A Lotka-Volterra symbiotic model with cross-diffusion

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Abstract

The main goal of this paper is to study the existence and non-existence of coexistence states for a Lotka-Volterra symbiotic model with cross-diffusion. We use mainly bifurcation methods and a priori bounds to give sufficient conditions in terms of the data of the problem for the existence of positive solutions. We also analyze the profiles of the positive solutions when the cross-diffusion parameter goes to infinity.

Key Words. Cross-diffusion, symbiotic model, bifurcation method, a priori bounds.

1 Introduction

In this paper we study the problem

$$\begin{cases} -\Delta u = u(\lambda - u + bv) & \text{in } \Omega, \\ -\Delta[(1+\beta u)v] = v(\mu - v + cu) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with a smooth boundary, $\beta, b, c > 0$ and $\lambda, \mu \in \mathbb{R}$. This system was introduced by Shigesada et al. [22] to model the segregation phenomenon of two species, where u and v are their densities, which are interacting and migrating in the same habitat Ω. Since b and c are positive, it is assumed in this model that both species cooperate. Here, b and c are the interaction rates between the species, λ and μ are the growth rates of the species, and the cross-diffusion parameter β describes the interference of the population u into v. So, v diffuses obeying, in addition to a random movement, a repulsive force due to the population pressure by u.

When $\beta = 0$, problem (1.1) is reduced to the classical Lotka-Volterra symbiotic model with linear diffusion which has been studied in [2], [5], [13], [17] and references therein.

When $\beta > 0$, that is, when the cross-diffusion is present, the competition ($b < 0$ and $c < 0$) and prey-predator ($bc < 0$) cases have been studied in more detail than the symbiotic case ($b > 0$ and $c > 0$), see for instance [4], [6], [9], [14], [15], [16], [19], [20], [21], [23]. Basically, in these papers the authors study existence, non-existence, uniqueness or multiplicity of positive solutions using fixed point index in positive cones, global and local bifurcation techniques; and also sub-supersolution methods in [18].

The symbiotic interaction has received less attention, in fact, to our knowledge, only Pao in [18] has analyzed the model, see also [11] for a different cross-diffusion nonlinearity.

Our attention here will be focused on the problem of analyzing the existence and non-existence of non-negative solution pairs (u, v) of (1.1). System (1.1) admits three types of non-negative componentwise solution pairs, namely:
(i) the trivial solution $(0, 0)$;
(ii) the semi-trivial solutions, that is, those with one positive component and the other zero, as $(u, 0)$ or $(0, v)$;
(iii) the *coexistence states*, those with both positive components (u, v).

We introduce some notations to show our main results and the differences with respect to the linear diffusion case. Given two functions $a, b \in C^{\nu}(\overline{\Omega})$, $\nu \in (0, 1)$, with a strictly positive (i.e. $a(x) \geq const > 0$), we denote by $\lambda_1(a; b)$ the principal eigenvalue of the problem

$$-\Delta[a(x)u] + b(x)u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Moreover, we observe that when one of the species is zero, the other one satisfies the logistic equation

$$-\Delta w = \gamma w - w^2 \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega.$$

It is well-known that this equation possesses a unique positive solution, denoted by θ_γ, if and only if, $\gamma > \lambda_1 := \lambda_1(1; 0)$. For $\gamma \leq \lambda_1$ we define $\theta_\gamma \equiv 0$.

Observe that the trivial solution $(0, 0)$ exists for all $(\lambda, \mu) \in \mathbb{R}^2$; and the semi-trivial solutions $(\theta_\lambda, 0)$ and $(0, \theta_\mu)$ if $\lambda > \lambda_1$ and $\mu > \lambda_1$, respectively. Hence, we focus our attention on the existence or non-existence of coexistence states of (1.1).

Roughly speaking, the presence of the species v is beneficial to u, due to the cooperative character of the system; however in the equation of v there is a balance between the cooperation (term $+cuv$) and the repulsive force in the diffusion (term $+\beta uv$). So, it is interesting to look at the necessary balance between both terms to obtain existence or non-existence of coexistence states to (1.1).

In order to show our results we need some notations:

$$F(\mu) = \begin{cases} \lambda_1(1; -b\theta_\mu) & \text{for } \mu > \lambda_1, \\ \lambda_1 & \text{for } \mu \leq \lambda_1, \end{cases} \tag{1.2}$$

$$H(\lambda) = \begin{cases} \lambda_1(1; -c\theta_\lambda) & \text{for } \lambda > \lambda_1, \\ \lambda_1 & \text{for } \lambda \leq \lambda_1, \end{cases} \tag{1.3}$$

and

$$G(\beta, \lambda) = \begin{cases} \lambda_1(1 + \beta\theta_\lambda; -c\theta_\lambda) & \text{for } \lambda > \lambda_1, \\ \lambda_1 & \text{for } \lambda \leq \lambda_1, \end{cases} \tag{1.4}$$

(see Section 2, Lemma 4.1, where we have studied in detail these curves). We state now our main result concerning existence and non-existence of coexistence states.

Theorem 1.1 *(1) If $\mu \leq \lambda_1$ and $\beta\lambda_1 \geq c$, (1.1) does not have coexistence states.*

(2) If $\mu, \lambda \leq \lambda_1$ and $b(c - \beta\lambda_1) < 1$, (1.1) does not have coexistence states.

(3) If $bc < 1$, then there exists at least one coexistence state of (1.1) if (λ, μ) verifies the following condition

$$\lambda > F(\mu) \quad \text{and} \quad \mu > G(\beta, \lambda). \tag{1.5}$$

(4) There exists $\beta_0 > 0$ such that for all $\beta > \beta_0$ problem (1.1) possesses at least one coexistence state if (λ, μ) verifies (1.5).

We point out that our existence results improve those of Pao in [18], where the existence is obtained only for $\lambda, \mu > \lambda_1$, $bc < 1$ and β small using the sub-supersolution method.

Here we mainly use the bifurcation method, showing that a continuum of coexistence states emanates from a semi-trivial solution at some specific values of the parameter λ and μ. For that matter, we need to prove a priori bounds for the coexistence states of (1.1). This is an easy task under weak cooperation interaction $bc < 1$, but more involved in the general cooperation case $bc \geq 1$. The general result of [10] can not be applied to (1.1). We prove that these a priori bounds are true for β large using a blow-up argument due to Gidas-Spruck [3]. The strong cooperation case ($bc > 1$) with β small will be studied elsewhere. Otherwise, if $bc < 1$ and the family (u_β, v_β) of coexistence states of (1.1) converges to a solution of system (1.1) with $\beta = 0$, see Remark 3.3.

We compare now the results in the case $\beta = 0$ and $\beta > 0$. Observe that relation (1.5) defines a *coexistence region* in the plane $\lambda - \mu$ (see Figure 1):

$$\mathcal{R}_\beta := \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda > F(\mu) \quad \text{and} \quad \mu > G(\beta, \lambda)\}.$$

This region $\mathcal{R}_\beta \subset \mathcal{R}_0$ and $\mathcal{R}_\beta \uparrow \mathcal{R}_0$ as $\beta \downarrow 0$ (see Section 2) being

$$\mathcal{R}_0 := \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda > F(\mu) \quad \text{and} \quad \mu > H(\lambda)\}.$$

Precisely $\mathcal{R}_0$ is a coexistence region when $bc < 1$ and $\beta = 0$, see [2] for example.

We have drawn the coexistence regions in Figure 1. In the Case a) we have represented only $\mathcal{R}_0$, in the other cases we have drawn $\mathcal{R}_\beta$ and $\mathcal{R}_0$ to compare them. In Case a) we present the coexistence region of (1.1) with $\beta = 0$ defined by $\mathcal{R}_0$. In Case b) we have drawn the case β small, specifically $0 < \beta < c/\lambda_1$; the particular case $\beta = c/\lambda_1$ is described in Case c); and finally the case β large ($\beta > c/\lambda_1$) is presented in Case d).

Hence, if $(\lambda, \mu) \in \mathcal{R}_\beta$ (and so there exists a coexistence state for (1.1)) then $(\lambda, \mu) \in \mathcal{R}_0$, and so there exists a coexistence state for (1.1) and $\beta = 0$. On the other hand, if $(\lambda, \mu) \in \mathcal{R}_0$ then there exists $\beta_0 > 0$ such that $(\lambda, \mu) \in \mathcal{R}_\beta$ for $\beta \leq \beta_0$. So, when $bc < 1$ the dynamics of the system in the cases $\beta = 0$ and β small are rather similar.

However, when β is large the behavior of the model is completely different. Indeed, when β is large and whatever value of $bc > 0$ is, the coexistence region is still $\mathcal{R}_\beta$ (see Figure 1 Cases c) and d)). However, when $bc > 1$ and $\beta = 0$ a coexistence region includes $\mathbb{R}^2 \setminus \overline{\mathcal{R}}_0$, see [13] and [2], and there does not exist coexistence states for $\lambda > \lambda_1$ and $\mu > \lambda_1$. We mention that in the case $\beta = 0$ and $bc > 1$ there is absence of a priori bounds for the coexistence states in high spatial dimensions $N > 6$ (see [13]), however for $N \leq 5$ system (1.1) possesses uniform a priori bounds in any compact subinterval of (λ, μ).

We give now some examples which reflect the difference between the cases $\beta = 0$ and $\beta > 0$. Fix $\lambda > \lambda_1$, $bc < 1$ and $(\lambda, \mu) \in \mathcal{R}_0$. In this case, for $\beta = 0$ the species coexist. On the other hand, by Theorem 1.2 item (1), there exists $\beta_0 > 0$ such that for $\beta \geq \beta_0$

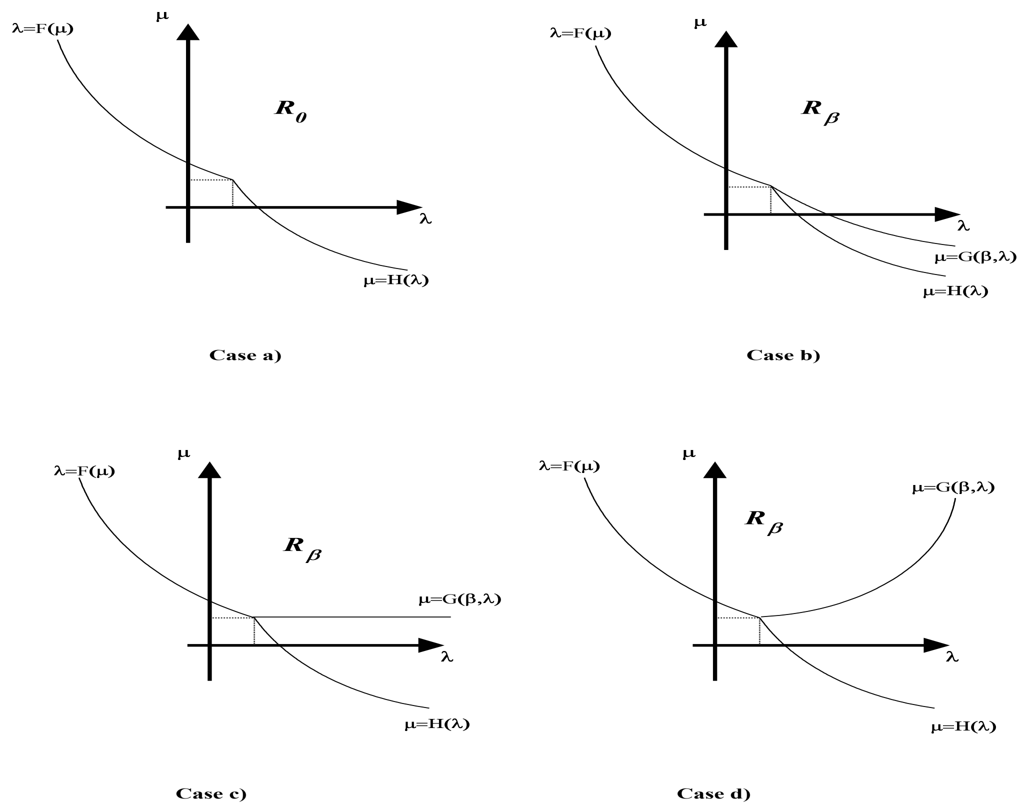

Figure 1: Coexistence regions in: Case a) $\beta = 0$; Case b) $0 < \beta < c/\lambda_1$; Case c) $\beta = c/\lambda_1$ and Case d) $\beta > c/\lambda_1$.

model (1.1) does not possess coexistence state. This fact has a biological interpretation: if the growth rate of u is large ($\lambda > \lambda_1$) and the repulsive force in the diffusion is large (β large), then v is driven to the extinction by u, hence the repulsive force is stronger than the cooperation between the species. This is translated into the fact that for $\beta \geq \beta_0$, there is only one stable semi-trivial solution, that is $(\theta_\lambda, 0)$, see Proposition 4.2.

Now fix $\lambda \leq \lambda_1$ and $\mu \leq \lambda_1$; then again by Theorem 1.1, there does not exist coexistence state if β is large or for any β if $bc \leq 1$, that is, if both growth rates are small, then the species do not coexist if the pressure produced by u is large or if the cooperation is too weak. This is completely different to the case $bc > 1$, $N \leq 5$ and $\beta = 0$, for which there exists a coexistence state for $\lambda, \mu \leq \lambda_1$, see [13].

In the second part of the paper, we have studied the profiles of the solutions when the cross-diffusion parameter β tends to $+\infty$, this type of study is made in a slight different problem in [7] and [8], see also [14]. We show the following result.

Theorem 1.2 *(1) Fix $(\lambda, \mu) \in \mathbb{R}^2$ with $\lambda > \lambda_1$. Then, (1.1) does not have coexistence states if $\beta > 0$ is large.*

(2) Assume now that $\lambda < \lambda_1$. Then any family of positive solutions (u_β, v_β) of (1.1) verifies that $(\beta u_\beta, v_\beta) \to (w, z)$ as $\beta \to \infty$ uniformly in $\overline{\Omega}$ where (w, z) is positive solution

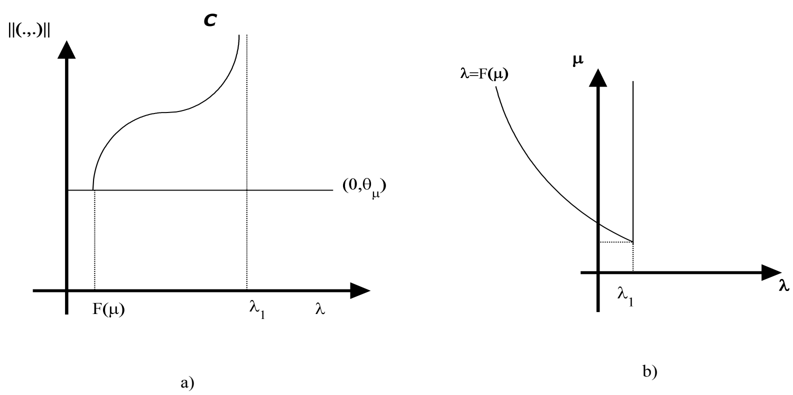

Figure 2: Bifurcation diagram and coexistence region of (1.6)

of

$$\begin{cases} -\Delta z = z(\lambda + bw) & in\ \Omega, \\ -\Delta[(1+z)w] = w(\mu - w) & in\ \Omega, \\ z = w = 0 & on\ \partial\Omega. \end{cases} \tag{1.6}$$

(3) System (1.6) does not possess any coexistence state if $\lambda \geq \lambda_1$ and has at least one coexistence state for

$$(\lambda, \mu) \in \mathcal{R}_\infty := \{(\lambda, \mu) \in \mathbb{R}^2 : F(\mu) < \lambda < \lambda_1\}. \tag{1.7}$$

In fact, for $\mu > \lambda_1$ fixed, an unbounded continuum $\mathcal{C}$ in $\mathbb{R} \times (C_0^1(\overline{\Omega}))^2$ bifurcates from the semi-trivial solution $(0, \theta_\mu)$ at $\lambda = F(\mu)$ and a bifurcation to infinity at $\lambda = \lambda_1$ appears when the parameter λ approaches to λ_1.

The paper is organized as follows. In Section 2 we present some preliminary results on the logistic equation and weighted eigenvalue problems. In Section 3 we prove parts (1) and (2) of Theorem 1.1. We also prove some a priori bounds. In Section 4 we show the stability of the trivial and semi-trivial solutions and a very general result of bifurcation of positive solutions from semi-trivial solutions and no assumption is made on the product bc, we just need $b, c > 0$. In Section 5 we prove the existence result for $bc < 1$ corresponding to item (3) of Theorem 1.1. Section 6 is devoted to show the existence result for β large and any $b, c > 0$, we then prove item (4) of Theorem 1.1. For that mater we perform a blow-up argument from [3]. Finally, in Section 7 we study the profiles of the positive solutions when $\beta \to \infty$. Theorem 1.2 is proved in this section.

2 Preliminaries

We are interested in non-negative solutions (u, v) of (1.1) in a classical sense, that is, $u, v \in C^2(\overline{\Omega})$. Recall that (1.1) has three kinds of solutions: the trivial one $(0, 0)$; the semi-trivial solutions $(u, 0)$ and $(0, v)$; and the solutions with both components non-negative and non-trivial. Thanks to the strong maximum principle, if a solution (u, v) of (1.1) is such that u and v are non-negative and non-trivial then both are positive in whole domain Ω. We call *coexistence state* this third type of solution.

Given $a, b \in C^\nu(\overline{\Omega})$, $\nu \in (0,1)$, with $a \geq const > 0$ we denote by $\lambda_1(a; b)$ the principal eigenvalue of the problem

$$\begin{cases} -\Delta[a(x)u] + b(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

This eigenvalue is simple and any positive eigenfunction ϕ associated to it, belongs to $C_0^{2,\nu}(\overline{\Omega})$. Moreover, $\lambda_1(a; b)$ is increasing in b. When $a \equiv 1$ we write $\lambda_1(b)$ instead of $\lambda_1(1; b)$ and $\lambda_1 := \lambda_1(1; 0)$. Finally, given a function $a \in C(\overline{\Omega})$ we denote

$$a_M := \max_{x \in \overline{\Omega}} a(x).$$

The change of variable $a(x)u = z$ transforms (2.1) into

$$\begin{cases} -\Delta z + \dfrac{b(x)}{a(x)} z = \lambda \dfrac{1}{a(x)} z & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

The equality

$$-\Delta z + \left(\frac{b(x) - \lambda}{a(x)} \right) z = 0$$

implies that $\lambda_1(a; b)$ is the unique root of the map

$$\lambda \mapsto \lambda_1 \left(\frac{b(x) - \lambda}{a(x)} \right),$$

that is, $\lambda_1(a; b)$ is the unique real number such that

$$\lambda_1 \left(\frac{b(x) - \lambda_1(a; b)}{a(x)} \right) = 0. \tag{2.3}$$

We will also need to know the properties of the problem

$$\begin{cases} -\Delta w = \gamma w - w^2 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

We remember that there exists a positive solution of (2.4) if, and only if,

$$\gamma > \lambda_1.$$

Moreover, the positive solution is unique, and denoted by θ_γ. We extend $\theta_\gamma \equiv 0$ as $\gamma \leq \lambda_1$. Furthermore, $\theta_\gamma / \gamma \to 1$ uniformly over compacts of Ω as $\gamma \to +\infty$.

3 Non-existence of coexistence states and a priori bounds

We begin by proving results of non-existence of coexistence states.

Proof of Theorem 1.1 items (1) and (2). Let φ_1 be a positive eigenfunction associated to λ_1. If we multiply the first equation of (1.1) by $k\varphi_1$ and the second one by φ_1, integrate and add both equations, we obtain

$$\int_\Omega \varphi_1 \left[(\lambda_1 - \lambda)ku + (\lambda_1 - \mu)v\right] = \int_\Omega \varphi_1 u^2 \left[-\left(\frac{v}{u}\right)^2 + (kb + c - \beta\lambda_1)\frac{v}{u} - k\right]. \tag{3.1}$$

Denote $f(r) = -r^2 + (kb + c - \beta\lambda_1)r - k$. Then $f(0) = -k,\ f'(0) = kb + c - \beta\lambda_1$, and the maximum is reached in the point $r_M = \frac{1}{2}(kb + c - \beta\lambda_1)$. Hence

(1) If $\beta\lambda_1 \geq c$, then for $k = 0,\ f(r) < 0\ \forall r > 0$. The first member of (3.1) is negative and so $\lambda_1 - \mu < 0$.

(2) We look for $k > 0$ such that $f(r) < 0,\ \forall r > 0$. Since $f(0) < 0$, it is enough to find $k > 0$ such that $(kb + c - \beta\lambda_1)^2 - 4k < 0$. It is easy to see that this is reached if $b(c - \beta\lambda_1) < 1$. So, with this condition, there exists $k_0 > 0$ such that

$$(\lambda_1 - \lambda)k_0 \int_\Omega \varphi_1 u + (\lambda_1 - \mu)\int_\Omega \varphi_1 v < 0,$$

and if $\lambda, \mu \leq \lambda_1$ there does not exist any coexistence state of (1.1). □

In the next sections we need estimates for the coexistence states, which also give other regions of non-existence of coexistence states of (1.1). We perform the change of variable

$$w := (1 + \beta u)v,$$

which transforms system (1.1) into

$$\begin{cases} -\Delta u = u\left(\lambda - u + \dfrac{bw}{1 + \beta u}\right) & \text{in } \Omega, \\ -\Delta w = \dfrac{w}{1 + \beta u}\left(\mu - \dfrac{w}{1 + \beta u} + cu\right) & \text{in } \Omega, \\ u = w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

For each $b \in L^\infty(\Omega)$ we denote by $\xi_{[b]}$ the unique solution of

$$\begin{cases} -\Delta \xi = b(x) & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega. \end{cases}$$

It is clear that the map $b \mapsto \xi_{[b]}$ is increasing and that for any positive constant $R > 0$ there holds $\xi_{[b]} = R\xi_{[b/R]}$.

The following result provides us a priori bounds of coexistence states of (3.2) for every $b > 0$ and $c > 0$.

Proposition 3.1 *Let (u, w) be a coexistence state of (3.2). Then*

(1) $\theta_\lambda \leq u \leq u_M \leq \lambda + \dfrac{bw_M}{1 + \beta u_M}$.

(2) $w_M \leq (1 + \beta u_M)(\mu + cu_M)$.

(3) $w(x) \leq \xi_{[(\mu + cu)^2/4]}(x)$ for all $x \in \overline{\Omega}$.

Proof. The inequality $\theta_\lambda \leq u$ follows because u is a supersolution of (2.4) with $\gamma = \lambda$ and the fact that (2.4) has a unique non-negative solution.

Let (u, w) be a coexistence state of (3.2) and x_u, $x_w \in \Omega$ the points such that

$$u(x_u) = \max_{x\in\overline{\Omega}} u(x) := u_M, \qquad w(x_w) = \max_{x\in\overline{\Omega}} w(x) := w_M.$$

Then, $-\Delta u(x_u) \geq 0$ and $-\Delta w(x_w) \geq 0$ and it is easy to obtain that

$$\begin{cases} u_M \leq \lambda + \dfrac{bw_M}{1+\beta u_M}, \\ w_M \leq (\mu + cu_M)\,(1+\beta u_M). \end{cases} \tag{3.3}$$

From these considerations (1) and (2) follow. To show (3), observe that

$$w\left(\frac{\mu+cu}{1+\beta u} - \frac{w}{(1+\beta u)^2}\right) \leq \frac{(\mu+cu)^2}{4}. \tag{3.4}$$

Since w is a solution of

$$-\Delta w = w\left(\frac{\mu+cu}{1+\beta u} - \frac{w}{(1+\beta u)^2}\right),$$

it follows that $\xi_{[(\mu+cu)^2/4]}$ is a supersolution of the above equation in w, whence the result follows. $\square$

The values (u_M, w_M) described by (3.3) verify

$$(1-bc)u_M \leq \lambda + b\mu, \qquad (1-bc)w_M \leq (\mu + c\lambda)\,(1+\beta u_M).$$

Therefore, because $u_M > 0$ and $w_M > 0$, we can establish the following region of non-existence and a priori bounds for coexistence states in the case $bc < 1$:

Proposition 3.2 *(1) If there exists a coexistence state of (3.2) and $bc < 1$, then*

$$\lambda + b\mu > 0 \qquad \text{and} \qquad \mu + c\lambda > 0.$$

(2) If $bc < 1$ and (u, w) is a coexistence state of (3.2), then the following estimates are true in $\overline{\Omega}$:

$$\theta_\lambda \leq u \leq \frac{\mu b + \lambda}{1-bc}, \qquad w \leq \frac{(\mu+c\lambda)[1-bc+\beta(\mu b+\lambda)]}{(1-bc)^2}. \tag{3.5}$$

Remark 3.3 *If $bc < 1$ and (u_β, v_β) is a family of coexistence states of (1.1). Then, $(u_\beta, v_\beta) \to (u_0, v_0)$ uniformly in $\overline{\Omega}$ as $\beta \to 0$, where (u_0, v_0) is a solution of*

$$\begin{cases} -\Delta u = u(\lambda - u + bv) & \text{in } \Omega, \\ -\Delta v = v(\mu - v + cu) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Indeed, by Proposition 3.2 item (2), u_β and w_β are bounded in $L^\infty(\Omega)$ independently of β, and by the elliptic regularity, are bounded in $W^{2,p}(\Omega)$, for all $p > 1$. We deduce that $(u_\beta, w_\beta) \to (\bar{u}, \bar{w})$ in $C^{2,\gamma}(\overline{\Omega})$, for $(\bar{u}, \bar{w})$ a non-negative and non-trivial solution of (3.6).

4 A general bifurcation result

First we need to study the functions F and G defined in (1.2) and (1.4).

Observe there exists $z > 0$ in Ω such that $G(\beta, \lambda)$ verifies

$$-\Delta[(1+\beta\theta_\lambda)z] - c\theta_\lambda z = G(\beta,\lambda)z \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \partial\Omega.$$

Multiplying by φ_1 and integrating, we obtain

$$(\lambda_1 - G(\beta,\lambda)) \int_\Omega z\varphi_1 = (c - \beta\lambda_1) \int_\Omega \theta_\lambda z \varphi_1$$

and so,

$$\beta\lambda_1 > c \Longrightarrow G(\beta,\lambda) > \lambda_1,\ \beta\lambda_1 = c \Longrightarrow G(\beta,\lambda) \equiv \lambda_1,\ \beta\lambda_1 < c \Longrightarrow G(\beta,\lambda) < \lambda_1. \quad (4.1)$$

In the following result, we prove the main properties of F and G, see Figure 1.

Lemma 4.1 *(1) F is a decreasing map and $\lim_{\mu\to+\infty} F(\mu) = -\infty$.*
(2) Fix $\beta \geq 0$. Then,
(a) If $\beta\lambda_1 > c$, then G is increasing in λ and $\lim_{\lambda\to+\infty} G(\beta,\lambda) = +\infty$.
(b) If $\beta\lambda_1 = c$, then $G(\beta,\lambda) = \lambda_1$.
(c) If $\beta\lambda_1 < c$, then G is decreasing in λ and $\lim_{\lambda\to+\infty} G(\beta,\lambda) = -\infty$.
(3) Fix $\lambda > \lambda_1$. Then G is increasing in β and $\lim_{\beta\to+\infty} G(\beta,\lambda) = +\infty$.

Proof. The properties (1) of F follow from [2]. With respect to item (2), see the Appendix of [9] and [16]. To prove (3), we fix $\lambda > \lambda_1$. As we mentioned before, see (2.3), $G(\beta,\lambda)$ is the unique solution in μ of

$$0 = \lambda_1\left(-\frac{c\theta_\lambda + \mu}{1+\beta\theta_\lambda}\right).$$

This map is increasing in β, and so $G(\beta,\lambda)$ is also increasing in β. On the other hand, assume that $G(\beta,\lambda) \leq C$ for β large. Then

$$0 = \lambda_1\left(\frac{-c\theta_\lambda - G(\beta,\lambda)}{1+\beta\theta_\lambda}\right) \geq \lambda_1\left(\frac{-c\theta_\lambda - C}{1+\beta\theta_\lambda}\right) \to \lambda_1 > 0,$$

as $\beta \to +\infty$, a contradiction. □

We state now a result showing the stability of the trivial and semi-trivial solutions of (1.1). Its proof is rather similar to Proposition 4.1 in [2], and so we omit it.

Proposition 4.2 *1. The trivial solution of (1.1) is linearly asymptotically stable if $\lambda < \lambda_1$ and $\mu < \lambda_1$ and unstable if $\lambda > \lambda_1$ or $\mu > \lambda_1$.*

2. Assume that $\lambda > \lambda_1$. The semi-trivial solution $(\theta_\lambda, 0)$ is linearly asymptotically stable if $\mu < G(\beta,\lambda)$ and unstable if $\mu > G(\beta,\lambda)$.

3. Assume that $\mu > \lambda_1$. The semi-trivial solution $(0, \theta_\mu)$ is linearly asymptotically stable if $\lambda < F(\mu)$ and unstable if $\lambda > F(\mu)$.

We analyze system (3.2) instead of (1.1). Observe that (3.2) has, similarly to (1.1), the trivial solution $(0,0)$ and the semitrivial solutions $(\theta_\lambda, 0)$ and $(0, \theta_\mu)$. Since we will apply repeatedly the bifurcation method to (3.2), we prove a general result which provides us with existence of coexistence states of (3.2), in fact the existence of a *continuum* $\mathcal{C}$ of positive solutions, that is, a maximal connected and closed set in the set of positive solutions of (3.2). Along the following result we denote $cl(\mathcal{C})$ the closure of $\mathcal{C}$ in the $\mathbb{R} \times C_0^2(\overline{\Omega})$ topology.

Proposition 4.3 *(1) Fix $\mu > \lambda_1$ and consider λ as bifurcation parameter. Then, a continuum $\mathcal{C}$ of coexistence states of (3.2) bifurcates from the semi-trivial solution $(0, \theta_\mu)$ at $\lambda = F(\mu)$. This is the unique point of bifurcation of positive solutions from $(0, \theta_\mu)$. Moreover, $\mathcal{C}$ satisfies some of the following alternative:*

(a) $\mathcal{C}$ is unbounded in $\mathbb{R} \times (C_0^1(\overline{\Omega}))^2$, or

(b) there exists $\lambda_* \in \mathbb{R}$ such that $(\lambda_*, \theta_{\lambda_*}, 0) \in cl(\mathcal{C})$.

(2) Fix $\lambda > \lambda_1$ and consider μ as bifurcation parameter. Then, an unbounded continuum $\mathcal{C}$ of coexistence states of (3.2) bifurcates from the semi-trivial solution $(\theta_\lambda, 0)$ at $\mu = G(\beta, \lambda)$. This is the unique point of bifurcation of positive solutions from $(\theta_\lambda, 0)$.

(3) Fix $\lambda < \lambda_1$ and consider μ as bifurcation parameter. Then, an unbounded continuum $\mathcal{C}$ of coexistence states of (3.2) bifurcates from the semi-trivial solution $(0, \theta_\mu)$ at $\mu = \mu_\lambda > \lambda_1$, the unique value such that $\lambda = F(\mu_\lambda)$. This is the unique point of bifurcation of positive solutions from $(0, \theta_\mu)$.

Proof. (1) Fix $\mu > \lambda_1$ and consider λ as a bifurcation parameter. We apply the Crandall-Rabinowitz theorem [1] (see also Section 2 of [9] and [20]) to conclude that $\lambda = F(\mu)$ is a simple bifurcation point from the semi-trivial solution $(0, \theta_\mu)$, in fact it is the unique bifurcation point of positive solutions of (3.2) from $(0, \theta_\mu)$. Moreover, from Theorem 4.1 in [12] there exists a continuum $\mathcal{C}$ of coexistence states of (3.2) emanating from $(0, \theta_\mu)$ at $\lambda = F(\mu)$ which verifies at least one of the following alternatives:

(A1) $\mathcal{C}$ is unbounded in $\mathbb{R} \times (C_0^1(\overline{\Omega}))^2$, or

(A2) there exists $\lambda_0 \in \mathbb{R}$, $\lambda_0 \neq F(\mu)$, such that $(\lambda_0, 0, \theta_\mu) \in cl(\mathcal{C})$, or

(A3) there exists $\lambda_* \in \mathbb{R}$ such that $(\lambda_*, \theta_{\lambda_*}, 0) \in cl(\mathcal{C})$, or

(A4) there exists $\lambda_2 \in \mathbb{R}$ such that $(\lambda_2, 0, 0) \in cl(\mathcal{C})$.

Since $\lambda = F(\mu)$ is the unique point of bifurcation from $(0, \theta_\mu)$, alternative (A2) is not possible. Assume (A4) and consider a sequence $(\lambda_n, u_n, w_n) \in \mathcal{C}$ such that $\lambda_n \to \lambda_2$ and $(u_n, w_n) \to (0,0)$ in $(L^\infty(\Omega))^2$. Then, denoting by

$$W_n := \frac{w_n}{\|w_n\|_\infty},$$

it is easy to show that $W_n \to W$ in $C^2(\overline{\Omega})$ for some $W \geq 0$ and non-trivial with

$$-\Delta W = \mu W \quad \text{in } \Omega, \quad W = 0 \quad \text{on } \partial\Omega,$$

a contradiction because $\mu > \lambda_1$. So, only (A1) or (A3) is possible. This shows item (1).

(2) Fix now $\lambda > \lambda_1$. Again, with a similar argument to the one employed to prove (1), it follows that the existence of a continuum $\mathcal{C}$ of coexistence states of (3.2) emanating from $(\theta_\lambda, 0)$ at the value μ such that

$$0 = \lambda_1 \left(-\frac{\mu + c\theta_\lambda}{1 + \beta\theta_\lambda} \right),$$

that is $\mu = G(\beta, \lambda)$.

Moreover, by Theorem 4.1 in [12] the continuum $\mathcal{C}$ verifies at least one of the following alternatives:

(A1) $\mathcal{C}$ is unbounded in $\mathbb{R} \times (C_0^1(\overline{\Omega}))^2$, or

(A2) there exists $\mu_0 \in \mathbb{R}$, $\mu_0 > \lambda_1$, such that $(\mu_0, 0, \theta_{\mu_0}) \in cl(\mathcal{C})$, or

(A3) there exists $\mu_1 \in \mathbb{R}$, $\mu_1 \neq G(\beta, \lambda)$, such that $(\mu_1, \theta_\lambda, 0) \in cl(\mathcal{C})$, or

(A4) there exists $\mu_2 \in \mathbb{R}$ such that $(\mu_2, 0, 0) \in cl(\mathcal{C})$.

Again, it is clear that (A3) and (A4) are not possible. Now, assume (A2) and so the existence of a sequence $(\mu_n, u_n, w_n) \in \mathcal{C}$ such that $\mu_n \to \mu_0$ and $(u_n, w_n) \to (0, \theta_{\mu_0})$ in $(L^\infty(\Omega))^2$. Then, denoting by

$$U_n := \frac{u_n}{\|u_n\|_\infty}$$

it is clear that $U_n \to U$ in $C^2(\overline{\Omega})$ with $U \geq 0$ and non-trivial and

$$-\Delta U = U(\lambda + b\theta_{\mu_0}) \quad \text{in } \Omega, \quad U = 0 \quad \text{on } \partial\Omega,$$

and so $\lambda = F(\mu_0) < \lambda_1$, a contradiction.

(3) Now fix $\lambda < \lambda_1$. By Lemma 4.1 it is clear that there exists a unique value $\mu_\lambda > \lambda_1$ such that $\lambda = F(\mu_\lambda)$.

Again, $\mu = \mu_\lambda$ is the unique point of bifurcation from $(0, \theta_\mu)$ and a continuum $\mathcal{C}$ of coexistence states of (1.1) emanates at $\mu = \mu_\lambda$ from $(0, \theta_\mu)$. This continuum $\mathcal{C}$ verifies at least one of the following alternatives:

(A1) $\mathcal{C}$ is unbounded in $\mathbb{R} \times (C_0^1(\overline{\Omega}))^2$, or

(A2) there exists $\mu_0 \neq \mu_\lambda$ such that $(\mu_0, 0, \theta_{\mu_0}) \in cl(\mathcal{C})$, or

(A3) there exists $\mu_1 \in \mathbb{R}$, such that $(\mu_1, 0, 0) \in cl(\mathcal{C})$.

Now, it is not hard to show that only (A1) is possible. □

Observe that only in case (2) of the above result, there exists the possibility of the existence of a continuum of coexistence states connecting the two semi-trivial solutions. We will show in the next section, that in fact this occurs.

5 Weak cooperation

This section is devoted to study the weak cooperation interaction, that is $bc < 1$. The results of Theorem 1.1 items (1) and (2) as well as Proposition 3.2 are valid for every $b > 0$ and $c > 0$, this allows us to conclude that there does not exist any coexistence state of (1.1) if one of the following properties hold: (i) $\lambda, \mu \leq \lambda_1$; (ii) $\lambda + b\mu \leq 0$; (iii) $\mu + c\lambda \leq 0$. If $\beta\lambda_1 \geq c$, then there does not exist any coexistence state if $\mu \leq \lambda_1$. The coexistence states have a priori bounds given by (3.5).

Next we prove the existence of coexistence states corresponding to item (3) of Theorem 1.1.

Proof of Theorem 1.1 item (3). We apply Proposition 4.3 item (1), taking into account the a priori bounds for the solutions. In fact, fix $\mu > \lambda_1$ and consider λ as bifurcation parameter. Then, from the semi-trivial solution $(0, \theta_\mu)$ bifurcates a continuum $\mathcal{C}$ of coexistence states of (3.2) at $\lambda = F(\mu)$. The continuum $\mathcal{C}$ verifies some of the following alternatives: $\mathcal{C}$ is unbounded or there exists λ_* such that $(\lambda_*, \theta_{\lambda_*}, 0) \in cl(\mathcal{C})$.

Assume that the second alternative occurs. This means that there exist $(\lambda_n, u_n, w_n) \in \mathcal{C}$ such that

$$\lambda_n \to \lambda_* \text{ in } \mathbb{R}, \qquad u_n \to \theta_{\lambda_*} \text{ in } C(\overline{\Omega}), \qquad w_n \to 0 \text{ in } C(\overline{\Omega}).$$

If we denote

$$W_n = \frac{w_n}{\|w_n\|_\infty},$$

it is not hard to check that $W_n \to W$ in $C^2(\overline{\Omega})$, with W the non-negative and non-trivial solution of

$$\begin{cases} -\Delta W = \dfrac{\mu + c\theta_{\lambda_*}}{1 + \beta\theta_{\lambda_*}} W & \text{in } \Omega, \\ W = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence

$$\lambda_1\left(-\frac{\mu + c\theta_{\lambda_*}}{1 + \beta\theta_{\lambda_*}}\right) = 0 \Longrightarrow \mu = \lambda_1(1 + \beta\theta_{\lambda_*}; -c\theta_{\lambda_*}) = G(\beta, \lambda_*). \tag{5.1}$$

Therefore, if $\beta\lambda_1 \leq c$, then $G(\beta, \lambda_*) \leq \lambda_1$. And (4.1) leads to a contradiction. Thus $\mathcal{C}$ is unbounded in $\mathbb{R} \times (C_0^1(\overline{\Omega}))^2$ and thanks to the a priori bounds of the solutions, the existence of coexistence states follows for all $\lambda > F(\mu)$ (see Figure 2 Case d)).

However, if $\beta\lambda_1 > c$, we can prove that system (3.2) has no nontrivial solution if λ is big enough. Indeed,

$$-\Delta w = w\left(\frac{\mu}{1 + \beta u} - \frac{w}{(1 + \beta u)^2} + \frac{cu}{1 + \beta u}\right) \leq w\left(\frac{\mu}{1 + \beta\theta_\lambda} + \frac{c}{\beta}\right).$$

This implies that

$$\lambda_1\left(-\frac{\mu}{1 + \beta\theta_\lambda} - \frac{c}{\beta}\right) \leq 0.$$

But

$$\lambda_1\left(-\frac{\mu}{1 + \beta\theta_\lambda} - \frac{c}{\beta}\right) \to \lambda_1 - \frac{c}{\beta} > 0, \quad \text{as } \lambda \to \infty.$$

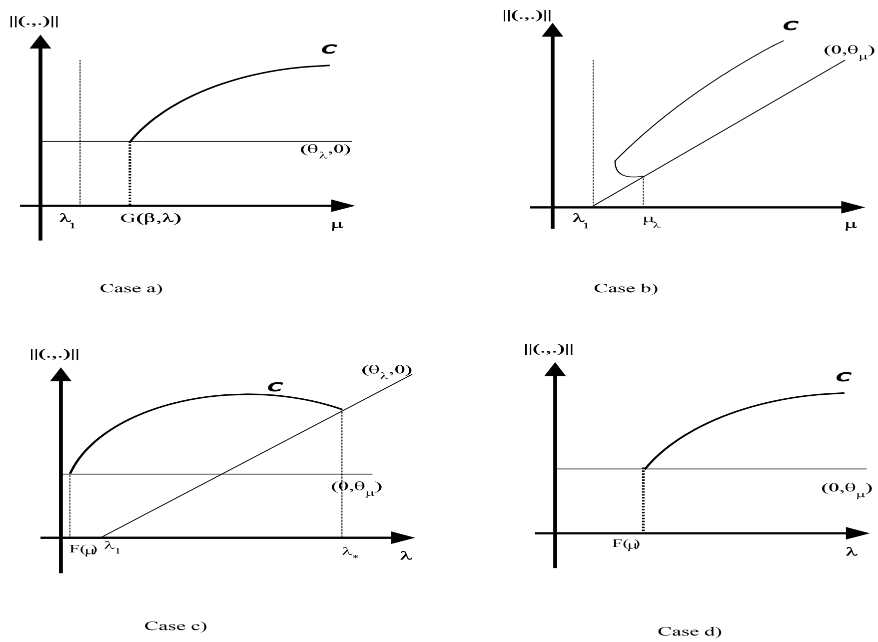

Figure 3: Bifurcation diagrams

In this case, again the a priori bounds, says that $\mathcal{C}$ cannot be unbounded, so there exists $\lambda_* \in \mathbb{R}$ such that $(\lambda_*, \theta_{\lambda_*}, 0) \in cl(\mathcal{C})$ (see Figure 2 Case c)). Moreover, by (5.1), λ_* is the unique value such that

$$\mu = \lambda_1(1 + \beta\theta_{\lambda_*}; -c\theta_{\lambda_*}) = G(\beta, \lambda_*),$$

which exists and it is unique by Lemma 4.1. Hence, we have coexistence states for $\lambda > F(\mu)$ and $\mu > G(\beta, \lambda)$ (see Figure 1).

This completes the study in the case $\mu > \lambda_1$. For $\mu \leq \lambda_1$ we fix $\lambda > \lambda_1$, and consider μ as bifurcation parameter. In this case, again by Proposition 4.3 the continuum $\mathcal{C}$ emanating from $(\theta_\lambda, 0)$ at $\mu = G(\beta, \lambda)$ is unbounded and by the a priori bounds and the non-existence of coexistence states for $\mu \leq \lambda_1$, which implies that there exists a coexistence state for all $\mu > G(\beta, \lambda)$ (see Figure 2 Case a)). $\square$

6 Large cross-diffusion effect

In this Section we show the existence of at least one coexistence state for the case $b > 0$, $c > 0$ and β large, thus proving Theorem 1.1 item (4). But first we need to show a priori bounds of the solutions.

Proposition 6.1 *Assume that for some $\alpha > 0$*

$$\max\{\, |\lambda|\,,\, |\mu|\,\} \leq \alpha\,.$$

Then, there exists $\beta_0 > 0$ such that for all $\beta \geq \beta_0$ a constant $C = C(\alpha, \Omega, b, c, \beta)$ exists such that

$$\|u\|_{L^\infty(\Omega)} \leq C\,, \qquad \|v\|_{L^\infty(\Omega)} \leq C\,,$$

for any coexistence state (u, v) of (1.1).

Proof. We are going to use a Gidas-Spruck argument [3]. Assume that there exist a sequence $(\beta_n, \lambda_n, \mu_n)$ with $|\lambda_n| \leq \alpha$, $|\mu_n| \leq \alpha$, $\beta_n \to \infty$ and a sequence of coexistence states (u_n, w_n) of (3.2) such that $\|u_n\|_\infty + \|w_n\|_\infty \to \infty$. Thanks to Proposition 3.1 we have that both $\|u_n\|_\infty \to \infty$ and $\|w_n\|_\infty \to \infty$. Indeed, it is clear that if $\|u_n\|_\infty \to \infty$ then $\|w_n\|_\infty \to \infty$ by Proposition 3.1 item (1). Now, suppose that $\|w_n\|_\infty \to \infty$ and $\|u_n\|_\infty \leq C$. Then, by Proposition 3.1

$$w_n \leq \xi_{[(\mu+cu_n)^2/4]} \leq \xi_{[(\mu+cC)^2/4]} \leq C$$

and so w_n is bounded, a contradiction.

Denote by

$$M_n := \|u_n\|_\infty = u(x_n) = \max_{x\in\Omega} u_n(x)$$

for some $x_n \in \Omega$, and so $M_n \to \infty$. By the compactness of $\overline{\Omega}$ we can assume that $x_n \to x_0 \in \overline{\Omega}$.

We distinguish two cases:

<u>Case 1: $x_0 \in \Omega$.</u> Define

$$\delta := \frac{dist(x_0, \partial\Omega)}{2} > 0.$$

We make now the following change of variable

$$U_n(y) := \frac{u_n(yM_n^{-1/2} + x_n)}{M_n}, \quad W_n(y) := \frac{w_n(yM_n^{-1/2} + x_n)}{M_n^2} \quad \text{in } \Omega_n,$$

where $\Omega_n := \{y \in \mathbb{R}^N : yM_n^{-1/2} + x_n \in \Omega\}$.

Observe that if $|y| < \delta M_n^{1/2}$ then $yM_n^{-1/2} + x_n \in \Omega$. So, given $R > 0$ there exists $n \in \mathbb{N}$ large enough such that $B(0, R) \subset B(0, \delta M_n^{1/2})$, where $B(0, S)$ stands for the ball of radius $S > 0$ centered at the origin.

Observe that

$$\|U_n\|_{L^\infty(B(0,R))} = 1 \quad \text{and} \quad U_n(0) = 1.$$

On the other hand, from Proposition 3.1 we get that

$$w_n \leq \xi_{[(\mu_n+cu_n)^2/4]} \leq \xi_{[(\alpha+cM_n)^2/4]},$$

and so,

$$\frac{w_n}{M_n^2} \leq \xi_{[(\alpha/M_n+c)^2/4]} \leq C$$

for some $C > 0$ and n large. Hence,

$$\|W_n\|_{L^\infty(B(0,R))} \leq C.$$

It is not hard to show that (U_n, W_n) satisfies in $B(0,R)$ the following system

$$\begin{cases} -\Delta U_n = F(U_n, W_n) := \lambda_n M_n^{-1} U_n - U_n^2 + \dfrac{b}{\beta_n} \dfrac{U_n W_n}{\frac{1}{\beta_n M_n} + U_n}, \\ -\Delta W_n = \mu_n M_n^{-2} \dfrac{W_n}{1+\beta_n M_n U_n} - M_n \dfrac{W_n^2}{(1+\beta_n M_n U_n)^2} + c \dfrac{U_n W_n}{1+\beta_n M_n U_n}. \end{cases} \tag{6.1}$$

Since U_n and W_n are bounded in $L^\infty(B(0,R))$, then

$$\|F(U_n, W_n)\|_{L^\infty(B(0,R))} \le C,$$

and so U_n is bounded in $C^{1,\nu}(\overline{B(0,R)})$ for some $0 < \nu < 1$, which provides bounds in $C^{2,\nu}(\overline{B(0,R)})$. Observe also that

$$\frac{b}{\beta_n} \frac{U_n W_n}{\frac{1}{\beta_n M_n} + U_n} = \frac{b}{\beta_n} W_n \Big(\frac{U_n}{\frac{1}{\beta_n M_n} + U_n} \Big) \le \frac{b}{\beta_n} W_n \to 0 \quad \text{as } n \to \infty.$$

We can pass to the limit in the first equation, or to a subsequence if necessary, and conclude that $U_n \to U$ in $C^{2,\nu}(\overline{B(0,R)})$ where $U \ge 0$, $U(0) = 1$ is solution of

$$-\Delta U = -U^2$$

in $B(0,R)$, for any $R > 0$. An standard argument shows that $U_n \to U$ in $C^2_{loc}(\mathbb{R}^N)$, then U is solution

$$-\Delta U = -U^2 \quad \text{in } \mathbb{R}^N, \tag{6.2}$$

with $0 \le U \le 1$, $U(0) = 1$. This implies that $U \equiv 0$, a contradiction.
Case 2: $x_0 \in \partial\Omega$. Observe that in this case, $\Omega_n \to \mathbb{R}^N_+$. After a linear change of variable (which straightens the boundary of Ω near x_0, see Theorem 1.1 in [3] and Step 2 in Lemma 4.3 in [13]) we arrive at the equation

$$\begin{cases} -\Delta U = -U^2 & \text{in } \mathbb{R}^N_+, \\ U = 0 & \text{in } \partial\mathbb{R}^N_+, \end{cases} \tag{6.3}$$

for some regular and bounded non-negative function with $U(0) = 1$, again a contradiction. □

We are ready to prove the main result of this section.

Proof of Theorem 1.1 item (4). We use Proposition 4.3. First, we fix $\lambda > \lambda_1$. Then, from the semi-trivial solution $(\theta_\lambda, 0)$ a unbounded continuum $\mathcal{C}$ of positive solutions of (1.1) bifurcates at

$$\mu = G(\beta, \lambda).$$

Thanks to Theorem 1.1 item (1), if $\beta \ge c/\lambda_1$ system (3.2) does not have coexistence states for $\mu \le \lambda_1$, and for Proposition 6.1 we have a priori bounds if $\beta \ge \beta_0$ for μ belonging to a bounded set. Hence, we have the existence of coexistence state for all $\mu > G(\beta, \lambda)$ (see again Figure 2 Case a)).

Now, assume $\lambda < \lambda_1$. Again, by Proposition 4.3 for $\mu = \mu_\lambda$, where μ_λ is the unique value such that

$$\lambda = \lambda_1(-b\theta_{\mu_\lambda}) = F(\mu)$$

an unbounded continuum $\mathcal{C}$ of positive solutions of (1.1) bifurcates. A similar argument to the used above shows that there exists a coexistence state for, at least, $\mu > \mu_\lambda$ (see Figure 2 Case b)).

Finally, assume that $\lambda = \lambda_1$ and $\mu > \lambda_1$. Take a sequence $\lambda_n > \lambda_1$, $\lambda_n \to \lambda_1$ and a sequence (u_n, w_n) of coexistence states of (3.2) with $\lambda = \lambda_n$, which exists thanks to the first part of the theorem. Then, thanks to the a priori bounds of (u_n, w_n) we can pass to the limit and conclude that $(u_n, w_n) \to (u, w)$ in $(C^2(\overline{\Omega}))^2$, with (u, w) a non-negative solution of (3.2). We show now that in fact (u, w) is a coexistence state of (3.2). We argue by contradiction. First, suppose that $(u, w) = (0, 0)$. Then denoting by

$$W_n = \frac{w_n}{\|w_n\|_\infty},$$

it is easy to prove that $W_n \to W$ in $C^2(\overline{\Omega})$ with $W \geq 0$, non-trivial and satisfies

$$-\Delta W = \mu W \quad \text{in } \Omega, \quad W = 0 \quad \text{on } \partial\Omega,$$

and so $\mu = \lambda_1$, a contradiction.

Now, suppose that $(u, w) = (u, 0)$ for some non-negative function u. In this case, it is easy to prove that $u = \theta_{\lambda_1} \equiv 0$ and so by the above reasoning we arrive at a contradiction.

Finally, assume that $(u, w) = (0, w)$ for some non-negative function w. In this case, $w = \theta_\mu > 0$ and denoting

$$U_n := \frac{u_n}{\|u_n\|_\infty}$$

we have that $U_n \to U$ in $C^2(\overline{\Omega})$, $U \geq 0$ and non-trivial and

$$-\Delta U = U(\lambda_1 + b\theta_\mu) \quad \text{in } \Omega, \quad U = 0 \quad \text{on } \partial\Omega,$$

and so $\lambda_1 = F(\mu)$, a contradiction. This completes the proof. $\square$

7 Profiles of the solutions when $\beta \to \infty$

In this section we analyze the behavior of the solutions when the parameter β goes to infinity and prove Theorem 1.2.

We begin by proving that (3.2) does not possesses coexistence state for β large and $\lambda > \lambda_1$.

Proof of Theorem 1.2 item (1). Assume that $\lambda > \lambda_1$ and consider (u_β, w_β) a coexistence state of (3.2). Observe that by Proposition 3.1 we get $\theta_\lambda \leq u_\beta$ and then $\beta\theta_\lambda \leq \beta u_\beta$. Thus,

$$\frac{\mu}{1 + \beta u_\beta} \to 0 \quad \text{uniformly on compacts of } \overline{\Omega} \text{ and} \quad \frac{u_\beta}{1 + \beta u_\beta} \leq \frac{1}{\beta} \to 0 \quad \text{in } L^\infty(\Omega).$$

Moreover, since

$$-\Delta w_\beta = w_\beta \left(\frac{\mu}{1 + \beta u_\beta} - \frac{w_\beta}{(1 + \beta u_\beta)^2} + b\frac{u_\beta}{1 + \beta u_\beta} \right) \quad \text{in } \Omega, \quad w_\beta = 0 \quad \text{on } \partial\Omega,$$

if $w_\beta > 0$ we conclude that

$$0 = \lambda_1\left(-\frac{\mu}{1+\beta u_\beta} + \frac{w_\beta}{(1+\beta u_\beta)^2} - b\frac{u_\beta}{1+\beta u_\beta}\right) > \lambda_1\left(-\frac{\mu}{1+\beta u_\beta} - b\frac{u_\beta}{1+\beta u_\beta}\right) \to \lambda_1 > 0$$

as $\beta \to \infty$, and then $w_\beta \equiv 0$ and consequently $u_\beta \equiv \theta_\lambda$. $\square$

Proof of Theorem 1.2 item (2). Denote by $z_\beta = \beta u_\beta$ and $w_\beta = (1+z_\beta)v_\beta$. Then, (z_β, w_β) verifies the system

$$\begin{cases} -\Delta z_\beta = z_\beta(\lambda - \dfrac{1}{\beta}z_\beta + b\dfrac{w_\beta}{1+z_\beta}) & \text{in } \Omega, \\ -\Delta w_\beta = \dfrac{w_\beta}{1+z_\beta}(\mu - \dfrac{w_\beta}{1+z_\beta} + \dfrac{c}{\beta}z_\beta) & \text{in } \Omega, \\ z_\beta = w_\beta = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

Observe that

$$\frac{c}{\beta}\frac{z_\beta}{1+z_\beta} \to 0 \quad \text{uniformly in } \overline{\Omega} \text{ as } \beta \to \infty. \tag{7.2}$$

Take $\varepsilon > 0$ such that $\lambda_1(-\varepsilon) = \lambda_1 - \varepsilon > 0$. For such $\varepsilon > 0$, using (7.2) there exists β_0 such that for $\beta \geq \beta_0$ we get

$$(-\Delta - \varepsilon)w_\beta \leq \frac{w_\beta}{1+z_\beta}(\mu - \frac{w_\beta}{1+z_\beta}) \quad \text{in } \Omega, \quad w_\beta = 0 \quad \text{on } \partial\Omega,$$

and so with a similar argument to the one used in Proposition 3.1 we get that

$$w_\beta \leq \hat{\xi}_{[\mu^2/4]},$$

where now $\hat{\xi}_{[b]}$ is the unique solution of

$$(-\Delta - \varepsilon)\hat{\xi} = b(x) \quad \text{in } \Omega, \quad \hat{\xi} = 0 \quad \text{on } \partial\Omega,$$

for $b \in L^\infty(\Omega)$. Hence, w_β is bounded in $L^\infty(\Omega)$.

Now, assume that $\|z_\beta\|_\infty \to \infty$ as $\beta \to \infty$. Then

$$\left\|\frac{w_\beta}{1+z_\beta}\right\|_\infty \to 0.$$

Hence

$$0 = \lambda_1\left(-\lambda + \frac{1}{\beta}z_\beta - b\frac{w_\beta}{1+z_\beta}\right) > \lambda_1\left(-\lambda - b\frac{w_\beta}{1+z_\beta}\right) \to \lambda_1 - \lambda > 0,$$

and then $z_\beta \equiv 0$, a contradiction.

We have that $\|z_\beta\|_\infty \leq C$. By elliptic regularity, z_β and w_β are bounded in $W^{2,q}(\Omega)$ for any $q > 1$. We can pass to the limit in (7.1) and conclude that $(z_\beta, w_\beta) \to (z, p)$ in $(C^2(\overline{\Omega})^2)$ with

$$\begin{cases} -\Delta z = z(\lambda + b\dfrac{p}{1+z}) & \text{in } \Omega, \\ -\Delta p = \dfrac{p}{1+z}(\mu - \dfrac{p}{1+z}) & \text{in } \Omega, \\ z = p = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.3}$$

After the change of variable $p = (1+z)w$, the result follows. $\square$

Proof of Theorem 1.2 item (3). We show first the non-existence result. Assume that (z, w) is a coexistence state of (1.6). Multiplying the first equation by φ_1, a positive eigenfunction associated to λ_1, and integrating we get

$$(\lambda_1 - \lambda) \int_\Omega z\varphi_1 = b \int_\Omega zw\varphi_1,$$

and so $\lambda < \lambda_1$.

With the change of variable $(1+z)w = p$, (1.6) is equivalent to (7.3). Observe that

$$p \le \xi_{[\mu^2/4]} \quad \text{in } \Omega. \tag{7.4}$$

Now, we show that there does not exist a coexistence state for λ very negative. Indeed, assume the contrary and denote by $x_\lambda \in \Omega$ such that $z(x_\lambda) = \max_{x\in\overline{\Omega}} z(x)$. Then, using (7.4) we get

$$0 \le \lambda + b\frac{p(x_\lambda)}{1+z(x_\lambda)} \le \lambda + b\xi_{[\mu^2/4]}(x_\lambda) \le \lambda + C(\mu),$$

and then $\lambda \ge -C(\mu)$.

Again, we apply the bifurcation method (see Figure 3 a)). Observe that (1.6) possesses the trivial solution and the semi-trivial one $(0, \theta_\mu)$. Then, fix $\mu > \lambda_1$ and regard λ as a bifurcation parameter. Again, it can be shown that an unbounded continuum $\mathcal{C}$ of coexistence states of (1.6) emanates from $(0, \theta_\mu)$ at $\lambda = F(\mu)$. Since, there are no coexistence states for $\lambda \le -C(\mu)$ and for $\lambda \ge \lambda_1$, there exists a sequence $\lambda_n \to \overline{\lambda} \le \lambda_1$ such that $\|(z_n, p_n)\|_\infty \to \infty$ as $n \to \infty$. We claim that $\overline{\lambda} = \lambda_1$. We know from (7.4) that p_n is bounded, and so $\|z_n\|_\infty \to \infty$. Then, denoting by

$$Z_n := \frac{z_n}{\|z_n\|_\infty}$$

we have that

$$-\Delta Z_n = Z_n(\lambda_n + b\frac{p_n}{1+z_n})$$

and so, $Z_n \to Z$, $Z \ge 0$ and non-trivial and

$$-\Delta Z = \overline{\lambda} Z \quad \text{in } \Omega, \quad Z = 0 \quad \text{on } \partial\Omega,$$

that is $\overline{\lambda} = \lambda_1$.

$\square$

In Figure 3 b) we have represented the coexistence region defined by (1.7).

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