Some non-local population models with non-linear diffusion

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Abstract

In this paper we present some theoretical results concerning to a non-local elliptic equation with non-linear diffusion arising from population dynamics.

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1 Introduction

There is a lot of phenomena that can be modelled by reaction-diffusion PDE of the general form

$$u_t - \Delta u = f(x, u(x,t))$$

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joint with initial and boundary conditions, $x \in \Omega$, Ω is a bounded and regular domain of $\mathbb{R}^N$, $N \geq 1$; $t \geq 0$, f a regular function and the unknown function $u : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}$. In this kind of equation the relation between u and its derivatives is *local*, that is, all the functions are taken at the same point x.

There are some phenomena where a *non-local* spatial term has to be included in this model. In this case, the equation has the form

$$u_t - \Delta u = f(x, u, B(u))$$

where B is a non-local operator, for instance

$$B(u) = \int_\Omega g(y, u(y, t))dy,$$

see [18] for a general survey of these equations.

In this paper we are interested in the stationary problem associated with the above problem. In fact, due to these motivations, we study the existence, uniqueness or multiplicity, and stability of positive solutions of the equation

$$\begin{cases} -\Delta u = f(x, u, B(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f is a regular function specified later and

$$B(u) = \int_\Omega b(x)u^\beta(x)dx, \qquad \beta > 0.$$

Roughly speaking, there are several difficulties that appear when one introduces a non-local term in (1.1). Let us point some of them:

a) In general, (1.1) has not a variational structure and so we can not apply the powerful tool of "variational methods" to attack (1.1). See [11] where a problem with a non-local term has a variational structure.

b) In general, the equation (1.1) does not satisfy a maximum principle, and as main consequences, we can not apply directly some classical methods as sub-supersolutions, see Section 4 for more details.

c) In general, the linearized operator of (1.1) at a stationary solution is an integral-differential operator and it will not be self-adjoint, see Section 2 for more details.

Specifically, in this paper we study the following equation, arising in some cases from the population dynamics, of the form

$$\begin{cases} -\Delta w^m = wf\left(x, \int_\Omega w^r\right) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with $r > 0$, f is a regular function and $m \geq 1$. Here, we are assuming that Ω is fully surrounded by an inhospitable area, since the population density is subject to homogeneous Dirichlet boundary conditions. The real parameter m represents the velocity of diffusion, the rate of movement of the species from high-density regions to low-density ones. In this context, $m > 1$ means that the diffusion is slower than in the linear case ($m = 1$), which seems to give more realistic models, see [22]. The term $m > 1$ was introduced in [22], see also [25], by describing the dynamics of biological population whose mobility depends upon their density. Finally, f denotes the crowding effect. Observe that this term includes a non-local term. Non-local terms have been introduced at least to our knowledge, in population dynamic models in [21]. The presence of the nonlocal terms in (1.2), from the biological point of view means that the crowding effect depends not only on their own point in space but also depends on the entire population.

The change $w^m = u$ transforms the problem (1.2) into

$$\begin{cases} -\Delta u = u^q\, f\left(x, \int_\Omega u^p\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with $0 < q < 1,\ p > 0$. Specifically, in this note, we are concerned with the the nonlocal elliptic problem

$$\begin{cases} -\Delta u = u^q \left(\lambda + a(x) \int_\Omega b(x) u^p\right) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded and regular domain of $\mathbb{R}^N$, $N \geq 1$, $a, b \in C(\overline{\Omega})$, $b \geq 0$, $b \not\equiv 0$,

$$\lambda \in \mathbb{R}, \quad 0 < q < 1, \quad p > 0,$$

and a verifies either $a > 0$ or $a < 0$.

The above equation is a nonlocal counterpart of the well known logistic equation, whose more general version is given by

$$\begin{cases} -\Delta u = u^q\left(\lambda + a(x)u^p\right) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where λ, p, q and a are as above.

Let us point an important fact on equations (1.5) and (1.4). When $q = 1$, the strong maximum principle implies that any *positive* solution u (positive means non-negative and non-trivial) is *strictly positive* ($u(x) > 0$ for all $x \in \Omega$.) This means that there are uniquely two kinds of solutions in this case: the trivial solution (the species is dead) and the strictly positive solution (the species survives in whole domain). However, when $q < 1$ a new type of solution appears: a non-negative and non-trivial solution u but vanishing in a part of the domain $\Omega_0 \subset \Omega$, that is $u(x) = 0$ for $x \in \Omega_0$. This set is called dead-core, see [26] and [12] where conditions on the coefficients are given to assure the existence of dead cores.

On the other hand, when $q < 1$ the nonlinear reaction term is not derivable at $u = 0$. This entails some theoretical problems to linearize: we can not linearize at $u \equiv 0$ and some singular terms appear when one linearizes at a positive solutions $u > 0$.

With respect to the mathematical analysis of (1.4) we consider two situations:

(i) The Homogeneous Case. Here we suppose that a is a constant and we use fixed point to obtain existence results. In this case we are able to describe exactly the set of positive solution of (1.4).

(ii) The Non-Homogeneous Case. Here we consider the situation in which a depends on $x \in \Omega$. In this case, bifurcation theory and sub-supersolution method plays a key role.

When $q = p = 1$ and $a = -1$ in [5] the authors proved the existence, uniqueness and stability of positive solution when $\lambda > \lambda_1$, λ_1 stands for the principal eigenvalue of the operator $-\Delta$ in Ω under homogeneous Dirichlet boundary conditions. In this case, the solution can be explicitly built, it is proportional to a positive eigenfunction associated to λ_1. More recently, and again with $q = p = 1$ but a a function such that $a \le a_0 < 0$, in [7] the authors proved the existence and uniqueness of positive solution for $\lambda > \lambda_1$. In this paper, the authors used bifurcation methods to prove the results. See also [8] for a related problem.

We study questions of stability of positive solutions by using heavily the results on nonlocal and singular eigenvalue problems contained in section 2 of this work. In sections 3 and 4 we study the local (1.5) and (1.4) equations, respectively. In the last section we discuss the main results of this paper and the differences between the local and the non-local equations.

2 Non-local eigenvalue problems

In this section we study a non-local and singular eigenvalue problem, which appears when one linearizes around a positive solution of (1.4). Specifically, we study the following problem

$$\begin{cases} -\Delta u + m(x)u - h(x)\displaystyle\int_\Omega g(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $m \in C^1(\Omega)$, $h \in C(\overline{\Omega})$ and $g \in C^1(\Omega)$ and verify: for some $\alpha \in (-1, 1)$ and $\beta < 1$

(Hm) $|\partial_i m| d(x, \partial\Omega)^{2-\alpha}$ are bounded for all $x \in \Omega$ and $i = 1, ..., N$;

(Hg) there exists $K > 0$ such that $g(x) \le K d(x, \partial\Omega)^{-\beta}$,

where $d(x, \partial\Omega) := \text{dist}(x, \partial\Omega)$.

Basically, in (2.1) there is a combination between a differential and an integral operator. Moreover, (2.1) has different difficulties: the existence of singular terms and that the operator is not self-adjoint (in fact it is self-adjoint if and only if h and g are proportional).

When the coefficients are bounded, in [17] (see also [19]) the authors proved the existence of a sequences $\{\lambda_i\}$ in the complex plane with finite multiplicity.

Since we are interested in the existence of positive solution of (1.4), with respect to (2.1) we want to prove the existence of a *principal eigenvalue* of (2.1), that is, a real and simple eigenvalue with positive eigenfunction associated to it. Moreover, it is less than all the real parts of the other eigenvalues. In order to prove the existence of a principal eigenvalue for a non self-adjoint operator, the Krein-Rutman Theorem is a powerful tool, see [1] for a general version of this result. In our setting, this theorem says: consider $f \geq 0$, $f \neq 0$, and consider the solution v of the linear integro-differential problem

$$\begin{cases} -\Delta v + m(x)v - h(x)\int_\Omega g(x)v = f(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

If v is strictly positive, then there exists the principal eigenvalue, $\lambda_1 \in \mathbb{R}$.

The following results are consequences of [4]:

a) If $h > 0$, then v is strictly positive, and the existence of a principal eigenvalue follows.

b) Given $f \geq 0$, $f \neq 0$ there exists $h < 0$ such that the solution v becomes negative in some part of Ω.

c) There exists $h_0 > 0$ such that if $\|h\|_\infty \in (0, h_0)$ then v is strictly positive.

What happens if $\|h\|_\infty$ large? In [4] the authors showed an example, with homogeneous Neumann boundary conditions, in which for $\|h\|_\infty$ large there are several eigenvalues less than an eigenvalue having a positive eigenfunction associated.

Using the Krein-Rutman Theorem and results from [23], the next theorems were proved in [9]:

Theorem 2.1. *Assume that m verifies (Hm), $h \in C^1(\Omega) \cap C(\overline{\Omega})$, a non-negative and non-trivial function, $g \in C^1(\Omega)$ is a non-negative and non-trivial function and verifies (Hg). Then, there exists a principal eigenvalue of (2.1), denoted by $\lambda_1(-\Delta + m; h; g)$, which has an associated positive eigenfunction $\varphi_1 \in C^2(\Omega) \cap C_0^{1,\delta}(\overline{\Omega})$ for some $\delta \in (0, 1)$. Moreover, $\lambda_1(-\Delta + m; h; g)$ is simple, and it is the unique eigenvalue having an associated eigenfunction without change of sign.*

As we said before, we need to study the sign of the principal eigenvalue in order to know the stability of a positive solution of (1.4).

In the following result we give a criteria to ascertain the sign of $\lambda_1(-\Delta + m; h; g)$.

Proposition 2.2. *Assume that there exists a positive function* $\overline{u} \in C^2(\Omega) \cap C_0^{1,\delta}(\overline{\Omega})$, $\delta \in (0,1)$, *such that*

$$-\Delta\overline{u} + m(x)\overline{u} - h(x)\int_\Omega g(x)\overline{u} > 0 \quad \text{in } \Omega \text{ (resp. } < 0 \text{ in } \Omega).$$

Then,

$$\lambda_1(-\Delta + m; h; g) > 0 \quad \text{(resp. } \lambda_1(-\Delta + m; h; g) < 0\ ).$$

3 The local problem

In this section we collect the main results concerning to the local equation (1.5). We employ the following notation

$$\sigma_1(-\Delta + m) := \lambda_1(-\Delta + m; 0; 0).$$

From the results of [26], [2], [13], [14], [15] and references therein, the results can be summarized in the following way:

Theorem 3.1. *a) Assume* $a < 0$. *Then, there exists a positive solution of (1.5) if and only if* $\lambda > 0$. *When* $\lambda > 0$ *the solution is strictly positive, unique and stable.*

b) Assume $a > 0$.

(a) Assume $p + q < 1$. *Then, there exists a value* $\underline{\lambda} < 0$ *such that there exists a positive solution of (1.5) if and only if* $\lambda \geq \underline{\lambda}$. *Moreover, if* $\lambda \geq 0$ *the solution is strictly positive, unique and stable.*

(b) Assume $p + q = 1$.

i. If $\sigma_1(-\Delta - a) > 0$ *there exists a positive solution of (1.5) if and only if* $\lambda > 0$. *The solution is strictly positive, unique and stable.*

ii. If $\sigma_1(-\Delta - a) = 0$ *there exists a positive solution of (1.5) if and only if* $\lambda = 0$. *Moreover, there exist infinite solutions and any positive solution is neutrally stable.*

iii. If $\sigma_1(-\Delta - a) < 0$ *there exists a positive solution of (1.5) if and only if* $\lambda < 0$.

(c) Assume $1 < p+q < (N+2)/(N-2)$. *Then there exists a value* $\overline{\lambda} > 0$ *such that (1.5) possesses a positive solution if and only if* $\lambda \leq \overline{\lambda}$. *Moreover, for* $\lambda \in (0, \overline{\lambda})$ *the problem (1.5) possesses at least two positive solution, one of them is the minimal solution and this is the unique stable solution.*

In Figure 1 we have represented the bifurcation diagrams of (1.5). The case $a < 0$ is shown in Case 1. Assume now that $a > 0$. Cases 1, 2 and 3 show the case $p + q = 1$ and $\sigma_1(-\Delta - a) > 0$, $\sigma_1(-\Delta - a) = 0$ and $\sigma_1(-\Delta - a) < 0$, respectively. In Cases 4 and 5 we have drawn the cases $p+q < 1$ and $1 < p+q < (N+2)/(N-2)$. We remark that in Cases 3, 4 and 5 the drawings are simple representations of the set of solutions, for example in Case 3 the solutions need not be unique in spite of the figure.

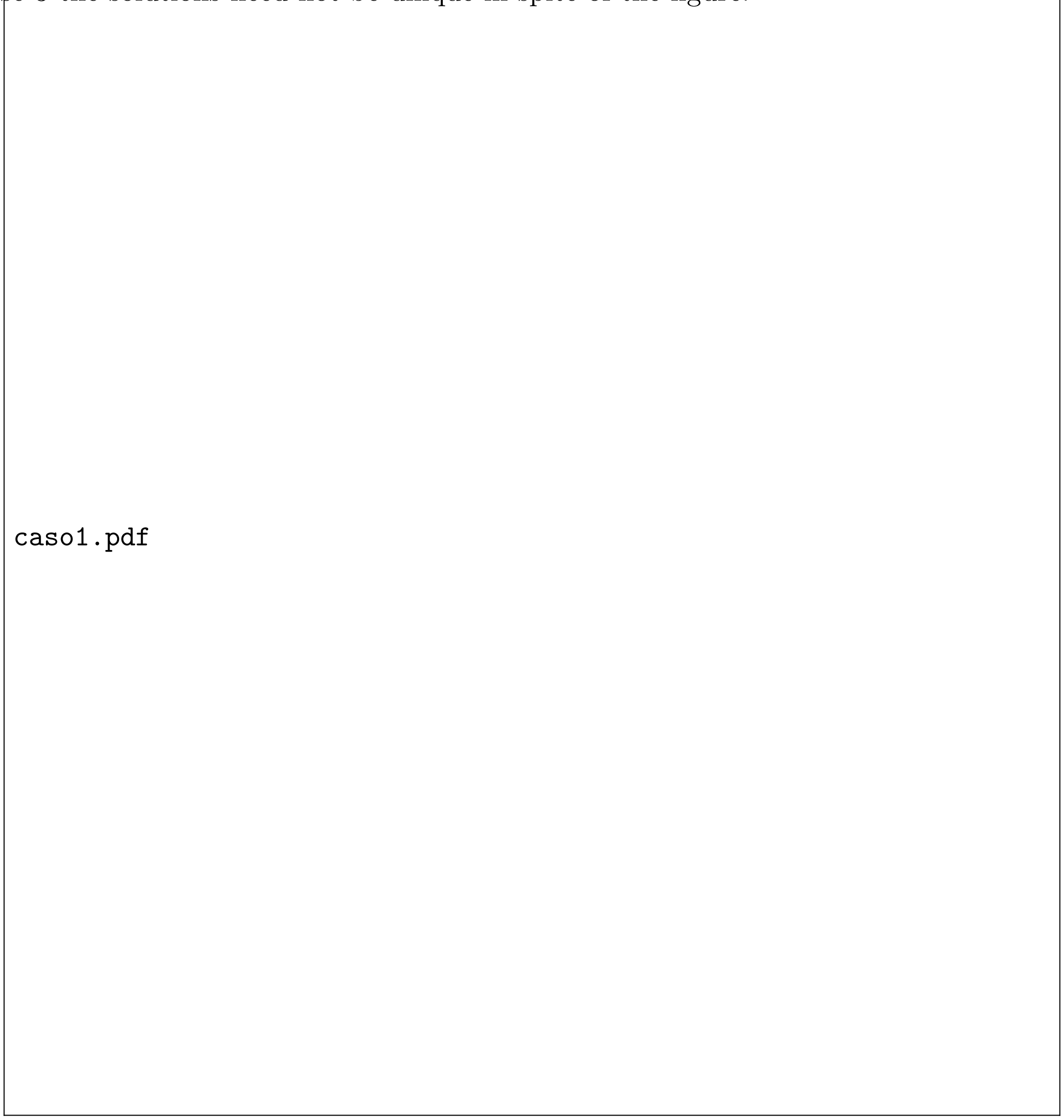

Figure 1: Bifurcation diagrams.

4 The non-local problem

Before we state the main results in this case we need some notations. Consider $a \in C(\overline{\Omega})$, $a > 0$ and denote by ω_a the unique positive solution of

$$\begin{cases} -\Delta u = a(x)u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

and

$$A := \int_\Omega b(x)\omega_a^p. \tag{4.2}$$

As we said in the introduction, in the non-local case the results and techniques used depend on the a. We distinguish two cases:

4.1 The homogeneous case

Assume that a is constant, that is $a \in \mathbb{R}$. In this case, we denote by

$$R := \int_\Omega b(x)u^p(x)dx.$$

We study the equation

$$\begin{cases} -\Delta w = w^q\,(\lambda + aR) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

It is well-known that (4.3) possesses a unique positive solution, denoted by u_R, if and only if $\lambda + aR > 0$ and in fact

$$u_R = ((\lambda + aR)^+)^{1/(1-q)}\omega_1.$$

Then, to find positive of (1.4) is equivalent to study the <u>unidimensional</u> equation

$$R = ((\lambda + aR)^+)^{p/(1-q)} \int_\Omega b(x)\omega_1^p = ((\lambda + aR)^+)^{p/(1-q)}A.$$

We prove in [9]:

Theorem 4.1. *a) Assume $a < 0$. Then, there exists a positive solution of (1.4) if and only if $\lambda > 0$. Moreover, the solution, u_λ, is unique and*

$$\lim_{\lambda\to 0}\|u_\lambda\|_\infty = 0, \qquad \lim_{\lambda\to+\infty}\|u_\lambda\|_\infty = +\infty.$$

b) Assume $a > 0$.

(a) *Assume* $p + q < 1$. *Then there exists a value* $\underline{\lambda} = \underline{\lambda}(q, p, a, A) < 0$ *such that (1.4) possesses a positive solution if and only if* $\lambda \geq \underline{\lambda}$. *Moreover, if* $\lambda = \underline{\lambda}$ *or* $\lambda \geq 0$ *the solution,* u_λ, *is unique and stable when* $\lambda \geq 0$ *and neutrally stable for* $\lambda = \underline{\lambda}$. *If* $\lambda \in (0, \underline{\lambda})$ *there exist exactly two positive solutions* $u_2 < u_1$; u_2 *is unstable and* u_1 *is stable. Furthermore,*

$$\lim_{\lambda \to +\infty} \|u_\lambda\|_\infty = +\infty.$$

(b) *Assume* $p + q = 1$.

i. *If* $A < 1$ *there exists a positive solution of (1.4) if and only if* $\lambda > 0$. *The solution,* u_λ, *is unique and stable, and*

$$\lim_{\lambda \to 0} \|u_\lambda\|_\infty = 0, \qquad \lim_{\lambda \to +\infty} \|u_\lambda\|_\infty = +\infty.$$

ii. *If* $A = 1$ *there exists a positive solution of (1.4) if and only if* $\lambda = 0$. *Moreover, there exist infinitely many solutions and any positive solution is neutrally stable.*

iii. *If* $A < 1$ *there exists a positive solution of (1.4) if and only if* $\lambda < 0$. *The solution,* u_λ, *is unique, unstable and*

$$\lim_{\lambda \to 0} \|u_\lambda\|_\infty = 0, \qquad \lim_{\lambda \to -\infty} \|u_\lambda\|_\infty = +\infty.$$

(c) *Assume* $p + q > 1$. *Then there exists a value* $\overline{\lambda} = \overline{\lambda}(q, p, a, A) > 0$ *such that (1.4) possesses a positive solution if and only if* $\lambda \leq \overline{\lambda}$. *Moreover, if* $\lambda = \overline{\lambda}$ *or* $\lambda \leq 0$ *the solution is unique and unstable when* $\lambda \leq 0$ *and neutrally stable for* $\lambda = \overline{\lambda}$. *If* $\lambda \in (0, \overline{\lambda})$ *there exist exactly two positive solutions* $u_2 < u_1$; u_2 *is stable and* u_1 *is unstable. Moreover,*

$$\lim_{\lambda \to -\infty} \|u_\lambda\|_\infty = +\infty.$$

Again, the set of positive solutions is represented in Figure 1. However, in this case the figures represent exactly the set of positive solutions.

4.2 The non-homogeneous case

In this case, the argument above can not be applied due to the equation (4.3) with a a variable function, it is not completely well-known. Hence, we need to apply another arguments. We have used a combination of bifurcation and sub-supersolution methods. Let us point some facts with respect to both methods. First, observe that, due to the presence of the term u^q, $q < 1$, we can not apply directly the classical bifurcation results of [27]. So, in order to apply the global bifurcation result (the existence of an unbounded continuum $\mathcal{C}$ of positive solutions of (1.4) bifurcating from the trivial solution $u \equiv 0$ at $\lambda = 0$) we have to compute the Leray-Schauder degree of the isolated solution $u \equiv 0$ by using appropriate homotopies, see [3] and [6]. Let us mention that bifurcation arguments have been used used previously in non-local problems in [16], [24], [7], [21] and [8].

On the other hand, it is well-known that the existence of a pair of sub-supersolution of (1.4), when this pair satisfies the classical definition of sub-supersolution, does not imply the existence of a solution of (1.4) between the sub and supersolution. Indeed, this result is only true when f and B satisfy specific monotony properties, see [20]. In [10] we introduce a modified definition of a pair of sub-supersolution, which coincides with the classical one under monotony of the function f and the non-local operator B, for which, if we have a pair of sub-supersolution it follows the existence of a solution between them.

With these two arguments, we show in [10] the following two results:

Theorem 4.2. *Assume $a < 0$. Then, there exists a positive solution of (1.4) if and only if $\lambda > 0$. Moreover,*

$$\lim_{\lambda \to 0} \|u_\lambda\|_\infty = 0.$$

Furthermore, there exist $0 < \underline{\lambda} < \overline{\lambda} < \infty$ such that if some of the following conditions holds:

a) $p + q < 1$ and $\lambda \geq \overline{\lambda}$ or;

b) $p + q = 1$ and $|\min_{x \in \overline{\Omega}} a(x)|$ small or;

c) $p + q > 1$ and $\lambda \leq \underline{\lambda}$;

problem (1.4) possesses a unique strictly positive solution.

Theorem 4.3. *Assume $a > 0$ and $0 < q < 1$.*

a) Assume $p+q<1$. *Then, there exists a value* $\underline{\lambda}<0$ *such that there exists a positive solution of (1.4) if and only if* $\lambda \geq \underline{\lambda}$. *Moreover, if* $\lambda \geq 0$ *the solution is strictly positive, unique and stable. Furthermore,*

$$\lim_{\lambda \to +\infty} \|u_\lambda\|_\infty = +\infty.$$

b) Assume $p+q=1$.

(a) If $A<1$ *there exists a positive solution of (1.4) if and only if* $\lambda>0$. *The solution is strictly positive, unique and stable and*

$$\lim_{\lambda \to 0} \|u_\lambda\|_\infty = 0, \qquad \lim_{\lambda \to +\infty} \|u_\lambda\|_\infty = +\infty.$$

(b) If $A=1$ *there exists a positive solution of (1.4) if and only if* $\lambda=0$. *Moreover, there exist infinite solutions and any positive solution is neutrally stable.*

(c) If $A>1$ *there exists a positive solution of (1.4) if and only if* $\lambda<0$. *Moreover,*

$$\lim_{\lambda \to 0} \|u_\lambda\|_\infty = 0, \qquad \lim_{\lambda \to -\infty} \|u_\lambda\|_\infty = +\infty.$$

c) Assume $p+q>1$. *Then there exists a value* $\overline{\lambda}>0$ *such that (1.4) possesses a positive solution if and only if* $\lambda \leq \overline{\lambda}$. *Moreover,*

$$\lim_{\lambda \to -\infty} \|u_\lambda\|_\infty = +\infty.$$

5 Concluding remarks

We have considered problem (1.4) for $0<q<1$, $p>0$ and $\lambda \in \mathbb{R}$, and we have distinguished two cases in our study.

The first case is the homogeneous one, i.e., when a is constant. In this case, we know that the solution of (1.4) must be of the form $u=\alpha\,\omega_1$, ω_1 is defined in (4.1) and α the solution of a nonlinear equation; so, the results of the existence and the uniqueness of the solution follow from the study of this equation. The stability of the solution depends on the sign of the principal eigenvalue of an eigenvalue problem which is singular due to the Dirichlet boundary condition; we are able to give the results of stability of the solution when $a>0$. In fact, we can describe exactly the set of positive solution in this case.

The second case is the nonhomogeneous one, i.e., when $a(x)$ is a function which does not change of sign and it can be zero on a subdomain with positive measure. In this case, we do not know the structure of the solution and the results of the existence and the uniqueness of the solution follow from the sub-supersolution method (whose validity we have checked) and the bifurcation method. We also obtain some results of stability with a method similar to the used in the first case.

This problem presents interesting differences with respect to the local problem, mainly in the case $a > 0$. For example, in the nonlocal problem, the strong maximum principle holds and any non-negative and non-trivial solution is a positive solution; in the local problem, the existence of dead cores in the solution can not be dismissed. We can also observe that in the nonlocal problem, we give exactly the number of the solutions unlike the local problem. And finally, observe that in the superlinear case $p + q > 1$, in order to assure the existence of solution we have to impose that $p + q < (N + 2)/(N - 2)$, to have a priori bounds of the solutions. However, this condition is not required in the non-local case. Finally, observe that is some cases we obtain results of the existence of a unique solution unstable, a not very usual result, in our knowledge, in non-linear elliptic equation.

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