# Existence of a positive solution for a singular system

Marcelo Montenegro
Universidade Estadual de Campinas, IMECC,
Departamento de Matemática,
Caixa Postal 6065 CEP 13083-970, Campinas, SP Brazil
E-mail: msm@ime.unicamp.br

Antonio Suárez
Dpto. Ecuaciones Diferenciales y Análisis Numérico
Fac. Matemáticas, C/ Tarfia s/n
C.P. 41012, Univ. Sevilla, Spain
E-mail: suarez@us.es

ABSTRACT. We show the existence and nonexistence of positive solutions to a system of singular elliptic equations with Dirichlet boundary condition. This system arises in studies of pattern formation in biology and in the activator-inhibitor model proposed by Gierer-Meinhardt.

AMS SUBJECT CLASSIFICATION 2000. 35J25, 35J55.

KEYWORDS. Singular equations, Gierer-Meinhardt model, Elliptic system, Sub-Supersolutions method

## 1 Introduction

In this paper we study the system

$$\begin{cases} -\Delta u = \lambda u^{q_1} - \dfrac{u^{p_1}}{v^{\beta_1}} & \text{in } \Omega, \\ -\Delta v = \mu v^{q_2} - \dfrac{u^{p_2}}{v^{\beta_2}} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$,

$$\lambda, \mu \in \mathbb{R}, \quad 0 < q_1, q_2, \beta_1, \beta_2 < 1 \quad \text{and} \quad p_1, p_2 > 0. \tag{1.2}$$

Our main goal in this paper is to show results about existence and nonexistence of positive solutions of (1.1) in terms of the parameters $\lambda$ and $\mu$. It is clear that, thanks to the maximum principle, if $\lambda \leq 0$ or $\mu \leq 0$ then (1.1) does not possess positive solutions. With respect to the existence, our main result is

**Theorem 1.1.** *(A) Assume that $q_1 < p_1$. There is a constant $\lambda^*(\Omega) > 0$ depending on $\Omega$ such that for*

$$\mu \geq \lambda^*(\Omega)\lambda^{\sigma} \quad \text{and} \quad \lambda > 0$$

*where*

$$\sigma = \frac{p_2(1-q_2)}{(1+\beta_2)(1-q_1)},$$

*there exists a positive $C^{1,\Upsilon}(\overline{\Omega})$, $0 < \Upsilon < 1$ solution of (1.1).*

*(B) Assume that $q_1 \geq p_1$. There is a constant $\lambda_*(\Omega) > 0$ depending on $\Omega$ such that for*

$$\lambda < \lambda_*(\Omega)\mu^{-r} \quad \text{and} \quad \mu > 0,$$

*where*

$$r = \frac{\beta_1(1-q_1)}{(1-p_1)(1-q_2)},$$

*then (1.1) does not possess a positive solution.*

Systems of singular equations like (1.1) are the stationary counterpart of general evolutionary problems of the form

$$\begin{cases} u_t = \eta\Delta u + \lambda u^{q_1} - \gamma \dfrac{u^{p_1}}{v^{\beta_1}} & \text{in } \Omega, \\ v_t = \delta\Delta v + \mu v^{q_2} - \theta \dfrac{u^{p_2}}{v^{\beta_2}} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In the original model proposed by Gierer-Meinhardt [10],

$$\eta,\delta > 0, \quad \lambda,\mu,\gamma,\theta < 0, \quad q_1 = q_2 = 1, \quad p_1,p_2,\beta_1,\beta_2 > 0, \quad 0 < (p_1-1)/\beta_1 < p_2/(\beta_2+1)$$

and the boundary conditions are of Neumann type. This system was motivated by biological experiments on hydra in morphogenesis, where $u$ represents the density of an activator chemical substance and $v$ is an inhibitor. The slow diffusion of $u$ and the fast diffusion of $v$ is translated into the fact that $\eta$ is small and $\delta$ is large, see also [11, 16, 18] for an account on biological applications of such systems. There are a few papers dealing with scalar equations [1, 4, 5, 8, 19] and references therein.

According to an observation made in [3], it is natural to study (1.3) with Dirichlet boundary conditions, since numerical experiments from [10] exhibit solutions approaching zero near the boundary of $\Omega$. Moreover, Neumann condition is not explicitly mentioned in the original paper [10]. Although, the majority of early papers deal with a system on a bounded domain with Neumann boundary conditions.

The stationary system with

$$\eta = \delta = 1, \quad \lambda = \mu = \gamma = \theta = -1 \quad \text{and} \quad p_1 = p_2 = q_1 = q_2 = \beta_1 = \beta_2 = 1.$$

was studied in [2]. Thus for the system

$$\begin{cases} -\Delta u = -u + \dfrac{u^{p_1}}{v^{\beta_1}} & \text{in } \Omega, \\ -\Delta v = -v + \dfrac{u^{p_2}}{v^{\beta_2}} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

they have shown existence and nonexistence of solutions and uniqueness of solution in one dimension. Another uniqueness result for (1.4) was proved in [3], in the situation

$$\eta = \delta = 1, \quad \lambda = \mu = \gamma = \theta = -1 \quad \text{and} \quad p_1 = p_2 > 1, \quad \beta_2 = 0, \quad \beta_1 = q_1 = q_2 = 1.$$

A study allowing more general singular nonlinearities was performed in [9, 13, 14].

We are interested in studying stationary states of (1.3) for a different range of parameters and constants (1.2). Notice that our results depend strongly on the size of $q_1$ and $p_1$. Indeed, in the existence part (A) of Theorem 1.1 we require $q_1 < p_1$, and the conclusion holds for $\lambda > 0$ and $\mu \geq C\lambda^{\sigma}$ for some positive constants $C$ and $\sigma$. Part (B) demands $q_1 \geq p_1$, thus the nonexistence of solution is inferred for $\lambda > 0$ and $\mu < C\lambda^{-r}$ for some positive constants $C$ and $r$. In order to obtain our main results we use an adequate sub-supersolution method, which will be detailed later.

The paper is organized as follows. In section 2 we show that the sub-supersolution method holds for our system, which has singular nonlinearities, generalizing classical results, see for instance [17]. In section 3 we study some auxiliary problems related to sublinear equations, singular equations and porous medium logistic equation. Section 4 is devoted to the proof of Theorem 1.1.

## 2 The sub-super method for singular systems

First of all we show that the sub-supersolution method works well for singular systems. We consider the general system

$$\begin{cases} -\Delta u = f(x,u,v) & \text{in } \Omega, \\ -\Delta v = g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f, g : \Omega \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are Caratheodory functions. On the other hand, we denote by

$$\rho_0(x) = dist(x, \partial\Omega),$$

and given $w \leq z$ a.e. in $\Omega$

$$[w,z] := \{u : w(x) \leq u(x) \leq z(x) \quad \text{a.e. } x \in \Omega\}.$$

The notions of solutions and sub-supersolutions of (2.1) are:

**Definition 2.1.** *We say that $(u,v) \in (L^1(\Omega))^2$ is a solution of (2.1) if*

1. $f(\cdot,u,v)\rho_0, g(\cdot,u,v)\rho_0 \in L^1(\Omega)$;

2.
$$-\int_\Omega u\Delta\xi = \int_\Omega f(x,u,v)\xi, \quad -\int_\Omega v\Delta\xi = \int_\Omega g(x,u,v)\xi, \quad \forall \xi \in C_0^2(\overline{\Omega}).$$

**Definition 2.2.** *We say that* $(\underline{u},\underline{v})$, $(\overline{u},\overline{v}) \in (L^1(\Omega))^2$ *is a pair of sub-supersolutions of (2.1) if*

1. $\underline{u} \le \overline{u}$ *and* $\underline{v} \le \overline{v}$ *in* $\Omega$*;*

2.
$$\begin{aligned} &f(\cdot,u,v)\rho_0, f(\cdot,u,v)\rho_0 \in L^1(\Omega) \quad \textit{for all } u \in [\underline{u},\overline{u}] \textit{ and } v \in [\underline{v},\overline{v}],\\ &g(\cdot,u,v)\rho_0, g(\cdot,u,v)\rho_0 \in L^1(\Omega) \quad \textit{for all } u \in [\underline{u},\overline{u}] \textit{ and } v \in [\underline{v},\overline{v}]; \end{aligned} \tag{2.2}$$

3. *for all* $\xi \in C_0^2(\overline{\Omega}), \xi \ge 0$,
$$-\int_\Omega \underline{u}\Delta\xi - \int_\Omega f(x,\underline{u},v)\xi \le 0 \le -\int_\Omega \overline{u}\Delta\xi - \int_\Omega f(x,\overline{u},v)\xi, \quad \forall v \in [\underline{v},\overline{v}];$$
*and*
$$-\int_\Omega \underline{v}\Delta\xi - \int_\Omega g(x,u,\underline{v})\xi \le 0 \le -\int_\Omega \overline{v}\Delta\xi - \int_\Omega g(x,u,\overline{v})\xi, \quad \forall u \in [\underline{u},\overline{u}].$$

Next we prove that the existence of a pair of sub-supersolutions implies the existence of a solution of the system.

**Theorem 2.3.** *Assume that there exists a pair of sub-supersolution* $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ *of (2.1). Then, there exists a solution* $(u,v)$ *of (2.1) such that* $\underline{u} \le u \le \overline{u}$ *and* $\underline{v} \le v \le \overline{v}$ *in* $\Omega$.

*Proof.* First, we define the truncations
$$Tu(x) := \begin{cases} \overline{u}(x) & \text{if } u(x) \ge \overline{u}(x),\\ u(x) & \text{if } \underline{u}(x) \le u(x) \le \overline{u}(x),\\ \underline{u}(x) & \text{if } u(x) \le \underline{u}(x), \end{cases} \tag{2.3}$$
and
$$Sv(x) := \begin{cases} \overline{v}(x) & \text{if } v(x) \ge \overline{v}(x),\\ v(x) & \text{if } \underline{v}(x) \le v(x) \le \overline{v}(x),\\ \underline{v}(x) & \text{if } v(x) \le \underline{v}(x). \end{cases} \tag{2.4}$$
We denote by
$$L^1(\rho_0,\Omega) := \{u : u\rho_0 \in L^1(\Omega)\}.$$
We define the Nemytskii operators (well defined by (2.2))
$$\begin{aligned} F : L^1(\Omega) \times L^1(\Omega) &\mapsto L^1(\rho_0,\Omega)\\ (u,v) &\mapsto F(u,v) := f(x,Tu,Sv) \end{aligned}$$

and similarly

$$\begin{aligned} G : L^1(\Omega) \times L^1(\Omega) &\mapsto L^1(\rho_0, \Omega) \\ (u, v) &\mapsto G(u, v) := g(x, Tu, Sv). \end{aligned}$$

We define the operator $K : L^1(\rho_0, \Omega) \mapsto L^1(\Omega)$ by $h \mapsto w := K(h)$, being $w$ the unique solution of

$$\begin{cases} -\Delta w = h & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

It can be proved:

1. $F$ and $G$ are continuous (Theorem 2.1 in [15], the notion of equi-integrability is not needed here).
2. $[F, G](L^1(\Omega))^2$ is bounded in $L^1(\rho_0, \Omega)$, since $T$ and $S$ defined by (2.3) and (2.4) are bounded.
3. $K \circ F$ and $K \circ G$ are continuos and compact operators from $(L^1(\Omega))^2$ to $L^1(\Omega)$ (Theorem 3.1 in [15]).

Then, by the Schauder's fixed point theorem, we can conclude the existence of a solution $(u, v) \in (L^1(\Omega))^2$ of

$$\begin{cases} -\Delta u = f(x, Tu, Sv) & \text{in } \Omega, \\ -\Delta v = g(x, Tu, Sv) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

We claim that $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ and so $(u, v)$ is solution of (2.1). Indeed, let

$$w := u - \overline{u}.$$

Then, for all $V \in [\underline{v}, \overline{v}]$ and all $\xi \in C_0^2(\overline{\Omega}), \xi \geq 0$, we get

$$-\int_\Omega w \Delta\xi \leq \int_\Omega (f(x, Tu, Sv) - f(x, \overline{u}, V))\xi$$

and then taking $V = Sv$

$$-\int_\Omega w \Delta\xi \leq \int_\Omega (f(x, Tu, Sv) - f(x, \overline{u}, Sv))\xi.$$

Then, applying the Kato's inequality (see Proposition 3.1 in [15]) we obtain

$$-\int_\Omega w^+ \Delta\xi \leq \int_{[u \geq \overline{u}]} (f(x, Tu, Sv) - f(x, \overline{u}, Sv))\xi = 0 \quad \forall \xi \in C_0^2(\overline{\Omega}), \xi \geq 0.$$

We deduce that $w^+ = 0$ a.e.; and conclude the proof. □

**Remark 2.4.** *Assuming more regularity to $f$, $g$ and the pair of sub-supersolution, we can obtain that the solution lies in a better space, see Section 5 in [15]. See also Remark 3.6.*

## 3 Some auxiliary problems

In order to find a pair of sub-supersolutions of (1.1) we need to study some scalar equations. First of all, given $\lambda \in \mathbb{R}$ and $0 < q < 1$, consider

$$\begin{cases} -\Delta u = \lambda u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

It is well-known that there exists a unique positive solution of (3.1) if, and only if, $\lambda > 0$. We denote this solution by $\omega_{[\lambda,q]}$; moreover

$$\omega_{[\lambda,q]} = \lambda^{1/(1-q)}\omega_{[1,q]}.$$

It is known that there exist constants $k$ and $K$ with $0 < k < K < +\infty$ such that

$$k\rho_0(x) \le \omega_{[\lambda,q]}(x) \le K\rho_0(x) \quad x \in \Omega. \tag{3.2}$$

We need to study the following problem

$$\begin{cases} -\Delta u = \lambda f(x,u) - \dfrac{a(x)}{u^\beta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

where $\beta \in (0,1)$ and

$$a : \Omega \to \mathbb{R} \quad \text{is a continuous positive function,} \tag{3.4}$$

$$\text{there is } 1 < \gamma < 2 \quad \text{such that} \quad \limsup_{x\to\partial\Omega} \frac{a(x)}{\rho_0(x)^{\gamma(1+\beta)-2}} < +\infty, \tag{3.5}$$

$$f : \Omega \times \mathbb{R} \to \mathbb{R} \quad \text{is a continuous function,} \tag{3.6}$$

$$f(x,s) > 0 \quad \text{for } s \neq 0, \tag{3.7}$$

$$\lim_{s\to+\infty} \frac{f(x,s)}{s} = 0 \quad \text{uniformly in } x. \tag{3.8}$$

In the following result we characterize the existence of positive solution of (3.3).

**Proposition 3.1.** *There exists $\lambda^* \in (0,+\infty)$ such that for all $\lambda \ge \lambda^*$, problem (3.3) has a positive a.e. weak solution and no positive solution for $\lambda < \lambda^*$.*

*Proof.* We are going to apply the sub-supersolution method from [15]. Take

$$\underline{u} := c\varphi_1^\gamma, \qquad \overline{u} := Ke,$$

for $c, K > 0$ such that $\underline{u} \le \overline{u}$ in $\Omega$, where $e$ is the unique positive solution of

$$\begin{cases} -\Delta e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\varphi_1 > 0$ is the first eigenfunction of the Laplacian in $H^1_0(\Omega)$ such that $\|\varphi_1\|_\infty = 1$. Recall that there exist positive constants $0 < c < C < \infty$ such that

$$0 < c\rho_0(x) \le e(x), \varphi_1(x) \le C\rho_0(x), \qquad \forall x \in \Omega.$$

First, observe that

$$\left|\lambda f(x,u) - \frac{a(x)}{u^\beta}\right| \rho_0 \in L^1(\Omega), \qquad \forall u \in [\underline{u}, \overline{u}].$$

Indeed, for $u \in [\underline{u}, \overline{u}]$ we have

$$|a(x)u^{-\beta}|\rho_0 \le Ca(x)\rho_0^{-\gamma\beta+1} \le C\rho_0^{\gamma-1} \in L^1(\Omega)$$

if $\gamma - 1 > -1$.

To show that $\underline{u}$ is subsolution, we need to verify

$$-\Delta\underline{u} + \frac{a(x)}{\underline{u}^\beta} = -c\gamma(\gamma-1)\varphi_1^{\gamma-2}|\nabla\varphi_1|^2 + c\gamma\lambda_1\varphi_1^\gamma + a(x)c^{-\beta}\varphi_1^{-\beta\gamma} \le \lambda f(x, c\varphi_1^\gamma) \quad \text{in } \Omega.$$

We distinguish two cases:
(i) Near the boundary $\partial\Omega$:

For every $M > 0$ there is a $\delta > 0$ such that for every

$$x \in \Omega_\delta := \{x \in \Omega : \rho_0(x) < \delta\}$$

one has by (3.5)

$$-c\gamma(\gamma-1)\varphi_1^{\gamma-2}|\nabla\varphi_1|^2 + a(x)c^{-\beta}\varphi_1^{-\beta\gamma} = c^{-\beta}\varphi_1^{\gamma-2}[-c^{1+\beta}\gamma(\gamma-1)|\nabla\varphi_1|^2 + \frac{a(x)}{\varphi_1^{\gamma-2+\beta\gamma}}]$$

$$\le c^{-\beta}\varphi_1^{\gamma-2}[-c^{1+\beta}\gamma(\gamma-1)|\nabla\varphi_1|^2 + M] \le \frac{-c}{2}\gamma(\gamma-1)\varphi_1^{\gamma-2}|\nabla\varphi_1|^2$$

for a sufficiently large $c > 0$.

In this way, taking $\delta$ smaller if necessary, we get

$$-\Delta\underline{u} + \frac{a(x)}{\underline{u}^\beta} \le c\gamma\varphi_1^{\gamma-2}[-\frac{(\gamma-1)}{2}|\nabla\varphi_1|^2 + \lambda_1\varphi_1^2] \le 0.$$

Notice that if $M = 0$, we can take $c > 0$ arbitrary.

(ii) Inner points $x \in \Omega \setminus \overline{\Omega}_\delta$.

Once $c$ has been fixed above, take $\lambda$ large enough in such a way that

$$c^{1+\beta}\gamma\lambda_1\varphi_1^\gamma + a(x)\varphi_1^{-\beta\gamma} \le \lambda c^\beta f(x, c\varphi_1^\gamma).$$

On the other hand, with respect to the supersolution we need that

$$-\Delta\overline{u} \ge \lambda f(x,\overline{u}) - \frac{a(x)}{\overline{u}^\beta},$$

for which it suffices that

$$K \ge \lambda f(x, Ke).$$

This is promptly verified for $K$ large enough thanks to (3.8).

We claim that there is no positive solution of (3.3) if $\lambda > 0$ is small. Indeed, if $u > 0$ is an existing solution, multiply the equation by $\varphi_1$ and integrate. Hence,

$$\int_\Omega \big(\lambda_1 \varphi_1 u + \frac{a(x)}{u^\beta}\varphi_1\big) = \lambda \int_\Omega f(x,u)\varphi_1 \tag{3.9}$$

Let $\delta > 0$ and $\Omega^\delta := \{x \in \Omega : \rho_0(x) > \delta\}$. Thus

$$c\int_{\Omega^\delta} \big(u + \frac{1}{u^\beta}\big)\varphi_1 \le \int_{\Omega^\delta} \big(\lambda_1 u + \frac{a(x)}{u^\beta}\big)\varphi_1 < \lambda \int_\Omega f(x,u)\varphi_1 \tag{3.10}$$

where $c$ is a constant depending on $\delta$, $\Omega$ and $\|a\|_{L^\infty(\Omega^\delta)}$. Since

$$\lambda \int_\Omega f(x,u)\varphi_1 \to 0 \quad \text{as } \lambda \to 0$$

we get a contradiction since $u + 1/u^\beta$ is bounded from below and $\int_\Omega f(x,u)\varphi_1$ is bounded. This last assertion follows from the fact that $u$ is a priori bounded independently from $\lambda$ by a bootstrap argument, since there is a constant $C > 0$ such that $-\Delta u \le C\lambda(1+u)$ for every $u$.

Setting

$$\lambda^* = \inf\{\,\lambda > 0 \mid \text{such that (3.3) has a positive a.e. solution}\,\}.$$

Then $\lambda^* < +\infty$ and for all $\lambda \ge \lambda^*$, problem (3.3) has a positive a.e. weak solution. □

**Remark 3.2.** *If* $\gamma - 2 + \beta\gamma > 0$*, then in view of* (3.5)*,* $a(x) \to 0$ *as* $x \to \partial\Omega$*. This is true if* $\beta \ge 1$ *for example.*

*If* $\gamma - 2 + \beta\gamma < 0$*, then eventually* $0 < \beta < 1$ *and* $a(x) \to 0$ *as* $x \to \partial\Omega$ *or* $a(x) \to +\infty$ *as* $x \to \partial\Omega$*. But with* (3.5) *satisfied.*

We now consider a particular case of (3.3),

$$\begin{cases} -\Delta u = \lambda u^q - a(x)\dfrac{1}{u^\beta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.11}$$

where $0 < q, \beta < 1$ and $a$ verifies (3.4) and (3.5).

**Proposition 3.3.** *There exists* $\lambda^*(a) > 0$ *such that a positive maximal solution of (3.11) exists if, and only if,*

$$\lambda \ge \lambda^*(a).$$

*We denote this maximal solution by* $\Theta_{[\lambda,q,\beta,a]}$*. Moreover, the map* $a \mapsto \lambda^*(a)$ *is increasing. Furthermore, if* $a \in C(\overline{\Omega})$*, there exist constants* $c$ *and* $C$ *such that*

$$c\rho_0(x) \le \Theta_{[\lambda,q,\beta,a]}(x) \le C\rho_0(x). \tag{3.12}$$

*Proof.* The existence of a positive solution as well as $\lambda^*(a)$ follow by Proposition 3.1. The maximality of the solution is due to the fact that any positive solution of (3.3) is a subsolution of (3.1).

The fact that $a \mapsto \lambda^*(a)$ is increasing is immediate.

The existence of the constant $c$ verifying (3.12) is due to the Hopf maximum principle and $C$ is due to the $C^1(\overline{\Omega})$ regularity of the solution, see also Remark 3.6. □

We need some properties of the porous medium logistic equation with a possibly singular weight

$$\begin{cases} -\Delta u = \lambda u^q - N(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.13}$$

where $0 < q < 1$, $p > 0$ with

$$0 < N \leq k\rho_0(x)^\beta, \quad k > 0, \tag{3.14}$$

$N \in C(\Omega)$ and $\beta \in \mathbb{R}$ (possibly negative).

**Proposition 3.4.** *Assume that $\beta + p > -1$.*

1. *If $q < p$, then there exists a unique $C^1(\overline{\Omega})$ positive solution if, and only if, $\lambda > 0$.*
2. *If $q \geq p$, then there exists $\lambda_*(N) \geq 0$ such that there exists a positive $C^1(\overline{\Omega})$ solution if, and only if, $\lambda \geq \lambda_*(N)$.*

   *Moreover, if $N \geq N_0 > 0$ for some $N_0 \in \mathbb{R}$ then $\lambda_*(N) > 0$.*

*Proof.* Take $\overline{u} := Ke$ and $\underline{u} := \varepsilon\varphi_1^r$, $r \geq 1$ and $K, \varepsilon > 0$ positive constants to be chosen later. In order to apply the sub-supersolution method we need that

$$|N(x)u^p|\rho_0 \in L^1(\Omega), \qquad \forall u \in [\underline{u}, \overline{u}].$$

Observe that (3.14) implies

$$|N(x)u^p|\rho_0 \leq K\rho_0^{\beta+p+1}$$

and so $|N(x)u^p|\rho_0 \in L^1(\Omega)$ if

$$\beta + p > -2.$$

First observe that $\underline{u}$ is subsolution of (3.13) provided that

$$r(1-r)\varepsilon^{1-q}\varphi_1^{r(1-q)-2}|\nabla\varphi_1|^2 + r\varepsilon^{1-q}\lambda_1\varphi_1^{r(1-q)} + C\varepsilon^{p-q}\varphi_1^{r(p-q)+\beta} \leq \lambda. \tag{3.15}$$

On the other hand, $\overline{u}$ is supersolution if $K$ is taken large. Take also $K$ large such that $\underline{u} \leq \overline{u}$ in $\Omega$. So, it suffices to verify (3.15). For that, we consider two cases:

1. Assume that $p > q$. Take $r > 1$ such that $r(p-q) + \beta > 0$. Then, recalling that $\|\varphi_1\|_\infty = 1$, (3.15) is satisfied if

$$r\varepsilon^{1-q}\lambda_1 + C\varepsilon^{p-q} \leq \lambda$$

for which it suffices to take $\varepsilon$ sufficiently small.

With respect to the uniqueness, the result follows applying Theorem 2.1 in [6], specifically taking $g(t) = t^q$.

2. Assume now that $p \leq q$. Take now $\varepsilon = 1$. Again we distinguish two cases:
(i) Near the boundary $\partial\Omega$:

Take in this case $r \geq 1$ and $r(1-q) - 2 < r(p-q) + \beta$, or equivalently, $r(1-p) < \beta + 2$. Then we need that $1 < (2+\beta)/(1-p)$ or equivalently $-1 < \beta + p$. In this case, (3.15) is equivalent to

$$\varphi_1^{r(1-q)-2}\left[r(1-r)|\nabla\varphi_1|^2 + r\lambda_1\varphi_1^2 + C\varphi_1^{r(p-1)+\beta+2}\right] \leq \lambda.$$

Take $\delta > 0$ small enough such that

$$r(1-r)|\nabla\varphi_1|^2 + r\lambda_1\varphi_1^2 + C\varphi_1^{r(p-1)+\beta+2} < 0$$

in $\Omega_\delta = \{x \in \Omega : \rho_0(x) < \delta\}$.

(ii) Inner points:

In the region $\Omega \setminus \overline{\Omega}_\delta$ we have that $\varphi_1 \geq c(\delta)$ for some $c(\delta) > 0$. Hence, for (3.15) it is sufficient that

$$r\lambda_1 + C(\delta) \leq \lambda,$$

for some $C(\delta)$. Fixed $\delta$, we can take $\lambda$ large.

Hence, we can define

$$\lambda_*(N) = \inf\{\,\lambda > 0 \mid \text{such that (3.13) has a positive a.e. solution}\,\}.$$

Then $\lambda_*(N) < +\infty$ and for all $\lambda \geq \lambda_*(N)$, problem (3.13) has a positive a.e. weak solution.

Finally, assume that $N \geq N_0 > 0$ and $q \geq p$. Then, multiplying the equation by $\varphi_1$ and integrating we have

$$0 = \int_\Omega \varphi_1 u^p(\lambda u^{q-p} - N - \lambda_1 u^{1-p}) \leq \int_\Omega \varphi_1 u^p(\lambda u^{q-p} - N_0 - \lambda_1 u^{1-p}).$$

Assuming $q > p$, the maximum of the function $f(x) := \lambda x^{q-p} - \lambda_1 x^{1-p}$ is attained at

$$x_M = \left(\frac{\lambda(q-p)}{\lambda_1(1-p)}\right)^{1/(1-q)}$$

and

$$f(x_M) = \lambda^{(1-p)/(1-q)}\left(\frac{q-p}{\lambda_1(1-p)}\right)^{(q-p)/(1-q)}\frac{1-q}{1-p}$$

and so if $\lambda$ is small we have that

$$\int_\Omega \varphi_1 u^p(\lambda u^{q-p} - N_0 - \lambda_1 u^{1-p}) < 0,$$

a contradiction. A similar argument can be used in the case $q = p$. This completes the proof. □

**Remark 3.5.** *Equations (3.3) and (3.11) have been studied in [5] and [19], but with different behavior of $a(x)$ or without $a(x)$. Also, equation (3.13) has been previously studied when $N$ is bounded, see [7] and references therein.*

**Remark 3.6.** *The solutions of Propositions 3.3, 3.4 and Theorem 1.1 (A) belong to $C^{1,\Upsilon}(\overline{\Omega})$, $0 < \Upsilon < 1$. This follows from the results in [12] which says that if $-\Delta u = h$ in $\Omega$ with $u = 0$ on $\partial\Omega$ and $\sup_{\overline{\Omega}} |h(x)|\rho_0^{\Upsilon}(x) < \infty$ for some $0 < \Upsilon < 1$, then $u \in C^{1,1-\Upsilon}(\overline{\Omega})$.*

# 4 Proof of Theorem 1.1

We are going to apply the sub-supersolution method to system (1.1). If we denote

$$f(u,v) := \lambda u^{q_1} - \frac{u^{p_1}}{v^{\beta_1}} \quad g(u,v) := \mu v^{q_2} - \frac{u^{p_2}}{v^{\beta_2}},$$

the third paragraph of the definition of sub-supersolution (Definition 2.2) is equivalent to

$$-\Delta \underline{u} \le f(\underline{u}, \underline{v}), \quad -\Delta \overline{u} \ge f(\overline{u}, \overline{v}),$$

and

$$-\Delta \underline{v} \le g(\overline{u}, \underline{v}), \quad -\Delta \overline{v} \ge g(\underline{u}, \overline{v}).$$

We start the proof of Theorem 1.1:

*Proof.* (A) Take

$$\overline{u} := \omega_{[\lambda,q_1]}, \quad \text{and} \quad \overline{v} := \omega_{[\mu,q_2]}. \tag{4.1}$$

A subsolution is

$$\underline{v} := \Theta_{[\mu,q_2,\beta_2,\omega^{p_2}_{[\lambda,q_1]}]}. \tag{4.2}$$

Observe that $\omega_{[\lambda,q_1]} = \lambda^{1/(1-q_1)}\omega_{[1,q_1]}$ and so $\underline{v}$ verifies

$$-\Delta v = \mu v^{q_2} - \lambda^{p_2/(1-q_1)} \frac{\omega^{p_2}_{[1,q_1]}}{v^{\beta_2}} \quad \text{in } \Omega. \tag{4.3}$$

Under the change of variable

$$V = Rv,$$

where

$$R = \frac{1}{\lambda^{p_2/((1-q_1)(1+\beta_2))}},$$

(4.3) transforms into

$$\begin{cases} -\Delta V = \Lambda V^{q_2} - \dfrac{\omega^{p_2}_{[1,q_1]}}{V^{\beta_2}} & \text{in } \Omega, \\ V = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

where

$$\Lambda = \mu \lambda^{-\sigma},$$

with

$$\sigma = \frac{p_2(1-q_2)}{(1-q_1)(1+\beta_2)}$$

Observe that (4.4) is in the setting of (3.11) by taking $a = \omega^{p_2}_{[1,q_1]}$. Indeed, (3.4) and (3.5) are verified for all $\gamma$ such that

$$\gamma \le \frac{p_2+2}{1+\beta_2},$$

which can be chosen $1 < \gamma$. Hence, applying Proposition 3.3, we conclude the existence of a positive solution of (4.4) if

$$\Lambda \ge \lambda^*(\Omega)$$

or equivalently,

$$\mu \geq \lambda^*(\Omega)\lambda^\sigma.$$

It is clear that $\underline{v} \leq \overline{v}$ and $\underline{v} > 0$ if $\mu \geq \lambda^*(\Omega)\lambda^\sigma$. It remains to check that there exists $\underline{u} > 0$ and satisfies

$$-\Delta\underline{u} \leq \lambda\underline{u}^{q_1} - \underline{v}^{-\beta_1}\underline{u}^{p_1} \quad \text{in } \Omega.$$

Let $u$ be the solution of

$$\begin{cases} -\Delta u = \lambda u^{q_1} - \underline{v}^{-\beta_1}u^{p_1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

Observe that in this case $N(x) = \underline{v}^{-\beta_1}$, being $\underline{v}$ defined in (4.2). Hence, taking into account (3.12) we obtain that $0 < N \leq C\rho_0^{-\beta_1}$ and so it is clear that

$$-\beta_1 + p_1 > -1.$$

Thus we can apply Proposition 3.4 to conclude that, if $q_1 < p_1$, there exists a positive solution of (4.5) provided $\lambda > 0$. Moreover, it is clear that $\underline{u} \leq \overline{u}$.

Finally, the second paragraph of Definition 2.2 is easy to verify.

In conclusion, if $q_1 < p_1$ there is a positive solution of (1.1) if $\lambda > 0$ and $\mu \geq \lambda^*(\Omega)\lambda^\sigma$.

(B) Finally, we assume that $q_1 \geq p_1$. Observe that if $(u, v)$ is a solution of (1.1), then

$$v \leq \omega_{[\mu,q_2]} = \mu^{1/(1-q_2)}\omega_{[1,q_2]}$$

and then,

$$-\Delta u \leq \lambda u^{q_1} - \mu^{-\beta_1/(1-q_2)}\omega_{[1,q_2]}^{-\beta_1}u^{p_1}.$$

Under the change of variable

$$U = Ru, \qquad R = \mu^{\beta_1/((1-q_2)(1-p_1))}$$

the above inequality is transformed into

$$\begin{cases} -\Delta U \leq \lambda\mu^r U^{q_1} - \omega_{[1,q_2]}^{-\beta_1}U^{p_1} & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, multiplying by $\varphi_1$, integrating and with a similar argument to the proof of Proposition 3.4, we can conclude that if

$$\lambda\mu^r < \lambda_*(\Omega),$$

there is no positive solution of (1.1). □

Acknowledgement. M. Montenegro was supported by CNPq-Brazil and A. Suárez was supported by MEC under grants MTM2006-07932.

## References

[1] Y. S. Choi, A. C. Lazer and P.J. McKenna, *Some remarks on a singular elliptic boundary value problem*, Nonlinear Anal. **32** (1998), 305–314.

[2] Y. S. Choi and P. J. McKenna, *A singular Gierer-Meinhardt system of elliptic equations*, Ann. Inst. Henri Poincaré, Analyse non linèaire **17** (2000), 503-522.

[3] Y. S. Choi and P. J. McKenna, *A singular Gierer-Meinhardt system of elliptic equations: the classical case*, Nonlinear Anal. **55** (2003), 521–541.

[4] M. G. Crandall, P. H. Rabinowitz and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations **2** (1977), 193–222.

[5] J. Dávila and M. Montenegro, *Positive versus free boundary solutions to a singular elliptic equation*, J. Anal. Math., **90** (2003), 303–335.

[6] M. Delgado and A. Suárez, *On the uniqueness of positive solution of an elliptic equation*, Applied Mathematics Letters **18** (2005), 10891093.

[7] M. Delgado and A. Suárez, *On the structure of the positive solutions of the logistic equation with nonlinear diffusion*, J. Math. Anal. Appl. **268** (2002) 200-216.

[8] J. I. Díaz, J. M. Morel and L. Oswald, *An elliptic equation with singular nonlinearity*, Comm. Partial Differential Equations **12** (1987), 1333–1344.

[9] M. Ghergu and V. Radulescu, *On a class of singular Gierer-Meinhardt systems arising in morphogenesis*, C. R. Acad. Sci. Paris, Ser. I **344** (2007), 163-168.

[10] A. Gierer and H. Meinhardt, *A theory of biological pattern formation*, Kyberntik **12** (1972), 30-39.

[11] A. Gierer and H. Meinhardt, *Generation and regeneration of sequence of structures during morphogenesis*, J. Theoret. Biol. **85** (1980), 429-450.

[12] C. Gui and F. H. Lin, *Regularity of an elliptic problem with a singular nonlinearity*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), 1021–1029.

[13] E. H. Kim, *A class of singular Gierer-Meinhardt systems of elliptic boundary value problems*, Nonlinear Anal. **59** (2004), 305-318.

[14] E. H. Kim, *Singular Gierer-Meinhardt systems of elliptic boundary value problems*, J. Math. Anal. Appl. **308** (2005), 1-10.

[15] M. Montenegro and A. Ponce, *The sub-supersolution method for weak solutions*, Proc. Amer. Math. Soc. **136** (2008), 2429–2438.

[16] W. M. Ni, *Diffusion, cross-diffusion, and their spike-layer steady states*, Notices Amer. Math. Soc. **45** (1998), 9-18.

[17] C. V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York, 1992.

[18] W. M. Ni, K. Suzuki and I. Takagi, *The dynamics of a kinetics activator-inhibitor system*, J. Differential Equations **229** (2006), 426-465.

[19] J. Shi and M. Yao, *On a singular nonlinear semilinear elliptic problem*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), 1389–1401.