Stability, instability, and bifurcation phenomena in non-autonomous differential equations

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Abstract. There is a vast body of literature devoted to the study of bifurcation phenomena in autonomous systems of differential equations. However, there is currently no well-developed theory that treats similar questions for the non-autonomous case. Inspired in part by the theory of pullback attractors, we discuss generalisations of various autonomous concepts of stability, instability, and invariance. Then, by means of relatively simple examples, we illustrate how the idea of a bifurcation as a change in the structure and stability of invariant sets remains a fruitful concept in the non-autonomous case.

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1. Introduction

The first step in the qualitative theory of autonomous differential equations is an analysis of fixed points and their stability. This leads on naturally to a study of how the behaviour near such fixed points can change as a parameter is varied, and this is the genesis of the whole theory of bifurcations. As one of the main tools in the study of autonomous systems, this theory is now extremely well developed (see for example Glendinning (1994) or Guckenheimer and Holmes (1983)). More generally one can start to discuss stability and bifurcation phenomena associated with more general invariant sets, for example periodic orbits (e.g. Chow and Hale, 1982).

Such local, and hence relatively "small scale" analysis can be supplemented with information of a coarser nature concerning the existence of globally attracting sets. Again, for the autonomous case this approach is well-developed in the theory of global attractors (Hale, 1988; Ladyzhenskaya, 1991; Robinson, 2001; Temam, 1988). These are compact, invariant, globally attracting subsets of the phase space which determine all the asymptotic dynamics. However, only in relatively simple cases are we able to understand the structure of this attractor in any detail (see Henry (1984) for example).

In this paper we investigate bifurcation phenomena in non-autonomous ordinary differential equations. Essentially we present a collection of examples which illustrate some of the problems and, we hope, the sense of the definitions of stability and instability that we have chosen.

It is clear that for the solution of the non-autonomous equation on $\mathbb{R}^m$

$$\dot{x} = f(x,t) \qquad x(s) = x_0 \tag{1}$$

the initial time (s) is as important as the final time (t). To treat these equations as dynamical systems we have to consider a family of solution operators $\{S(t,s)\}_{t\geq s}$ (termed a "process" by Sell (1967)) that depend on both the final and initial times. We can then denote the solution of (1) at time t by $S(t,s)x_0$. If f is sufficiently smooth then it is clear that $S(t,s) : \mathbb{R}^m \to \mathbb{R}^m$ must satisfy

(i) $S(t,t)$ is the identity for all $t \in \mathbb{R}$,

(ii) $S(t,\tau)S(\tau,s) = S(t,s)$ for all t, τ, and $s \in \mathbb{R}$†, and

(iii) $S(t,s)x_0$ is continuous in t, s, and x_0.

Note that for an autonomous equation the solution only depends on $t - s$, so we have $S(t,s) = T(t-s)$ for some appropriate family $\{T(t)\}_{t\in\mathbb{R}}$.

If we try to develop a qualitative theory for non-autonomous equations by following the same route as we would for autonomous systems we immediately find that there are problems. Indeed, for a generic non-autonomous system we would not expect to find

† When there are solutions of (1) that do not exist for all time some restrictions to the possible values of s and t are necessary here: we could define a local process such that for every s there exists a $t_-(s) \leq 0$ and $t_+(s) > 0$ such that $S(t,s)$ is defined for all $t_- \leq t \leq t_+$. Although we do in fact deal with equations that generate only local processes in what follows, we will not be too careful about the distinction between global and local.

any stationary points: if x_0 is stationary then this would require that $f(x_0, t) = 0$ for all $t \in \mathbb{R}$. The only candidate we have to replace stationary points is the notion of a complete trajectory.

Definition 1.1 *The continuous map $x : \mathbb{R} \to \mathbb{R}^m$ is a complete trajectory if*

$$S(t,s)x(s) = x(t) \qquad \textit{for all} \qquad t, s \in \mathbb{R}.$$

However, in general there will be many complete trajectories, since they are just solutions that exist for all $t \in \mathbb{R}$. For example, if $f(x,t)$ is bounded then any trajectory of (1) is a complete trajectory. Thus we would expect any complete trajectory that plays an important role in the dynamics to have very particular stability properties (although we do not require the concept in this paper, the notion of a *hyperbolic trajectory* seems the appropriate one here: see, for example, Malhotra & Wiggins (1998) and (for another application) Langa *et al.* (2001)).

A complete trajectory is a particular example of an *invariant set* in a non-autonomous equation. We will find this idea particularly useful.

Definition 1.2 *A time-varying family of sets $\{\Sigma(t)\}$ is invariant (we say "$\Sigma(\cdot)$ is invariant") if*

$$S(t,s)\Sigma(s) = \Sigma(t) \qquad \textit{for all} \qquad t, s \in \mathbb{R}.$$

This formal definition simply says that if $x(s) \in \Sigma(s)$ then $S(t,s)x(s) \in \Sigma(t)$.

This paper is to our knowledge the first attempt to begin a systematic development of a theory of bifurcations for non-autonomous equations‡. However, there is already a substantial body of theory treating attractors for this case (see Kloeden and Schmalfuss (1997 and 1998), Kloeden and Stonier (1998), and references at the end of this paragraph). In particular, this theory makes much play with the concept of "pullback attraction", where we consider not the asymptotic behaviour of $S(t,s)$ as $t \to \infty$ for fixed s, but as $s \to -\infty$ for fixed t. This is discussed in detail in section 2.1. (In this paper we restrict our discussion to attractors in $\mathbb{R}^m$, although a similar theory is possible in a general Banach space, see for example Cheban *et al.* (2000), Chepyzhov and Vishik (1994), Crauel *et al.* (1997), and Schmalfuss (2000)).

‡ There is some previous work couched in the language of skew product flows due to Johnson (1989) and Johnson and Yi (1994) that deals with a generalised notion of Hopf bifurcation about bounded trajectories of autonomous systems, and Shen and Yi (1998) have discussed the general behaviour of almost periodic scalar differential equations (but make no particular reference to bifurcation phenomena). Both these approaches make use of the fact that the problem can be recast as a skew-product flow over a *compact* base space, and are not directly applicable to the problems that we consider here. It may nonetheless be possible to treat the kind of questions that we wish to investigate here within this framework by defining an appropriate topology on this base space, as is discussed briefly in the conclusion.

More immediately relevant is the recent work of Siegmund (2002a & b) who treats the problem of finding normal forms for non-autonomous equations. Just as normal form theory is central for the theory of bifurcations in autonomous systems, these results could well prove crucial in the non-autonomous case.

Throughout section 2 we make use of this "pullback" idea to define various notions of stability and instability which generalise those used in the autonomous case. We can then begin to discuss non-autonomous bifurcations, where – as in the autonomous case – we understand this notion as a change in the structure and stability of the invariant sets of the system.

We present three examples which illustrate our definitions and demonstrate some of the rich behaviour which can occur in these systems. The first example (section 3) is a relatively straightforward generalisation of the canonical ODE example of a pitchfork bifurcation,

$$\dot{x} = ax - b(t)x^3 \qquad x \in \mathbb{R},$$

and, by means of an exact solution, we obtain very similar behaviour. Next (in section 4) we consider how a change of linear stability near the origin effects the behaviour of the system

$$\dot{u} = \mathbb{A}u + F(u;t) \qquad u \in \mathbb{R}^m,$$

assuming that $\mathbb{A}u$ contains all the linear terms: for the particular case when F contributes to the dissipation we outline a bifurcation from $\{0\}$ to a non-trivial attractor. Finally in section 5 we investigate a non-autonomous version of the canonical saddle-node example,

$$\dot{x} = a - b(t)x^2 \qquad x \in \mathbb{R}.$$

This equation requires an analysis that is "essentially non-autonomous" since there is no stationary solution, and exhibits some features unique to such models (for example, we obtain an asymptotically unstable set which is not a subset of the attractor). The dynamics can be described by means of a pair of complete trajectories with well-defined stability properties.

In the conclusion we summarise and consider how one might develop the theory further in a systematic way.

2. Notions of attraction and stability

We wish to define the non-autonomous equivalents of various concepts of stability familiar from the theory of autonomous systems. We start with the notion of attraction, since it is the theory of non-autonomous attractors (see section 2.3) that has inspired much of our approach.

In what follows we make constant use of the Hausdorff semidistance between two sets A and B, $\text{dist}(A, B)$, which is defined as

$$\text{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b) :$$

note that this only measures how far A is from B ($\text{dist}(A, B) = 0$ only implies that $A \subseteq B$). We also use the notation $N(X, \epsilon)$ to denote the ϵ-neighbourhood of a set X:

$$N(X, \epsilon) = \{y : \; y = x + z, \; x \in X, \; z \in \mathbb{R}^m \text{ with } |z| \leq \epsilon\}.$$

2.1. Attraction

In autonomous systems, an invariant set Σ is (locally) attracting (or "quasi-asymptotically stable", Glendinning (1994)) if there exists a neighbourhood $\mathcal{O}$ of Σ such that

$$\mathrm{dist}(S(t,0)x_0,\Sigma)\to 0 \qquad \text{as} \qquad t\to\infty, \qquad \text{for all} \qquad x_0\in\mathcal{O}. \tag{2}$$

(Of course, in an autonomous system the initial time is not important, but we make the dependence explicit here to facilitate our discussion.)

Indeed, we are accustomed to considering "time-asymptotic" behaviour as the limiting behaviour as $t \to \infty$. The "practical" implication of this idea is, however, that if we run an experiment for long enough and then consider the state of the system at some future time, it is well approximated by one of the states in the attractor.

In the autonomous case, where $S(t,s) = T(t-s)$, the concept of attraction in (2) is entirely equivalent to the existence of a neighbourhood $\mathcal{O}$ of Σ such that for each fixed $t \in \mathbb{R}$,

$$\mathrm{dist}(S(t,s)x_0,\Sigma)\to 0 \qquad \text{as} \qquad s\to-\infty, \qquad \text{for all} \qquad x_0\in\mathcal{O}. \tag{3}$$

This idea of "pullback" attraction (cf. Kloeden and Schmalfuss, 1998; Schmalfuss, 2000) does not involve running time backwards: rather, we consider taking measurements in an experiment *now* (at time t) which began some time in the past (at time $s < t$). If the experiment has been running long enough, we expect once again that the state of the system is well approximated by one of the "attracting states".

It turns out that for non-autonomous systems this idea of "pullback attraction" is much more natural than the more familiar "forward" attraction of (2). One advantage is that the pullback procedure allows us to consider "time-asymptotic behaviour" without having to consider sets $\Sigma(t)$ that are moving (and perhaps become unbounded), since the final time is fixed. A clear example of this is given in proposition 3.1, where the attracting orbit (given explicitly in (10)) is unbounded as $t \to +\infty$.

This approach has proved extremely fruitful, particular in the study of stochastic differential equations (Crauel and Flandoli, 1994; Crauel *et al.*, 1997; Schmalfuss, 1992) and has already found application is the study of non-autonomous ODEs (Kloeden and Schmalfuss, 1998; Kloeden and Stonier, 1998) and PDEs (Langa and Robinson 2001; Langa and Suárez, 2001; Cheban *et al.*, 2000).

We will therefore make all our definitions "in the pullback sense". For some similar definitions for non-autonomous systems, but made "forwards in time" see Hale (1969).

Definition 2.1 *We say that $\Sigma(\cdot)$ is (locally) pullback attracting if for every $t \in \mathbb{R}$ there exists a $\delta(t) > 0$ such that if*

$$\lim_{s\to-\infty}\mathrm{dist}(x(s),\Sigma(s)) < \delta(t)$$

then

$$\lim_{s\to-\infty}\mathrm{dist}[S(t,s)x(s),\Sigma(t)] = 0. \tag{4}$$

Note that it is crucial that in the definition here, δ is not allowed to depend on s, otherwise – due to continuous dependence on initial conditions – every invariant set would be pullback attracting.

There are many possible definitions of what it might mean to be "*globally* pullback attracting" in a non-autonomous system, since the pullback procedure allows the "initial condition" to vary with s. All such definitions can be put into a coherent framework by defining a "universe" of sets which we require to lie in the basin of attraction of any set which we would like to call "an attractor" (the details of this idea are given in Flandoli and Schmalfuss (1996), Schenk-Hoppé (1998), and Schmalfuss (1992) in the stochastic case, and by Schmalfuss (2000) for non-autonomous systems). However, the following definition, which fixes the initial condition, seems to be the most appropriate.

Definition 2.2 *An invariant set $\Sigma(\cdot)$ is globally pullback attracting if for every $t \in \mathbb{R}$ and every $x_0 \in \mathbb{R}^m$,*

$$\lim_{s \to -\infty} \mathrm{dist}[S(t,s)x_0, \Sigma(t)] = 0.$$

[As an illustration of the many other possible definitions, let us call a set "uniformly pullback attracting" if (4) in definition 2.1 holds for any $\delta > 0$. The simple autonomous ODE $\dot{x} = -x$ serves to illustrate the distinction between "uniform" and "global" pullback attraction. Since the solution of this equation that satisfies $x(s) = x_s$ is

$$x(t,s;x_s) = x_s \mathrm{e}^{-(t-s)} = e^{-(t-s)}[x_s - \alpha e^{-s}] + \alpha e^{-t}$$

for any $\alpha \in \mathbb{R}$, it follows that every solution αe^{-t} is uniformly pullback attracting (we take an x_s with $|x_s - \alpha \mathrm{e}^{-s}| \leq \delta$), while only the zero solution is globally pullback attracting (we fix x_s).]

2.2. Stability

Standard definitions of asymptotic stability have two components. One is attraction, which we have already discussed in detail above, and the other is "Lyapunov stability", which constrains trajectories to stay close to the invariant set. We now give these definitions in the non-autonomous case.

Definition 2.3 *$\Sigma(\cdot)$ is pullback Lyapunov stable if for every $t \in \mathbb{R}$ and $\epsilon > 0$ there exists a $\delta(t) > 0$ such that for any $s < t$, $x_s \in N(\Sigma(s), \delta(t))$ implies that $S(t,s)x_s \in N(\Sigma(t), \epsilon)$.*

It is easy to verify that this agrees with the usual definition in the autonomous case.

Combining this with pullback attraction we obtain a definition of pullback asymptotic stability.

Definition 2.4 *We say that $\Sigma(\cdot)$ is locally (globally) pullback asymptotically stable if it is both pullback Lyapunov stable and locally (globally) pullback attracting.*

2.3. Pullback attractors

We now give a definition* of a "pullback attractor" (Cheban *et al.*, 2000; Kloeden and Schmalfuss, 1998; Schmalfuss, 2000). These have also been called "cocycle attractors" in the literature (Kloeden and Schmalfuss, 1997; Kloeden and Stonier, 1998) and correspond to the "kernel sections" of Chepyzhov and Vishik (1994).

Definition 2.5 *An invariant set* $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ *is said to be the (global) pullback attractor of the process* S *if it is*

(i) compact,

(ii) globally pullback attracting (in the sense of definition 2.2), and

(iii) minimal in the sense that if $\{C(t)\}_{t\in\mathbb{R}}$ *is another family of closed globally attracting sets then* $\mathcal{A}(t)\subset C(t)$ *for all* $t\in\mathbb{R}$.

If all the sets in the definition are time independent then this is the standard definition of a global attractor familiar from the autonomous case.

To ensure the existence of a pullback attractor we can appeal to a generalisation of a standard autonomous result (Babin and Vishik, 1992; Hale, 1988; Ladyzhenskaya, 1991; Robinson, 2001; Temam, 1988). This requires some notion of dissipation, which we make precise in the definition of an absorbing set.

Definition 2.6 *A family of sets* $\{B(t)\}_{t\in\mathbb{R}}$ *is said to be pullback absorbing if for each* t_0 *and each bounded set* $D\subset\mathbb{R}^m$ *there exists a* $T=T(t_0,D)$ *such that*

$$S(t_0,s)D\subset B(t_0) \qquad \text{for all} \qquad s\le T.$$

The existence of a compact pullback absorbing set is enough to ensure the existence of a pullback attractor (see Crauel *et al.*, 1997; Schmalfuss, 2000).

Theorem 2.7 *Assume that there exists a family of compact pullback absorbing sets. Then there is a pullback attractor* $\mathcal{A}(t)$, *and*

$$\mathcal{A}(t)=\overline{\bigcup_{B\ \text{bounded}}\Omega_B(t)}, \tag{5}$$

where $\Omega_B(t)$ *is the omega limit set of* B *at time* t,

$$\Omega_B(t)=\{x:\ \text{for some sequence } s_n\to-\infty \text{ and } x_n\in B \text{ we have } x=\lim_{n\to\infty}S(t,s_n)x_n\}. \tag{6}$$

Furthermore $\mathcal{A}(t)$ *is a connected set for each* $t\in\mathbb{R}$.

* The global attractor (autonomous or non-autonomous) is in fact defined to be the minimal compact invariant set that attracts all *bounded* sets of initial conditions. Since here our phase space is finite-dimensional, attraction of all one point sets in fact implies attraction of all bounded sets (see Robinson (2001) for example). When the phase space is infinite-dimensional attraction of all bounded sets is a stronger property, and needs to be made explicit in the definition.

Although there are essential differences between the concepts of autonomous global attractors and pullback attractors, they are related: it is shown in Caraballo and Langa (2001) that if a family of non-autonomous processes $S_\sigma(t,s)$ converge to an autonomous flow $S(t-s)$ as $\sigma\to 0$ and there is some uniformity in the size of the absorbing set for $0\le\sigma\le\sigma_0$, then $\mathrm{dist}(\mathcal{A}_\sigma(t),\mathcal{A})\to 0$ as $\sigma\to 0$, where $A_\sigma(t)$ is the pullback attractor for $S_\sigma(t,s)$ and $\mathcal{A}$ is the global attractor for $S(t-s)$. Indeed, this result is a further piece of evidence that the pullback definition of attraction (and, we hope, stability) is entirely consistent with more familiar autonomous ideas. (A special case of this result is discussed at the end of section 3.)

2.4. Instability

We define local pullback instability as the converse of pullback Lyapunov stability.

Definition 2.8 *We say that $\Sigma(\cdot)$ is locally pullback unstable if it is not pullback Lyapunov stable, i.e. if there exists a $t\in\mathbb{R}$ and an $\epsilon>0$ such that, for each $\delta>0$, there exists an $s<t$ and an $x_0\in N(\Sigma(s),\delta)$ such that*

$$\mathrm{dist}(S(t,s)x_0,\Sigma(t))>\epsilon.$$

However, a more natural concept from a dynamical point of view is the idea of an "unstable set", defined by Crauel (2001) for the case of random dynamical systems (which also necessitate the pullback concept of attraction) just as for autonomous deterministic systems (see Temam (1988) for example).

Definition 2.9 *If $\Sigma(\cdot)$ is an invariant set then the unstable set of Σ, $U_\Sigma(\cdot)$, is defined as*

$$U_\Sigma(s)=\{u:\lim_{t\to-\infty}\mathrm{dist}\,(S(t,s)u,\Sigma(t))=0\}.$$

We say that $\Sigma(\cdot)$ is asymptotically unstable if for some t we have

$$U_\Sigma(t)\neq\Sigma(t). \qquad (7)$$

Since we always have $\Sigma(t)\subset U_\Sigma(t)$ when $\Sigma(\cdot)$ is invariant, (7) says that $\Sigma(t)$ is a strict subset of $U_\Sigma(t)$ – in this case we will say that $U_\Sigma(t)$ is "non-trivial".

Note that the instability of $\Sigma(t)$ is equivalent to the stability (in the usual, forward, sense) of $\Sigma(-t)$ for the time-reversed system. This introduces a lack of symmetry in the definitions, since the pullback procedure is not involved here. However, if $\Sigma(\cdot)$ has a non-trivial unstable set then it is "unstable" in a strong sense, as the following result shows.

Proposition 2.10 *If $\Sigma(\cdot)$ is asymptotically unstable then it is also locally pullback unstable and cannot be locally pullback attracting.*

Proof. Since $\Sigma(\cdot)$ is asymptotically unstable there exists an s and an element $x\in U_\Sigma(s)$ such that

$$\mathrm{dist}(x,\Sigma(s))>\epsilon>0$$

for some $\epsilon > 0$. However, $x_t = S(t,s)x$ satisfies

$$\lim_{t\to-\infty} \mathrm{dist}(x_t, \Sigma(t)) = 0.$$

Given a $\delta > 0$, it follows that we can choose t_0 small enough that

$$\mathrm{dist}(x_t, \Sigma(t)) < \delta, \qquad \text{for all} \qquad t \le t_0.$$

Since this holds for all $t \le t_0$ it follows that $\Sigma(\cdot)$ is not locally pullback attracting, and choosing any particular $t < t_0$ shows that $\Sigma(\cdot)$ is locally pullback unstable. □

We observe here that in an autonomous or random dynamical system, the unstable set of any invariant set must be a subset of the attractor (see Robinson (2001), Stuart and Humphries (1996), or Temam (1988) for the autonomous case and Crauel (2001) for the random case). For non-autonomous systems we can only prove a similar result when $\Sigma(t)$ is bounded on each semi-infinite time interval $(-\infty, t]$. An example of a system with an asymptotically unstable invariant set which is not a subset of the attractor is given in section 5.

Proposition 2.11 *Suppose that* $\Sigma(\cdot)$ *is invariant and that*

$$\bigcup_{s\in(-\infty,t]} \Sigma(s)$$

is contained in a bounded set $B(t)$ *for each* $t \in \mathbb{R}$. *Then* $U_\Sigma \subset \mathcal{A}$.

Proof. If $x \in U_\Sigma(s)$ then it follows that there exists a $t_0(\epsilon, x)$ such that for all $t \ge t_0$

$$\mathrm{dist}(u(-t,s;x), \Sigma(-t)) \le \epsilon.$$

In particular it follows that for all such t the solution $u(-t,s;x)$ is contained in the bounded set $\tilde{B} = N(B(s),\epsilon)$. Thus we can write $x = S(s,-t_n)y_n$, where $y_n = S(-t_n,s)x \in \tilde{B}$ and $t_n \to \infty$. It follows (see (6)) that $x \in \Omega_{\tilde{B}}(s)$, and hence is an element of $\mathcal{A}(s)$. □

2.5. The concept of bifurcation in non-autonomous systems

Given the above definitions, we can discuss in a meaningful way the idea of a bifurcation in a non-autonomous system. Exactly as in the autonomous case, a bifurcation is *a change in the stability and structure of the invariant sets of the system.* This covers a multitude of sins, but in what follows we will concentrate on non-autonomous versions of the most simple bifurcation phenomena.

In our first two examples we will observe only local bifurcation phenomena in a neighbourhood of the origin: indeed, we will make the very strong assumption that zero is a stationary point for all values of the parameter. We give a non-autonomous version of the pitchfork bifurcation, and then a more general bifurcation scenario based on the linearised equation near zero. In some cases this second bifurcation can be understand in terms of the appearance of a non-trivial attractor.

Then we consider a non-autonomous version of the saddle-node bifurcation, which illustrates some of the rich behaviour possible in the non-autonomous case.

3. A non-autonomous pitchfork bifurcation

The canonical autonomous example of an equation exhibiting a pitchfork bifurcation (see Glendinning, 1994) is

$$\dot{x} = ax - bx^3 \qquad b > 0. \tag{8}$$

When $a < 0$ the origin is locally stable, while for $a > 0$ the origin becomes unstable and two new stable fixed points appear at $\pm\sqrt{b/a}$. For $a < 0$ the global attractor is just $\{0\}$, while for $a > 0$ the global attractor is the interval $[-\sqrt{b/a}, \sqrt{b/a}]$.

In this section we study a non-autonomous version of (8). (This was also studied in Caraballo and Langa (2001) with the emphasis on its attractor.)

Proposition 3.1 *Consider the equation*

$$\dot{x} = ax - b(t)x^3 \qquad x(s) = x_s, \tag{9}$$

under the assumption that $0 < b(t) \leq B$. *For* $a < 0$ *the origin is globally asymptotically pullback stable, while for* $a > 0$ *the origin becomes asymptotically unstable and two new locally asymptotically pullback stable complete trajectories* $\pm\alpha(t; a)$ *appear, where*

$$\alpha^2(t; a) = \frac{e^{2at}}{2\int_{-\infty}^{t} e^{2a\tau} b(\tau) d\tau}. \tag{10}$$

Note that if $b(t) \to 0$ then $\alpha(t; a) \to \infty$ as $t \to +\infty$.

[One could use this result directly to give a simple example of a non-autonomous version of the Hopf bifurcation by considering the system

$$\dot{r} = ar - b(t)r^3 \qquad \dot{\theta} = \omega,$$

where (r, θ) are polar coordinates in $\mathbb{R}^2$ (cf. Glendinning (1994) in the autonomous case).]

Proof. By means of the substitution $y = x^{-2}$ this equation admits an exact solution: for $x_s > 0$

$$x(t)^2 = \frac{\mathrm{e}^{2at}}{\mathrm{e}^{2as} x_s^{-2} + 2\int_s^t \mathrm{e}^{2ar} b(r)\, \mathrm{d}r}. \tag{11}$$

For $a < 0$ we can define a global attractor as in the autonomous case: when t tends to $+\infty$ all solutions are attracted to the point $\{0\}$. The pullback attractor is also just $\{0\}$, since this is the limit of (11) when s tends to $-\infty$.

On the other hand when $a > 0$ all solutions are unbounded as $t \to \infty$ if $b(t) \to 0$ as $t \to \infty$. However, we can always define the pullback attractor. Indeed, taking the limit as $s \to -\infty$ in (11) yields

$$\alpha^2(t; a) = \frac{e^{2at}}{2\int_{-\infty}^{t} e^{2a\tau} b(\tau) d\tau}, \tag{12}$$

and it is easy to check that $\alpha(\cdot; a)$ is a complete trajectory of (9) (see definition 1.1).

The construction of $\alpha(t;a)$ ensures that it is pullback attracting. If we rearrange (11) as

$$x^2(t) = \frac{\mathrm{e}^{2at}}{\mathrm{e}^{2as}[x^{-2}(s) - \alpha^{-2}(s;a)] + 2\int_{-\infty}^{t} \mathrm{e}^{2ar} b(r)\,\mathrm{d}r}$$

then pullback Lyapunov stability follows easily and it is clear that any solution $S(t,s)x_s$ with $x_s < \alpha(s;a)$ converges to 0 as $t \to -\infty$. □

We could rephrase this result in terms of attractors (the equation is treated this way in Caraballo and Langa, 2001): for $a < 0$ the pullback attractor is just the origin, while for $a > 0$ it is the interval $[-\alpha(t;a), \alpha(t;a)]$. In this context it is interesting to observe that if we consider a family of non-autonomous problems

$$\dot{x} = ax - b_\sigma(t)x^3$$

where for each $\sigma > 0$ the function $b_\sigma(t)$ satisfies the conditions of proposition 3.1, but with $b_\sigma(t) \to b$ (constant) uniformly on bounded sets, it is easy to see that $\alpha(t;a,\sigma) \to \sqrt{a/b}$ for each t and in the limit we recover the global attractor for the autonomous problem

$$\dot{x} = ax - bx^3.$$

(A general result along these lines, due to Caraballo and Langa (2001), was discussed briefly at the end of section 2.3.)

4. A bifurcation deduced from the linearised equation

We now consider a more complicated example which we cannot solve explicitly, and investigate what happens when the linearised flow at the origin (which we assume is autonomous) changes its stability. We consider the system for $u \in \mathbb{R}^m$

$$\dot{u} = \mathbb{A}u + F(u;t), \tag{13}$$

where

- $\mathbb{A}$ is a real $m \times m$ matrix of the form

  $$\mathbb{A} = \begin{pmatrix} \lambda & 0 \\ 0 & -A \end{pmatrix},$$

  with A a real $(m-1)\times(m-1)$ matrix that satisfies

  $$y^T A y \geq \mu |y|^2 \qquad \text{for all} \qquad y \in \mathbb{R}^{m-1},$$

- $F(0,t) = 0$ for all $t \in \mathbb{R}$,
- for some $p > 0$

  $$|F(u;t) - F(v;t)| \leq a(t)[|u|^{2p} + |v|^{2p}]|u - v|, \tag{14}$$

  where for each $t \in \mathbb{R}$

  $$\sup_{s\in(-\infty,t]} a(s) = \alpha(t) < \infty. \tag{15}$$

Note that the two properties of F above together imply that

$$|F(u;t)| \leq a(t)|u|^{2p+1}. \tag{16}$$

4.1. Local pullback asymptotic stability when $\lambda < 0$.

First we consider the case when $\lambda < 0$, for which the origin is linearly stable, and prove that it must also be locally pullback asymptotically stable.

Proposition 4.1 *If $\lambda < 0$ the origin is locally pullback asymptotically stable.*

Proof. Setting $\kappa = \min(\mu, -\lambda)$ it follows that $x^T \mathbb{A} x \leq -\kappa |x|^2$. Using this bound and (16) it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 \leq -\kappa|u|^2 + a(t)|u|^{2p+2}.$$

With the substitution $\theta(t) = |u(t)|^{-2p}$ the corresponding differential inequality

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} \geq 2\kappa p\theta - 2pa(t)$$

can be solved explicitly to give

$$\theta(t) \geq \mathrm{e}^{2\kappa pt}\left[\mathrm{e}^{-2\kappa ps}\theta(s) - 2p\int_s^t \mathrm{e}^{-2\kappa pr}a(r)\,\mathrm{d}r\right].$$

Using condition (15) it follows that

$$\int_s^t \mathrm{e}^{-2\kappa pr}a(r)\,\mathrm{d}r \leq \frac{\alpha(t)}{2\kappa p}\mathrm{e}^{-2\kappa ps},$$

and so provided that $\theta(s) = \theta_0 > \alpha(t)/\kappa$, we have $\theta(t) \to \infty$ as $s \to -\infty$ and hence $\{0\}$ is locally pullback attracting. The origin is also pullback Lyapunov stable, since choosing $\theta(s) \geq \epsilon^{-2p} + (\alpha(t)/\kappa)$ ensures that $\theta(t) \geq \epsilon^{-2p}$ for all $t \geq s$. □

4.2. Asymptotic instability when $\lambda > 0$

To show asymptotic instability when $\lambda > 0$ we follow the idea of the proof in Caraballo *et al.* (2001) and prove the existence of a local unstable manifold near the origin. In what follows we consider (13) in its equivalent form for $(x, y) \in \mathbb{R} \times \mathbb{R}^{m-1}$,

$$\begin{aligned}\dot{x} &= \lambda x + f(x, y; t)\\ \dot{y} &= -Ay + g(x, y; t).\end{aligned}$$

Proposition 4.2 *If $\lambda > 0$ then the origin is asymptotically unstable. In particular, there exists a Lipschitz function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{m-1}$ and a $\varrho(t) > 0$ such that the set*

$$\mathcal{M}(s) = \{(x_s, \phi(x_s, s)) : \; |x_s| \leq \varrho(s)\}$$

forms part of the unstable set of $\{0\}$.

Proof. Using a contraction mapping argument we find an invariant manifold in a small neighbourhood on the origin. First we truncate the nonlinear term to ensure that its Lipschitz constant is uniformly small.

We take a smooth function $\theta : [0,\infty) \to [0,1]$ satisfying

$$\theta(r) = 1 \quad 0 \le r \le 1, \quad |\theta'(r)| \le 2 \quad 1 \le r \le 2 \quad \theta(r) = 0 \quad r \ge 2$$

we truncate the nonlinear term in a ball of radius $\delta(t)$,

$$F_\delta(u;t) = \theta\left(\frac{|u|}{\delta(t)}\right) F(x,y;t).$$

It follows (cf. Temam, 1988) that

$$|F_\delta(u;t) - F_\delta(v;t)| \le Ca(t)\delta(t)^{2p}|u-v|.$$

We now take $\delta(t)^{2p} = \epsilon[C\alpha(t)]^{-1}$ (note that $\delta(t)$ is non-increasing in t) so that

$$|F_\delta(u;t) - F_\delta(v;t)| \le \epsilon|u-v| \qquad \text{and} \qquad |F_\delta(u;t)| \le \epsilon|u|. \tag{17}$$

We look for an invariant manifold for the truncated equation

$$\dot{u} = \mathbb{A}u + F_\delta(u;t). \tag{18}$$

We will write $(f_\delta(u;t), g_\delta(u;t))$ for the corresponding nonlinear terms in the separated (x,y) equations.

We now define a mapping T on the collection $\mathcal{L}$ of Lipschitz functions $\phi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{m-1}$ that satisfy

$$|\phi(x,t) - \phi(y,t)| \le |x-y|,$$

equipped with the usual supremum norm ($\|\phi\|_\infty = \sup_{(x,t)\in\mathbb{R}\times\mathbb{R}} |\phi(x,t)|$). For $\phi \in \mathcal{L}$ we define $[T\phi](x_s,s)$ as the solution at time s of the y equation in the partially coupled system

$$\begin{aligned} \dot{x} &= \lambda x + f_\delta(x,\phi(x,t);t) \qquad && x(s) = x_s \\ \dot{y} &= -Ay + g_\delta(x,\phi(x,t);t) \qquad && \lim_{t\to-\infty} y(t) = 0. \end{aligned}$$

¿From the explicit form for the solution of the y equation between s an t,

$$y(t) = \mathrm{e}^{-A(t-s)}y(s) + \int_s^t \mathrm{e}^{-A(t-r)} g_\delta(x(r),\phi(x(r),r))\,\mathrm{d}r, \tag{19}$$

and the bound on g_δ as in (17) it follows that

$$[T\phi](x_s,s) = \int_{-\infty}^s e^{-A(s-r)} g_\delta(x(r),\phi(x(r),r);r)\ dr. \tag{20}$$

The graph of a fixed point ϕ of T,

$$\{(x,\phi(x)) :\ x \in \mathbb{R}\}$$

corresponds to an invariant manifold for the truncated equation (18) – this can be shown by substituting the expression $(x(t), \phi(x(t), t))$ into (19) and then using the fact that $T\phi = \phi$.

First we check that $T\phi \in \mathcal{L}$. We have

$$\begin{aligned}
&|[T\phi](x_s, s) - [T\phi](\hat{x}_s, s)| \\
&\quad = \left| \int_{-\infty}^{s} e^{-A(s-r)} [g_\delta(x(r), \phi(x(r), r); r) - g_\delta(\hat{x}(r), \phi(\hat{x}(r), r); r)] \, \mathrm{d}r \right| \\
&\quad \leq \int_{-\infty}^{s} \epsilon e^{-\mu(s-r)} 2|x(r) - \hat{x}(r)| \, \mathrm{d}r.
\end{aligned}$$

Easy estimates on the difference of two solutions of

$$\dot{x} = \lambda x + f_\delta(x, \phi(x, t); t)$$

yield

$$|x(r) - \hat{x}(r)| \leq e^{(\lambda - 2\epsilon)(r-s)} |x_s - \hat{x}_s|,$$

and so we require

$$\epsilon \int_{-\infty}^{s} e^{-[\mu + \lambda - 2\epsilon](s-r)} \, \mathrm{d}r \leq 1.$$

This follows for every s if ϵ is sufficiently small.

To show that T is a contraction we consider

$$\begin{aligned}
&|[T\phi](x_s) - [T\hat{\phi}](x_s)| \\
&\quad = \left| \int_{-\infty}^{s} \mathrm{e}^{-A(s-r)} [g_\delta(x(r), \phi(x(r), r); r) - g_\delta(\hat{x}(r), \hat{\phi}(\hat{x}(r), r); r)] \, \mathrm{d}r \right| \\
&\quad \leq \int_{-\infty}^{s} \mathrm{e}^{-\mu(s-r)} \epsilon \left[2|x(r) - \hat{x}(r)| + \|\phi - \hat{\phi}\|_\infty \right] \, \mathrm{d}r.
\end{aligned}$$

Again, relatively straightforward estimates comparing the solutions of

$$dx/dt = \lambda x + f_\delta(x, \phi(x, t); t) \quad \text{and} \quad d\hat{x}/dt = \lambda \hat{x} + f_\delta(\hat{x}, \hat{\phi}(\hat{x}, t); t)$$

yield, provided that $\lambda > 2\epsilon$,

$$|x(r) - \hat{x}(r)| \leq \frac{\epsilon \|\phi - \hat{\phi}\|_\infty}{\lambda - 2\epsilon},$$

so that

$$|[T\phi](x_s, s) - [T\hat{\phi}](x_s, s)| \leq \frac{\lambda \epsilon}{\mu(\lambda - 2\epsilon)} \|\phi - \hat{\phi}\|_\infty,$$

which shows that T is a contraction for ϵ sufficiently small.

We now show that if $|x_s| \leq \varrho(s) = \delta(s)/2\sqrt{2}$ then $u_s = (x_s \, \phi(x_s)) \in \Sigma[0](s)$. Indeed, on the graph of $\phi(x, t)$ the equation reduces to

$$\dot{x} = \lambda x + f_\delta(x, \phi(x, t); t).$$

Then, if $t = -\tau$, we have

$$d|x|/\mathrm{d}\tau \leq -\lambda|x| + 2\epsilon|x| = -(\lambda - 2\epsilon)|x|.$$

It follows that

$$|x(t)| \leq e^{(\lambda-2\epsilon)(t-s)}|x(s)|.$$

Since $y(t) = \phi(x(t),t)$ we have $|u(t)| \leq \sqrt{2}|x(t)|$. Since $\delta(t)$ is non-increasing, it follows that if $|x_s| \leq \delta(s)/2\sqrt{2}$ then $|u(t)| \leq \delta(t) < 2$ for all $t \leq s$: in particular the solution $u(t)$ is also a solution of the original, untruncated equation, and so $x_s \in \Sigma[0](s)$. $\square$

4.3. A bifurcation to a non-trivial pullback attractor

We note here that if the nonlinear term in fact contributes to the dissipation, so that

$$(F(u;t),u) \leq -a(t)|u|^{2p+2},$$

then for $\lambda < 0$ the origin is *globally* pullback asymptotically stable. However for $\lambda > 0$ proposition 4.2 still applies. It follows that while for $\lambda < 0$ the pullback attractor is just the origin, for $\lambda > 0$ this attractor is non-trivial since (using proposition 2.11 and the fact that $\{0\}$ is bounded) it must contain $\mathcal{M}(\cdot)$ which is a subset of the unstable set of $\{0\}$. It should be relatively simple to demonstrate this kind of bifurcation behaviour in complicated systems where a more detailed description is extremely difficult.

5. A non-autonomous saddle-node bifurcation

Our final example is a non-autonomous version of the saddle-node bifurcation. The simplest autonomous example which exhibits this bifurcation is

$$\dot{x} = a - Bx^2$$

(see Glendinning (1994) for example). Clearly this equation has no fixed points for $a < 0$, while for $a > 0$ there are two fixed points $\pm\sqrt{a/B}$: if $x_0 < -\sqrt{a/B}$ then $x(t) \to -\infty$ in finite time, while if $x_0 > -\sqrt{a/B}$ then $x(t;x_0) \to \sqrt{a/B}$ as $t \to +\infty$.

In this section we consider the non-autonomous equation

$$\dot{x} = a - b(t)x^2 \tag{21}$$

where $0 < b(t) \leq B$, $b(t) \to 0$ as $|t| \to \infty$, and

$$\int_{-\infty}^{0} b(s)\,\mathrm{d}s = \int_{0}^{\infty} b(s)\,\mathrm{d}s = \infty. \tag{22}$$

The invariance of these conditions when $t \mapsto -t$ greatly simplifies the analysis.

5.1. Behaviour for $a \leq 0$

First we consider the simple cases $a \leq 0$ where the behaviour is essentially the same as in the autonomous case.

Lemma 5.1 *If $a < 0$ then every solution $S(t,s)x_0$ tends to $-\infty$ in a finite time, both "forwards" (fix s and x_0 and let $t \to t^* < +\infty$) and "pullback" (fix t and x_0 and let $s \to s* > -\infty$). If $a = 0$ then solutions with $x_0 > 0$ tend to zero (forwards and pullback) while if $x_0 < 0$ then $S(t,s)x_0$ tends to $-\infty$ in finite time (as in the case $a < 0$).*

Proof. When $a = 0$ the equation can be solved explicitly to yield

$$S(t,s)x_s = \frac{x_s}{1 + x_s \int_s^t b(r)\,\mathrm{d}r}:$$

the behaviour described in the statement of the lemma is immediate. □

5.2. Saddle-node type behaviour for $a > 0$

We now show that when $a > 0$ there are two complete trajectories which we can identify as being "stable" and "unstable".

Theorem 5.2 *For $a > 0$ there exist two complete trajectories of (21) $\alpha(t)$ and $\beta(t)$ which are bounded below (resp. above) by $\sqrt{a/B}$ (resp. $-\sqrt{a/B}$) and diverge to $+\infty$ (resp. $-\infty$) as $|t| \to \infty$. The solution $\alpha(t)$ is globally pullback asymptotically stable and the solution $\beta(t)$ is asymptotically unstable:*

$$\begin{aligned} \lim_{s\to-\infty} x(t,s;x_0) &= \alpha(t) && \forall\ x_0 \\ \lim_{t\to-\infty}[x(t,s;x_0) - \beta(t)] &= 0 && \forall\ x_0 < \alpha(s). \end{aligned} \tag{23}$$

We also have††

$$\begin{aligned} \lim_{t\to+\infty}[x(t,s;x_0) - \alpha(t)] &= 0 && \forall\ x_0 > \beta(s) \\ \lim_{s\to+\infty} x(t,s;x_0) &= \beta(t) && \forall\ x_0. \end{aligned} \tag{24}$$

Note that the global pullback stability of $\alpha(t)$ in (23) shows that $\alpha(t)$ is the pullback attractor. Nevertheless, the complete orbit $\beta(t)$ plays an important role in the dynamics if we consider the asymptotic behaviour as $t \to \infty$. Indeed, the system provides an example in which there exists an invariant set with a non-trivial unstable set which is nevertheless not a subset of the attractor (here the trajectory $\beta(t)$ is unbounded as $t \to -\infty$, and so falls outside the conditions of proposition 2.11).

Proof. We investigate the behaviour of solutions $S(t,s)x_0$ when $s \to -\infty$. First note that if t and x_0 are fixed then there exists an $s_0(t,x_0)$ such that the solution is bounded below,

$$S(t,s)x_0 > \sqrt{a/B} \qquad \text{for all} \qquad s \leq s_0. \tag{25}$$

††We have so far resisted the temptation of defining a notion of "pullback instability" as "pullback backwards stability", cf. $\lim_{s\to+\infty} x(t,s;x_0) = \beta(t)$.

Indeed, choose $\epsilon(x_0) < B$ small enough that $x_0 > -\sqrt{a/\epsilon}$. Then there exists a time $\tau(\epsilon)$ such $|b(t)| \leq \epsilon$ for all $t \leq \tau(\epsilon)$, and so

$$\dot{x} \geq a - \epsilon x^2 \qquad \text{for all} \qquad t \leq \tau(\epsilon).$$

Now, there exists a time T such that the solution of

$$\dot{y} = a - \epsilon y^2 \qquad y(0) = x_0$$

satisfies $y(t) \geq \sqrt{a/B}$ for all $t \geq T$. Thus if we take $s \leq s_0 \equiv \tau(\epsilon) - T$ we obtain (25).

Now set

$$\beta = \inf_{r \in [t-1,t]} b(r).$$

Then on the time interval $[t-1, t]$ we have

$$\dot{x} \leq a - \beta x^2.$$

It follows (see Temam (p87 in 1988 1st edition or p89 in 1996 2nd edition) that for all $s \leq s_0 - 1$ the solution $S(t, s)x_0$ is also bounded above independently of the value of x_0, so we have

$$\sqrt{a/B} \leq S(t, s)x_0 \leq \sqrt{a/\beta} + \beta^{-1} \qquad \text{for all} \qquad s \leq s_0 - 1.$$

It follows from theorem 2.7 that there exists a pullback attractor $\mathcal{A}(t)$, which for each $t \in \mathbb{R}$ is a connected set. Since $\mathcal{A}(t)$ is a subset of $\mathbb{R}$

$$\mathcal{A}(t) = [\alpha(t), \alpha^+(t)]$$

for some $\alpha(t)$ and $\alpha^+(t)$. Since the phase space is one-dimensional the process is order-preserving and so $\alpha(t)$ and $\alpha^+(t)$ are trajectories of (21).

Note that it follows from (25) that $\alpha(t) \geq \sqrt{a/B}$ for all t. In consequence

$$\alpha(t) \to +\infty \qquad \text{as} \qquad t \to -\infty. \tag{26}$$

To show this consider the time-reversed problem with $\tau - t$: writing $\alpha' = \mathrm{d}\alpha/\mathrm{d}\tau$ we have

$$\alpha' = -a + b(-\tau)\alpha^2.$$

Suppose that (26) does not hold, so that there exists a k such that for each τ_0 there exists a $\tau \geq \tau_0$ with $\alpha(\tau) \leq k$. Choose ϵ small enough that $k < \sqrt{a/\epsilon}$, and then T such that for all $\tau \geq T$, $b(-\tau) \leq \epsilon$. It follows that for all $\tau \geq T$ we have

$$\alpha' \leq -a + \epsilon\alpha^2,$$

and so in particular, since $\alpha(\tau) \leq k < \sqrt{a/\epsilon}$ for some $\tau \geq T$,

$$\lim_{\tau \to \infty} \alpha(\tau) \leq -\sqrt{a/\epsilon} < \sqrt{a/B},$$

and we obtain (26) by contradiction.

In particular for all x_0 we have $x_0 < \alpha(s)$ for s small enough. Since $S(t,s)$ is order-preserving and $\alpha(t)$ is a complete trajectory, it is a consequence of the pullback attraction property of $\mathcal{A}(t)$,

$$\lim_{s\to-\infty} \mathrm{dist}[S(t,s)x_0, \mathcal{A}(t)] = 0,$$

that we must have

$$\lim_{s\to-\infty} S(t,s)x_0 = \alpha(s),$$

and this implies that $\mathcal{A}^+(t) = \{\alpha(t)\}$.

If we now consider the system with both time and space reversed ($y = -x$ and $\tau = -t$) the symmetry of the conditions on $b(t)$ allows us to use exactly the same arguments to prove the existence a complete trajectory $\beta(t)$ such that $\beta(t) \leq -\sqrt{a/B}$, $\beta(t) \to -\infty$ as $|t| \to \infty$, and

$$\lim_{s\to\infty} S(t,s)x_0 = \beta(t) \tag{27}$$

for all x_0.

It only remains to prove the second part of (23) or (24): we choose to prove the second part of (24), taking an $x_0 > \beta(s)$. Note that if we take any two initial conditions x_0 and y_0 and consider the two solutions of (21) $x(t) = S(t,s)x_0$ and $y(t) = S(t,s)y_0$ then their difference $z(t) = x(t) - y(t)$ satisfies

$$\dot{z} = -b(t)[x(t) + y(t)]z,$$

so that

$$z(t) = |x_0 - y_0| \exp\left(-\int_s^t [x(r) + y(r)]b(r)\,\mathrm{d}r\right). \tag{28}$$

In particular, since we know from (22) that the integral $\int_s^t b(r)\,\mathrm{d}r$ diverges as $t \to +\infty$, any two trajectories that are bounded below by a positive quantity will converge as $t \to +\infty$. Taking $x_0 = \alpha(s)$ we know that $\alpha(t) \geq \sqrt{a/B}$, so we only need to show that eventually $S(t,s)x_0 > 0$ for our choice of $x_0 > \beta(s)$. But this is an almost immediate consequence of (27), since we know that for t sufficiently large we have $S(s,t)0 < x_0$ and the order-preserving property of $S(t,s) = [S(s,t)]^{-1}$ gives $0 < S(t,s)x_0$. $\Box$

6. Conclusion

Taking our cue from the theory of non-autonomous attractors we have defined notions of stability and instability in non-autonomous systems, and have used these to describe various simple bifurcations in example systems.

Of course, this is very much a first step towards a general theory of non-autonomous bifurcations. We would, at the very least, like to be able to categorise one-dimensional bifurcations as Crauel *et al.* (1999) have managed to do for random dynamical systems.

Even this (apparently) simple problem seems hard, since non-autonomous systems do not have the ergodic properties that greatly aid the study of random dynamical systems by "tying together" the dynamics (see Arnold (1998) for more details).

However, all one-dimensional ordinary differential equations are order-preserving (we have used this fact repeatedly in our analysis of the saddle-node bifurcation above), and Chueshov (2001) (see also Arnold and Chueshov, 1998) has developed an extensive theory for monotone non-autonomous systems which in certain cases gives more information on the structure of the attractor than is available for general systems.

Another possible approach that goes some way to recapturing the constraint on the dynamics that the ergodicity imposes in the stochastic case is to treat the non-autonomous system as a skew-product flow over a suitable base space (e.g. Sell, 1967 & 1971). This device that makes the dynamics appear like those of an autonomous system, although it requires some restrictions on the nonlinearity. The simplest such formulation would be to include another fictitious time τ satisfying the equation $\dot{\tau} = 1$ and to consider the autonomous system

$$\begin{aligned} \dot{x} &= f(x, \tau) \\ \dot{\tau} &= 1. \end{aligned}$$

However τ is asymptotically unbounded in any sense one might choose. To remedy this the approach has generally been to take a subspace of $X = C^0(\mathbb{R}; \mathbb{R}^n)$ (continuous functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the supremum norm) as the base space: if one restricts to almost periodic functions $f(x, t)$ then the set of functions

$$\overline{\cup_{s \in \mathbb{R}} f(\cdot, s)} \tag{29}$$

is a compact subset of X (see Hale (1969) for example). The dynamics on the base space X is simply given by the shift $f(\cdot, t) \mapsto f(\cdot, t+s)$. It should be possible to circumvent the requirement that $f(x, t)$ is almost periodic in t by taking $X = C^0(\mathbb{R}; \mathbb{R}^n)$ equipped with the (metrisable) topology of uniform convergence on compact intervals. Then the set given in (29) will be compact provided that the set of functions $f(\cdot, s)$ is equicontinuous, allowing a much larger class of functions to be considered (cf. Johnson and Kloeden, 2001). Such ideas should be extremely useful in completing the above programme.

We hope that this short paper will serve to stimulate research into this area, which still seems to be essentially uncharted mathematical territory.

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References

L. Arnold. *Random Dynamical Systems*, Springer Verlag, New York, 1998.

——and I.D. Chueshov. Order-preserving random dynamical systems: equilibria, attractors, applications. *Dynamics and Stability of Systems* **13** (1998) pp. 265–280.

A.V. Babin and M.I. Vishik. *Attractors of Evolution Equations.* North Holland, Amsterdam, 1992.

T. Caraballo and J.A. Langa. On the upper semicontinuity of cocycle attractors for non-autonomous and random dynamical systems. *Dynamics of Continuous and Discrete Impulsive Systems*, to appear.

——, ——, and J.C. Robinson. A stochastic pitchfork bifurcation in a reaction-diffusion equation. *Proceedings A of the Royal Society of London*, to appear.

D.N. Cheban, P.E. Kloeden and B. Schmalfuss. The relationship between pullback, forwards and global attractors of nonautonomous dynamical systems. Preprint (2000).

V.V. Chepyzhov and M.I. Vishik. Attractors of non-autonomous dynamical systems and their dimension. *J. Math. Pures Appl.* **73** (1994) pp. 279–333.

S.-N. Chow and J.K. Hale. *Methods of bifurcation theory*, Springer Verlag, New York 1982.

I.D. Chueshov. Order-preserving skew-product flows and nonautonomous parabolic systems. *Acta Appl. Math.*, to appear.

H. Crauel. Random point attractors versus random set attractors. *J. London Math. Soc.* **63** (2001) pp. 413–427.

——and F. Flandoli. Attractors for random dynamical systems. *Prob. Th. Rel. Fields* **100** (1994) pp. 365–393.

——, A. Debussche and F. Flandoli. Random attractors. *J. Dynamics Differential Equations* **9** (1997) pp. 397—341.

——, P. Imkeller and M. Steinkamp. Bifurcations of one-dimensional stochastic differential equations. In H. Crauel and V.M. Gundlach (eds.). *Stochastic dynamics*, New York: Springer (1999) pp. 27–47.

F. Flandoli and B. Schmalfuss. Random attractors for the stochastic 3-D Navier-Stokes equation with multiplicative white noise. *Stochastics and Stoch. Reports* **59** (1996) pp. 21–45.

P.A. Glendinning, *Stability, instability and chaos: an introduction to the theory of nonlinear differential equations*, Cambridge Texts in Applied Mathematics, CUP 1994.

J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Applied Mathematical Sciences 42, Springer-Verlag, New York 1983.

J.K. Hale *Ordinary Differential Equations*, Wiley, Baltimore, 1969.

——*Asymptotic Behavior of Dissipative Systems*, Providence: Math. Surveys and Monographs, Amer. Math. Soc. 1988.

D. Henry *Geometric Theory of Semilinear Parabolic Equations*, Springer Lecture Notes in Mathematics Volume 840 , Springer Verlag, Berlin 1984.

R.A. Johnson. Hopf bifurcation from nonperiodic solutions of differential equations. I. Linear theory. *J. Dyn. Diff. Eq.* **1** (1989) pp. 179–198.

——and P.E. Kloeden. Nonautonomous attractors of skew product flows with digitized driving systems. *Electronic J. Diff. Eq.*, to appear (2001).

——and Y. Yi. Hopf bifurcation from nonperiodic solutions of differential equations, II. *J. Dyn. Diff. Eq.* **107** (1994) pp. 310–340.

P.E. Kloeden and B. Schmalfuss. Cocyle attractors of variable time step discretizations of Lorenzian systems. *J. Difference Eqns. Applns.* **3** (1997) pp. 125–145.

——and B. Schmalfuss. Asymptotic behaviour of nonautonomous difference inclusions. *Systems and Control Letters* **33** (1998), pp. 275-280.

——and D. Stonier. Cocyle attractors of nonautonomously perturbed differential equations. *Dynamics of Discrete, Continuous and Impulsive Systems* **4** (1998) pp. 221–226.

O.A. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Accademia Nazionale dei Lincei, Cambridge University Press, Cambridge 1991.

J.A. Langa and J.C. Robinson. A finite number of point observations which determine a non-autonomous fluid flow. *Nonlinearity* **14** (2001) pp. 673–682.

——and A. Suárez. Pullback permanence for non-autonomous partial differential equations. Preprint (2001).

——, J.C. Robinson and A. Suárez. Forwards and pullback behaviour of a non-autonomous Lotka-Volterra system. Preprint (2001).

N. Malhotra and S. Wiggins. Geometric structures, lobe dynamics, and Lagrangian transport in flows with aperiodic time-dependence, with applications to Rossby wave flow. *J. Nonlinear Sci.* **8** (1998) pp. 401–456.

J.C. Robinson. *Infinite-dimensional dynamical systems*, Cambridge University Press, Cambridge 2001.

K.R. Schenk-Hoppé. Random attractors – general properties, existence and applications to stochastic bifurcation theory. *Discrete and Continuous Dynamical Systems* **4** (1998) pp. 99–130.

B. Schmalfuß. Backward cocycles and attractors of stochastic differential equations. In V. Reitmann, T. Riedrich and N. Koksch (eds.). *International Seminar on Applied Mathematics - Nonlinear Dynamics: Attractor Approximation and Global Behaviour*, Technische Universität, Dresden (1992) pp. 185–192.

——. Attractors for the nonautonomous dynamical systems. In B. Fiedler, K. Gröger and J. Sprekels (eds.). *Proceedings of Equadiff 99 Berlin*, World Scientific, Singapore (2000) pp. 684–689.

G.R. Sell. Non-autonomous differential equations and dynamical systems. *Trans. Amer. Math. Soc.* **127** (1967), pp. 241-283.

G.R. Sell. *Lectures on topological dynamics and differential equations*, Van-Nostrand-Reinhold, Princeton NJ, 1971.

W. Shen and Y. Yi. Almost automorphic and almost periodic dynamics in skew-product semiflows. *Mem. Amer. Math. Soc.* **647** (1998).

S. Siegmund. Dichotomoy spectrum for nonautonomous differential equations. *J. Dyn. Diff. Eq.* **14** (2002a), pp. 243–258.

S. Siegmund. Normal forms for nonautonomous differential equations. *J. Diff. Eq.* **178** (2002b), pp. 541–573.

A.M. Stuart and A.R. Humphries. *Dynamical Systems and Numerical Analysis*, Cambridge University Press, Cambridge, 1996.

R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York 1988 (1st edition) and 1996 (2nd edition).