \end{cases}$$

in such a way that for all $t_0 \in \mathbb{R}$ the integral system (3.34)-(3.35) is satisfied in the L^{r^*}-norm.

Firstly, we study the uniqueness of that solution. Let $(\mathbf{v}_1, w_1)$ another solution of (3.34)-(3.35), in the class given by Proposition 3.11, with the same initial condition. Let $(\mathbf{V}, W) = (\mathbf{v} - \mathbf{v}_1, w - w_1)$. Note that $(\mathbf{V}(t), W(t)) \in \mathbf{L}^n_\sigma(\Omega) \times L^n(\Omega)$ for all $0 < t < T$, with

$$\sup_{t_0 < t < T} \|\mathbf{V}(t)\|_n < \infty, \quad \sup_{t_0 < t < T} \|W(t)\|_n < \infty. \tag{3.65}$$

In fact, taking $n < q < \min\{2n, r^*\}$ we have for all $\phi \in \mathbf{C}^\infty_{0,\sigma}$ and all $t_0 < t < T$,

$$\begin{aligned} |(\mathbf{V}(t), \phi)| &\leq \left| \int_{t_0}^t (\mathbf{v} - \mathbf{v}_1 \otimes \mathbf{v}_1, \nabla e^{-(t-s)\mathbf{A}} \phi) \, ds \right| + \left| \int_{t_0}^t (e^{-(t-s)\mathbf{A}} P(W\mathbf{g}), \phi) ds \right| \\ &\leq C(n,q) \left\{ \left(\sup_{t_0 < s < T} s^{n/2(1/n - 1/q)} \|\mathbf{v}\|_q \right)^2 + \left(\sup_{t_0 < s < T} s^{n/2(1/n - 1/q)} \|\mathbf{v}_1\|_q \right)^2 \right\} \|\phi\|_{n'} \\ &\quad + C(n,q) \left\{ \left(\sup_{t_0 < s < T} s^{n/2(1/n - 1/q)} \|w(s)\|_q + \sup_{t_0 < s < T} s^{n/2(1/n - 1/q)} \|w_1(s)\|_q \right) \right. \\ &\quad \left. \times \left(\sup_{t_0 < s < T} s^{1/2} \|\mathbf{g}\|_{(n,\infty)} \right) \right\} \|\phi\|_{n'}. \end{aligned}$$

Analogously, for all $\varphi \in C^\infty_0$ and all $t_0 < t < T$

$$\begin{aligned} |(W(t), \varphi)| &\leq C(n,q) \left\{ \sup_{t_0 < s < T} s^{n/2(1/n - 1/q)} \|\mathbf{v}(s)\|_q \sup_{t_0 < s < T} s^{n/2(1/n - 1/q)} \|w(s)\|_q \right. \\ &\quad + \left. \sup_{t_0 < s < T} s^{n/2(1/n - 1/q)} \|\mathbf{v}_1(s)\|_q \sup_{t_0 < s < T} s^{n/2(1/n - 1/q)} \|w_1(s)\|_q \right\} \|\varphi\|_{n'}. \end{aligned}$$

By duality we can conclude (3.65). We define $K_{\mathbf{V}}(t) \equiv \sup\limits_{t_0 < t < T} \|\mathbf{V}(t)\|_{(n,\infty)}$ and $K_W(t) \equiv \sup\limits_{t_0 < t < T} \|W(t)\|_{(n,\infty)}$. Let p satisfying $1/p = 1 - 1/n - 1/r^*$. Then

$$
\begin{aligned}
|(\mathbf{V}(t), \boldsymbol{\phi})| \;\leq\;& \left| \int_{t_0}^t (\mathbf{V} \otimes \mathbf{v}(s) - \mathbf{v}_1, \nabla e^{-(t-s)\mathbf{A}} \boldsymbol{\phi})\, ds \right| + \left| \int_{t_0}^t (e^{-(t-s)\mathbf{A}} P(W\mathbf{g})(s), \boldsymbol{\phi})\, ds \right| \\
\leq\;& \int_{t_0}^t (\|\mathbf{v}(s)\|_{r^*} + \|\mathbf{v}_1(s)\|_{r^*}) \|\mathbf{V}(s)\|_n \|\nabla e^{-(t-s)L} \boldsymbol{\phi}\|_p\, ds \\
&+ \int_{t_0}^t (t-s)^{-1/2} \|W(s)\|_n \|\mathbf{g}\|_{(n,\infty)} \|\boldsymbol{\phi}\|_{n'}\, ds \\
\leq\;& C\, K_{\mathbf{V}}(t) \left(\sup_{t_0 < s < t_0+t} (s-t_0)^{n/2(1/n - 1/r^*)} \|\mathbf{v}(s)\|_{r^*} \right. \\
&+ \left. \sup_{t_0 < s < t_0+t} (s-t_0)^{n/2(1/n - 1/r^*)} \|\mathbf{v}_1(s)\|_{r^*} \right) \\
&\times \left(\int_{t_0}^t (t-s)^{-n/2(1/n'-1/p)-1/2} (s-t_0)^{-n/2(1/n-1/r^*)}\, ds \right) \|\boldsymbol{\phi}\|_{n'} \\
&+ K_W \sup_{t_0 < s < t_0+t} (s-t_0)^{1/2} \|\mathbf{g}\|_{(n,\infty)} \left(\int_{t_0}^t (t-s)^{-1/2} (s-t_0)^{-1/2}\, ds \right) \|\boldsymbol{\phi}\|_{n'},
\end{aligned}
$$

for all $\boldsymbol{\phi} \in \mathbf{C}_{0,\sigma}^\infty$ and for all $t_0 < t < T$. Then, using duality we obtain

$$
\|\mathbf{V}(t)\|_n \leq K_{\mathbf{V}}(t) \left(\sup_{t_0 < s < t_0+t} (s-t_0)^{n/2(1/n-1/r^*)} \|\mathbf{v}(s)\|_{r^*} + \right.
$$

$$
+ \left. \sup_{t_0 < s < t_0+t} (s-t_0)^{n/2(1/n-1/r^*)} \|\mathbf{v}_1(s)\|_r^* \right) + C_2\, K_W(t) \sup_{t_0 < s < t_0+t} (s-t_0)^{1/2} \|\mathbf{g}\|_{(n,\infty)}. \quad (3.66)
$$

Analogously,

$$
\begin{aligned}
|(W(t), \varphi)| \;\leq\;& C\, K_{\mathbf{V}}(t) \sup_{t_0 < s < t_0+t} (s-t_0)^{n/2(1/n-1/r^*)} \|w_1(s)\|_{r^*} \|\varphi\|_{n'} \\
&+ C\, K_W(t) \sup_{t_0 < s < t_0+t} (s-t_0)^{n/2(1/n-1/r^*)} \|\mathbf{v}(s)\|_{r^*} \|\varphi\|_{n'},
\end{aligned}
$$

for all $\varphi \in C_0^\infty$ and all $t_0 < t < T$. By duality,

$$
\begin{aligned}
\|W(t)\|_n \;\leq\;& C_3\, K_{\mathbf{V}}(t) \sup_{t_0 < s < t_0+t} (s-t_0)^{n/2(1/n-1/r^*)} \|w_1(s)\|_{r^*} \\
&+ C_4\, K_W(t) \sup_{t_0 < s < t_0+t} (s-t_0)^{n/2(1/n-1/r^*)} \|\mathbf{v}(s)\|_{r^*}. \quad (3.67)
\end{aligned}
$$

Let $K(t) = \max\{K_{\mathbf{V}}(t), K_W(t)\}$. Then we have that

$$
\begin{aligned}
\max\left\{\|\mathbf{V}(t)\|_n, \|W(t)\|_n\right\} \leq\;& C_*\, K(t) \left\{ \sup_{t_0 < s < t_0+t} (s-t_0)^{n/2(1/n-1/r^*)} \|\mathbf{v}(s)\|_{r^*} \right. \\
&+ \sup_{t_0 < s < t_0+t} (s-t_0)^{n/2(1/n-1/r^*)} \|\mathbf{v}_1(s)\|_{r^*} + \sup_{t_0 < s < t_0+t} (s-t_0)^{n/2(1/n-1/r^*)} \|w(s)\|_{r^*} \\
&+ \left. \sup_{t_0 < s < t_0+t} (s-t_0)^{n/2(1/n-1/r^*)} \|w_1(s)\|_{r^*} \right\} + C_{**}\, K(t) \sup_{t_0 < s < t_0+t} (s-t_0)^{1/2} \|\mathbf{g}\|_{(n,\infty)}.
\end{aligned}
$$

We define the constant $\widetilde{C} = C_* + C_{**}$. From hypothesis of this theorem, there exists $t_0 < t_1 \leq T$, such that

$$
\sup_{t_0 < s < t_0+t} (s-t_0)^{n/2(1/n-1/r^*)} (\|w(s)\|_{r^*} + \|w_1(s)\|_{r^*} + \|\mathbf{v}(s)\|_{r^*} + \|\mathbf{v}_1(s)\|_{r^*})
$$

$$
+ \sup_{t_0 < s < t_0+t} (s-t_0)^{1/2} \|\mathbf{g}\|_{(n,\infty)} < \frac{1}{2\widetilde{C}}, \qquad \text{for } t_0 < t < t_1.
$$

Therefore, we have $\max\{\|\mathbf{V}(t)\|_n, \|W(t)\|_n\} \le \frac{1}{2}K(t)$, for $t_0 < t < t_1$. As $K(t)$ is not increasing, we conclude that $\|\mathbf{V}(t)\|_n \equiv 0, \quad \|W(t)\|_n \equiv 0$, for $t_0 < t \le t_1$ and hence $\mathbf{v}(t) = \mathbf{v}_1(t), w(t) = w_1(t)$, on the interval $[t_0, t_1]$.

Now, it remains to show that

$$\mathbf{v}(t) = \mathbf{v}_1(t), \quad w(t) = w_1(t), \text{ on } [t_0, T). \tag{3.68}$$

By definition, we have that

$$\sup_{t_1 < s < T} \|\mathbf{v}(s)\|_{r^*} + \sup_{t_1 < s < T} \|\mathbf{v}_1(s)\|_{r^*} \equiv M_1 < \infty,$$
$$\sup_{t_1 < s < T} \|w(s)\|_{r^*} + \sup_{t_1 < s < T} \|w_1(s)\|_{r^*} \equiv M_2 < \infty.$$

Thus, in order to conclude (3.68) we use the following proposition

Proposition 3.17 *There exists a constant $\xi = \xi(n, r, M_1, M_2)$ such that, if $\mathbf{v} = \mathbf{v}_1$ and $w = w_1$ on $[t_0, \tau]$ for all $\tau \in [t_1, T)$, then $\mathbf{v} = \mathbf{v}_1$ and $w = w_1$ in $[t_0, \tau + \xi]$.*

Proof. We define $D_{\mathbf{V}}(t) \equiv \sup_{\tau < s < t} \|\mathbf{V}(s)\|_n$ and $D_W(t) \equiv \sup_{\tau < s < t} \|W(s)\|_n$. Working similarly as the calculus in (3.66)-(3.67), we have

$$
\begin{aligned}
|(\mathbf{V}(t), \boldsymbol{\phi})| &\le \int_\tau^t (\|\mathbf{v}(s)\|_{r^*} + \|\mathbf{v}_1(s)\|_{r^*})\|\mathbf{V}(s)\|_n \|\nabla e^{-(t-s)\mathbf{A}}\boldsymbol{\phi}\|_p \, ds \\
&+ \int_\tau^t (t-s)^{-1/2} \|W(s)\|_n \|\mathbf{g}\|_{(n,\infty)} \|\boldsymbol{\phi}\|_{n'} \\
&\le CM_1 D_{\mathbf{V}}(t-\tau)^{\frac{r^*-n}{2r^*}} \|\boldsymbol{\phi}\|_{n'} + CD_W(t)(t-\tau)^{1/2}\|\mathbf{g}\|_{(n,\infty)}\|\boldsymbol{\phi}\|_{n'},
\end{aligned}
$$

for all $\boldsymbol{\phi} \in \mathbf{C}_{0,\sigma}^\infty$ and all $\tau < t < T$. Analogously, for any $\varphi \in C_0^\infty$ and any $t, \tau < t < T$, we obtain: $\|W(t)\|_n \le C(M_2 D_{\mathbf{V}}(t) + M_1 D_W(t))(t-\tau)^{\frac{r^*-n}{2r^*}}\|\varphi\|_{n'}$. Let $D(t) \equiv \max\{D_{\mathbf{V}}(t), D_W(t)\}$ and $M \equiv \max\{M_1, M_2\}$. Then we have

$$\max(\|\mathbf{V}(t)\|_n, \|W(t)\|_n) \le \widetilde{C} M D(t) (t-\tau)^{\frac{r^*-n}{2r^*}},$$

for $\widetilde{C} = \max\{C M \|\boldsymbol{\phi}\|_{n'}, C M \|\varphi\|_{(n,\infty)}, C \|\mathbf{g}\|_{n'} T^{n/2r^*}\|\boldsymbol{\phi}\|_{n'}\}$.

Let we define ξ as $\xi \equiv \dfrac{1}{(4\widetilde{C} M)^{\frac{2r^*}{r^*-n}}}$. Then, $D(\tau + \xi) \le \frac{1}{2}D(\tau + \xi)$. Consequently, $D(\tau + \xi) = 0$. This implies that $\mathbf{v} = \mathbf{v}_1$ and $w = w_1$ on $[t_0, \tau + \xi]$. ∎

Now we prove the existence part of Theorem 3.10. Being $(\mathbf{v}, w)$ the integral solution of (3.34)-(3.35), we can prove the time Hölder continuity of

$$\widetilde{\mathbf{F}}(\mathbf{v}, w) \equiv -P(\mathbf{v} \cdot \nabla \mathbf{v}) + P(w \, \mathbf{g}), \quad \widetilde{G}(\mathbf{v}, w) \equiv -(\mathbf{v} \cdot \nabla w) + f,$$

in the L^{r^*}-space. Indeed, we follow the ideas of Kozono and Ogawa [12] (Lemma A.4), and we use the hypothesis of Hölder continuity of function f (that follows using a standard argument).

Finally, using Theorem 3.11 of [17], we conclude that the integral solution $(\mathbf{v}, w)$ verifies Definition 3.9. Hence we conclude the proof of Theorem 3.10. $\qquad\square$

Proof of Theorem 2.3. Let $(\mathbf{u}, \theta)$ be the periodic solution of (2.8),(2.9) given by Theorem 2.2. As $\mathbf{u} \in Y, \theta \in X$, from Lemma 3.4 we have that $\mathbf{u} \in BC(\mathbb{R}; \mathbf{L}_\sigma^{(n,\infty)}(\Omega) \cap \mathbf{L}^{(q^*,\infty)}(\Omega))$, $\theta \in BC(\mathbb{R}; L^{(n,\infty)}(\Omega) \cap L^{(q^*,\infty)}(\Omega))$. Let T defined by (3.33). By Theorem 3.10, for each $t_0 \in \mathbb{R}$, there exists a unique strong solution $(\mathbf{v}, w)$ of (1.1)-(1.5) on $(t_0, t_0 + T)$ with the initial data $(\mathbf{u}(t_0), \theta(t_0))$. From (3.25), (3.32) we have

$$\sup_{t_0 < t < t_0+T} \|\mathbf{v}(t)\|_{(q^*,\infty)} + \sup_{t_0 < t < t_0+T} \|\nabla \mathbf{u}(t)\|_{(q,\infty)} \le C_{7,1}, \tag{3.69}$$

$$\sup_{t_0 < t < t_0+T} \|w(t)\|_{(q^*,\infty)} + \sup_{t_0 < t < t_0+T} \|\nabla \theta(t)\|_{(q,\infty)} \le C_{7,2}, \tag{3.70}$$

where $C_{7,1}, C_{7,2}$ are independents of t_0. Replacing $(\mathbf{a}, b)$ by $(\mathbf{u}(t_0), \theta(t_0))$ in (3.34)-(3.35), and using estimates similar to (2.8) and (2.9) (but with integrals defined in (t_0, t)), we can see for t, $t_0 < t < t_0 + T$,

$$\begin{aligned} \mathbf{u}(t) - \mathbf{v}(t) &= -\int_{t_0}^t e^{-(t-s)\mathbf{A}} P[((\mathbf{u} - \mathbf{v}) \cdot \nabla)\mathbf{u}]\, ds + \int_{t_0}^t e^{-(t-s)\mathbf{A}} P((\theta - w)\, \mathbf{g})\, ds \\ &\quad - \int_{t_0}^t e^{-(t-s)\mathbf{A}} P[(\mathbf{v} \cdot \nabla)(\mathbf{u} - \mathbf{v})](s)\, ds \equiv I_1(t) + I_2(t) + I_3(t), \end{aligned} \tag{3.71}$$

$$\begin{aligned} \theta(t) - w(t) &= -\int_{t_0}^t e^{-(t-s)B}((\mathbf{u} - \mathbf{v}) \cdot \nabla \theta)\, ds - \int_{t_0}^t e^{-(t-s)B}(\mathbf{v} \cdot \nabla(\theta - w))\, ds \\ &\equiv I_4(t) + I_5(t). \end{aligned} \tag{3.72}$$

Note that, for every t, $t_0 < t < t_0 + T$,

$$\|I_1(t)\|_{(n,\infty)} \le C \sup_{s \in \mathbb{R}} \|\nabla \mathbf{u}(s)\|_{(q,\infty)} \sup_{t_0 < s < t_0+t} \|(\mathbf{u} - \mathbf{v})(s)\|_{(n,\infty)} (t - t_0)^{(1-n/2q)}, \tag{3.73}$$

where $C = C(n, q)$ is independent of t_0. Moreover, for any $\phi \in \mathbf{C}_{0,\sigma}^\infty$ and t, $t_0 < t < t_0 + T$:

$$|(I_2(t), \phi)| \le C \sup_{t_0 < s < t_0+t} \|\mathbf{v}(s)\|_{(q^*,\infty)} \sup_{t_0 < s < t_0+t} \|\mathbf{u}(s) - \mathbf{v}(s)\|_{(n,\infty)} (t - t_0)^{1-n/2q} \|\phi\|_{(n',1)},$$

being C independent of t_0. By duality, we have that for any t, $t_0 < t < t_0 + T$,

$$\|I_2(t)\|_{(n,\infty)} \le C \sup_{t_0 < s < t_0+t} \|\mathbf{v}(s)\|_{(q^*,\infty)} \sup_{t_0 < s < t_0+t} \|\mathbf{u}(s) - \mathbf{v}(s)\|_{(n,\infty)} (t - t_0)^{1-n/2q}, \tag{3.74}$$

$$\begin{aligned} \|I_3(t)\|_{(n,\infty)} &\le C \int_{t_0}^t (t - s)^{-n/2b} \|\theta(s) - w(s)\|_{(n,\infty)} \|\mathbf{g}\|_{(b,\infty)} \\ &\le C \sup_{t_0 < s < t_0+t} \|\theta(s) - w(s)\|_{(n,\infty)} (t - t_0)^{1-n/2b}, \end{aligned} \tag{3.75}$$

with C independent of t_0. Analogously we have that for any $t \in (t_0, t_0 + T)$,

$$\|I_4(t)\|_{(n,\infty)} \le C \sup_{t_0 < s < t_0+t} \|\nabla \theta(s)\|_{(\tilde{q},\infty)} \sup_{t_0 < s < t_0+t} \|(\mathbf{u} - \mathbf{v})(s)\|_{(n,\infty)} (t - t_0)^{(1-n/2\tilde{q})}, \tag{3.76}$$

$$\|I_5(t)\|_{(n,\infty)} \le C \sup_{t_0 < s < t_0+t} \|\mathbf{v}(s)\|_{(q^*,\infty)} \sup_{t_0 < s < t_0+t} \|\theta(s) - w(s)\|_{(n,\infty)} (t - t_0)^{1-n/2q}, \tag{3.77}$$

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with C independent of t_0. From (3.69)-(3.77) follow that for all $t \in (t_0, t_0 + T)$,

$$
\begin{aligned}
\|\mathbf{u}(t) - \mathbf{v}(t)\|_{(n,\infty)} \ \leq \ & C_1 \Big(\sup_{t_0 < s < t_0 + t} \|\mathbf{u}(s) - \mathbf{v}(s)\|_{(n,\infty)} (t - t_0)^{1-n/2q} \\
& + \sup_{t_0 < s < t_0 + t} \|\theta(s) - w(s)\|_{(n,\infty)}) (t - t_0)^{1-n/2b} \Big), \qquad (3.78) \\
\|\theta(t) - w(t)\|_{(n,\infty)} \ \leq \ & C_2 \Big(\sup_{t_0 < s < t_0 + t} \|\mathbf{u}(s) - \mathbf{v}(s)\|_{(n,\infty)} (t - t_0)^{1-n/2\widetilde{q}} \\
& + \sup_{t_0 < s < t_0 + t} \|\theta(s) - w(s)\|_{(n,\infty)} (t - t_0)^{1-n/2q} \Big), \qquad (3.79)
\end{aligned}
$$

where C_1, C_2 are independents of t_0. Taking $E(t) = \max\{\|\mathbf{u}(t) - \mathbf{v}(t)\|_{(n,\infty)}, \|\theta(t) - w(t)\|_{(n,\infty)}\}$, for all $t \in (t_0, T_0 + T)$, from (3.78)-(3.79) it follows that for all $t_0 < t < t_0 + T$

$$
E(t) \leq C_3 \sup_{t_0 < s < t_0 + t} E(s)(t - t_0)^{1-n/2p},
$$

where $p = \max(b, q)$. Therefore, $E(t) \leq C_3 \sup_{t_0 < s < t_0 + t} E(s) T^{1-n/2p}$.

Taking $\varsigma \equiv \min\{(1/2C_3)^{2p/(2p-n)}, T\}$, we conclude that: for every t, $t_0 < t < t_0 + T$,

$$
E(t) \leq C_3 \varsigma^{1-n/2p} \sup_{t_0 < s < t_0 + t} E(T) \leq 1/2 \sup_{t_0 < t < t_0 + t} E(s),
$$

and hence we obtain $E(t) \equiv 0$ on $[t_0, t_0 + \varsigma)$. Since ς can be taken independently of t_0, we have $E(t) \equiv 0$ on $[t_0, t_0 + T)$. This implies that $\mathbf{u} = \mathbf{v}$ on $[t_0, t_0 + T)$ and $\theta = w$ on $[t_0, t_0 + T)$. Finally, as t_0 is an arbitrary time in Theorem 3.10, we conclude that $(\mathbf{u}, \theta)$ is the required solution in Theorem 2.3.

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