Study of an elliptic system arising from angiogenesis with chemotaxis and flux at the boundary

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Abstract

The main goal of this paper is to study a stationary problem arising from angiogenesis, including terms of chemotaxis and flux at the boundary of the tumor. We give sufficient conditions on terms of the data of the problems assuring the existence of positive solutions.

Key Words. Angiogenesis, chemotaxis, bifurcation methods, coexistence state.

1 Introduction

In this paper we analyze a stationary-state system arising from a crucial step of the growth process tumor: the angiogenesis. The interested reader is suggested to read the paper [29] about multiple aspects of angiogenesis. We are only interested in the behaviour of two populations: the endothelial cells (CEs) which move and reproduce to generate a new vascular net attracted by the chemical substance generated by the tumor (TAF). We represent them by u and v respectively. They live together in a region $\Omega \subset \mathbb{R}^d$, $d \geq 1$ (generically $d = 3$) that is assumed to be bounded and connected and with a regular boundary $\partial\Omega$. Specifically, we consider the case in which

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, being Γ_i closed and open in the relative topology of $\partial\Omega$. We assume that Γ_2 is the boundary of the tumor, and Γ_3 the boundary of the blood vessel. Finally, Γ_1 is the exterior boundary, such that the tumor and the primary vessel blood are inside of Γ_1, see Figure 1 where we have represented a particular situation. We assume Dirichlet and Neumann homogeneous boundary conditions in both variables at Γ_1 and Γ_3 respectively. However, in the boundary of the tumor we assume that there does not exist flux of CEs, that is, $\partial u/\partial n = 0$, n denotes the outward unit normal on Γ_2, but

$$\frac{\partial v}{\partial n} = \mu v$$

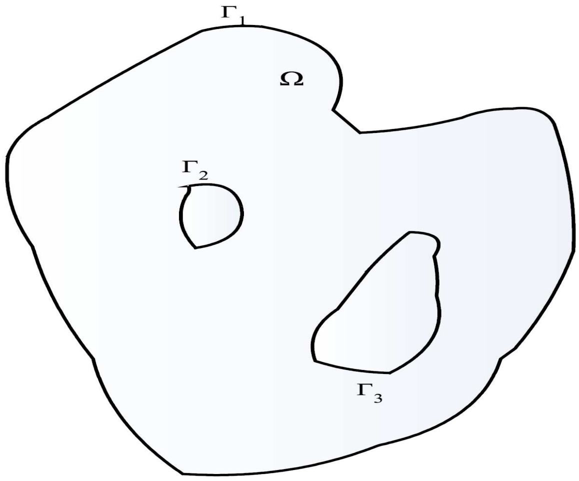

Figure 1: A particular example of domain Ω.

where $\mu \in \mathbb{R}$. So, μ represents the amount of TAF that the tumor is generating, and it will play an important role in the paper. We note that this is a Robin condition with a negative coefficient when $\mu > 0$. In summary, we have the boundary conditions $B_1 u = 0$ and $B_2(\mu)v = 0$ being

$$B_1 u := \begin{cases} u & \text{on } \Gamma_1, \\ \dfrac{\partial u}{\partial n} & \text{on } \Gamma_2 \cup \Gamma_3, \end{cases} \qquad \text{and} \qquad B_2(\mu)v := \begin{cases} v & \text{on } \Gamma_1, \\ \dfrac{\partial v}{\partial n} - \mu v & \text{on } \Gamma_2, \\ \dfrac{\partial v}{\partial n} & \text{on } \Gamma_3. \end{cases}$$

We study the existence of positive solutions of the following system

$$\begin{cases} -\Delta u = -div(\alpha u \nabla v) + \lambda u - u^2 & \text{in } \Omega, \\ -\Delta v = -v^2 - cuv & \text{in } \Omega, \\ B_1 u = B_2(\mu) v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

$\lambda, \mu \in \mathbb{R}$, $\alpha \geq 0$ and $c > 0$. So, we are assuming that u is affected by a chemotaxis term, that is, CEs move toward the higher concentration of TAF and that its growth follows a logistic law. On the other hand, the decay of the TAF, v, is modelled by a term logistic and a proportional term of competition with u, see [8] for a similar model considering a logistic behaviour on v. Observe that for $\alpha = 0$ the CEs are free of the chemotaxis effect, and so they can live independently of the TAF.

Although recently there is a great attention to systems with a chemotaxis term, this study is based mainly in the parabolic problem associated to (1.1), and there are not many papers dedicated to the stationary case including a nonlinear reaction term. We cite for example the papers [15] and [14] and references therein, where existence results are obtained using topological index theory. Although the models considered in the cited papers have several species of cells or bacteria, and so their study is more difficult, we will give easily computable conditions which assure the existence of positive solutions, unlike the obtained in the above papers. In [33] a similar system is studied with a linear equation in v and homogeneous Neumann boundary conditions in u and v. See also [34] for a one dimensional problem.

We can summarize our main results as follows: it is clear that there exists three kinds of solutions of (1.1): the trivial one, the semi-trivial solutions $(u, 0)$ and $(0, v)$ and the solutions with both components positive, the *coexistence states* (u, v). Basically, the trivial solution always exists, and:

1. There exists a value $\lambda_1 > 0$ such that the semi-trivial solution $(u, 0)$ exists if, and only if $\lambda > \lambda_1$.
2. There exists a value $\mu_1 > 0$ such that the semi-trivial solution $(0, v)$ exists if, and only if $\mu > \mu_1$.

Moreover, there exist two curves $\mu = F(\lambda)$ and $\lambda = \Lambda(\alpha, \mu)$ in the (λ, μ)-plane such that: there exists at least a coexistence state of (1.1) if (λ, μ) belongs to the region limited by the two curves, specifically if

$$(\mu - F(\lambda))(\lambda - \Lambda(\alpha, \mu)) > 0.$$

Finally, with respect to the stability of the semi-trivial solutions, we show that

1. $(u, 0)$ is stable if $\mu < F(\lambda)$, and unstable if $\mu > F(\lambda)$.
2. $(0, v)$ is stable if $\lambda < \Lambda(\alpha, \mu)$, and unstable if $\lambda > \Lambda(\alpha, \mu)$.

So, when both semi-trivial solutions are stable or unstable, there exists at least one coexistence state. Hence, these curves are crucial in the study of existence of positive solutions and we will study in detail both maps.

In order to prove these results we use mainly bifurcation methods, sub and supersolution, homogenization techniques and a deep study of different eigenvalue problems.

In section 2 we collect some results related mainly with eigenvalue problems. In section 3 we study the semi-trivial solutions, in Section 4 we study the stability of the semi-trivial solutions. Section 5 is devoted to the case that the chemotaxis is not present, in section 6 we analyze the existence of coexistence state and the curves $\mu = F(\lambda)$ and $\lambda = \Lambda(\alpha, \mu)$. Finally in the last section we briefly discuss some biological implications of our results.

2 Preliminaries and notations

Along the work we are going to use the following notation: for $\gamma \in (0, 1)$ we denote

$$C_0^{2,\gamma}(\overline{\Omega}) := \{u \in C^{2,\gamma}(\overline{\Omega}) : u = 0 \text{ on } \Gamma_1\},$$

$$X_1 := \{u \in C_0^{2,\gamma}(\overline{\Omega}) : \partial u/\partial n = 0 \text{ on } \Gamma_2 \cup \Gamma_3\}, \quad X_2 := \{u \in C_0^{2,\gamma}(\overline{\Omega}) : \partial u/\partial n = 0 \text{ on } \Gamma_3\}$$

and finally

$$X := X_1 \times X_2.$$

Moreover, given a function $c \in C(\overline{\Omega})$ we denote by

$$c_M := \max_{\overline{\Omega}} c(x), \qquad c_L := \min_{\overline{\Omega}} c(x).$$

We are interested in solutions $(u,v) \in X$ of (1.1) with both components non-negative and non-trivial. Observe that thanks to the strong maximum principle, any component, u or v, of a non-negative and non-trivial solution is in fact positive in all the domain Ω and at $\Gamma_2 \cup \Gamma_3$.

Finally, for a solution U_0 of a nonlinear equation, we say that is *linearly asymptotically stable* (l. a. s.) if the first eigenvalue of the linearization around U_0 is positive, and *unstable* if it is negative.

We collect also in this section some eigenvalue problems which will be useful in the work. Consider functions $m \in C^{\gamma}(\overline{\Omega})$, $h \in C^{1,\gamma}(\Gamma_2), g \in C^{1,\gamma}(\Gamma_3)$ and the eigenvalue problem

$$\begin{cases} -\Delta\phi + m\phi = \lambda\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial\phi}{\partial n} + h\phi = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial\phi}{\partial n} + g\phi = 0 & \text{on } \Gamma_3. \end{cases} \tag{2.1}$$

We are interested only in the principal eigenvalue of (2.1), i.e., the eigenvalues which have an associated positive eigenfunction. In the following result we recall its main properties, see [2], [6] and [20].

Lemma 2.1. *Problem* (2.1) *admits a unique principal eigenvalue, which will be denoted by $\lambda_1(-\Delta + m; D, N + h, N + g)$. Moreover, this eigenvalue is simple, and any positive eigenfunction, ϕ, verifies $\phi \in C_0^{2,\gamma}(\overline{\Omega})$. In addition, $\lambda_1(-\Delta + m; D, N + h, N + g)$ is separately increasing in m, h and g; when $h = K$ constant, it verifies*

$$\begin{aligned} &\lim_{K\to-\infty} \lambda_1(-\Delta + m; D, N + K, N + g) = -\infty, \\ &\lim_{K\to+\infty} \lambda_1(-\Delta + m; D, N + K, N + g) = \lambda_1(-\Delta + m; D, D, N + g), \end{aligned} \tag{2.2}$$

where $\lambda_1(-\Delta + m; D, D, N + g)$ stands for the principal eigenvalue of $-\Delta + m$ with homogeneous Dirichlet boundary conditions on $\Gamma_1 \cup \Gamma_2$ and $\partial\phi/\partial n + g\phi = 0$ on Γ_3.

Also, it will appear eigenvalue problems with a potential blowing up on Γ_2. To be more specific, consider $m \in C(\Omega \cup \Gamma_1 \cup \Gamma_3)$ and $m\rfloor_{\Gamma_2} = +\infty$ (in the sense that $\lim_{dist(x,\Gamma_2)\to 0} m(x) = +\infty$) and the following eigenvalue problem

$$\begin{cases} -\Delta\phi + m\phi = \lambda\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma_1 \cup \Gamma_2, \\ \dfrac{\partial\phi}{\partial n} = 0 & \text{on } \Gamma_3. \end{cases} \tag{2.3}$$

This kind of eigenvalues has been studied in [13] (Section 3.2) and [27] (Section 8) with homogeneous Dirichlet boundary conditions, but their results can be easily extrapolated to our case. Let us recall some properties in the following result.

Lemma 2.2. *Take $m \in C(\Omega \cup \Gamma_1 \cup \Gamma_3)$ and $m\rfloor_{\Gamma_2} = +\infty$. There exists the principal eigenvalue $\lambda_1(-\Delta + m; D, D, N)$ which has associated a positive eigenfunction $\phi \in \mathcal{H}$, where*

$$\mathcal{H} := \{\phi \in H^1(\Omega) : \phi = 0 \ \text{on } \Gamma_1 \cup \Gamma_2, \ \text{and } \partial\phi/\partial n = 0 \ \text{on } \Gamma_3\},$$

and satisfies the equation (2.3) in the weak sense, that is,

$$\int_\Omega \nabla\phi\cdot\nabla\psi + \int_\Omega m\phi\psi = \lambda_1(-\Delta+m; D, D, N)\int_\Omega \phi\psi, \qquad \forall\psi\in\mathcal{H}.$$

Moreover, if there exists $\phi_0\in\mathcal{H}$, $\phi_0>0$ in Ω such that

$$-\Delta\phi_0 + m(x)\phi_0 = \mu\phi_0,$$

in the weak sense, then $\mu = \lambda_1(-\Delta+m; D, D, N)$.

Finally, given $a\in C^1(\overline{\Omega})$, $b\in C^\gamma(\overline{\Omega})$ both positive and $d\in C^{1,\gamma}(\Gamma_2)$, denote by $\Gamma(a; b, D, N+d, N)$ the principal eigenvalue of

$$\begin{cases} -div(a(x)\nabla\varphi) = \lambda b(x)\varphi & \text{in } \Omega,\\ \varphi = 0 & \text{on } \Gamma_1\\ \dfrac{\partial\varphi}{\partial n} + d(x)\varphi = 0 & \text{on } \Gamma_2,\\ \dfrac{\partial\varphi}{\partial n} = 0 & \text{on } \Gamma_3, \end{cases} \tag{2.4}$$

that is,

$$\Gamma(a; b, D, N+d, N) = \inf_{\varphi\in\mathcal{S},\varphi\neq 0} \frac{\displaystyle\int_\Omega a(x)|\nabla\varphi|^2 + \int_{\Gamma_2} d(x)\varphi^2}{\displaystyle\int_\Omega b(x)\varphi^2}, \tag{2.5}$$

with $\mathcal{S} := \{\varphi\in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_1\}$. It is clear from (2.5) that $\Gamma(a; b, D, N+d, N)$ is increasing in a and d and decreasing in b.

3 Study of the semi-trivial solutions

In this section we study the semi-trivial solutions of (1.1). First, for $v=0$ the system (1.1) has the form

$$\begin{cases} -\Delta u = \lambda u - u^2 & \text{in } \Omega,\\ B_1 u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

This equation has been analyzed in [5], see also [16] when $\partial\Omega$ has only one component. Their results can be generalized in our case:

Proposition 3.1. *There exists a positive solution of (3.1) if, and only if,*

$$\lambda > \lambda_1(-\Delta; D, N, N) := \lambda_1. \tag{3.2}$$

In the case that the solution exists, it is unique and we denote it by ϑ_λ. Moreover, the following estimate holds

$$\frac{(\lambda-\lambda_1)}{\|\varphi_1\|_\infty}\varphi_1 \le \vartheta_\lambda, \tag{3.3}$$

where φ_1 is a positive eigenfunction associated to λ_1, that is

$$-\Delta\varphi_1 = \lambda_1\varphi_1 \quad \text{in } \Omega, \qquad B_1\varphi_1 = 0 \quad \text{on } \partial\Omega. \tag{3.4}$$

Furthermore, the map $\lambda \in (\lambda_1, +\infty) \mapsto \vartheta_\lambda \in X_1$ *is regular, increasing and*

$$\vartheta_\lambda = n_1\varphi_1(\lambda - \lambda_1) + O((\lambda - \lambda_1)^2), \quad \text{as } \lambda \downarrow \lambda_1, \tag{3.5}$$

where

$$n_1 = \frac{\displaystyle\int_\Omega \varphi_1^2}{\displaystyle\int_\Omega \varphi_1^3}. \tag{3.6}$$

Finally, ϑ_λ *is l. a. s. for* $\lambda > \lambda_1$*, that is*

$$\lambda_1(-\Delta - \lambda + 2\vartheta_\lambda; D, N, N) > 0. \tag{3.7}$$

Proof. The existence, uniqueness and (3.7) follow by Lemma 3.1 and Theorem 3.5 of [16], see also Theorem 1.1 in [5]. Estimate (3.3) follows showing that $((\lambda - \lambda_1)/\|\varphi_1\|_\infty)\varphi_1$ is a subsolution of (3.1). Finally, (3.5) is deduced in a similar way to Lemma 4.3 in [11]. $\square$

When in system (1.1) the function $u \equiv 0$, we have the following equation

$$\begin{cases} -\Delta v = -v^2 & \text{in } \Omega, \\ B_2(\mu)v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

In the following result the eigenvalue $\lambda_1(-\Delta; D, N - \mu, N)$ will play an important rolle. Thanks to Lemma 2.1, the map $\mu \mapsto \lambda_1(-\Delta; D, N - \mu, N)$ is decreasing, when $\mu = 0$ its value is $\lambda_1(-\Delta; D, N, N) > 0$ and by (2.2)

$$\lim_{\mu\to-\infty} \lambda_1(-\Delta; D, N-\mu, N) = \lambda_1(-\Delta; D, D, N) > 0 \quad \text{and} \lim_{\mu\to+\infty} \lambda_1(-\Delta; D, N-\mu, N) = -\infty.$$

Hence, there exists a unique value $\mu_1 > 0$ such that

$$\lambda_1(-\Delta; D, N - \mu_1, N) = 0$$

and

$$\lambda_1(-\Delta; D, N - \mu, N) < 0 \text{ (resp. } > 0) \Longleftrightarrow \mu > \mu_1 \text{ (resp. } \mu < \mu_1).$$

We have the following result:

Theorem 3.2. *1. There exists a positive solution of (3.8) if, and only if,*

$$\mu > \mu_1. \tag{3.9}$$

Moreover, if the solution exists, it is the unique positive solution, and we denote it by θ_μ*. Furthermore,* θ_μ *is l. a. s. for* $\mu > \mu_1$*, i.e. ,*

$$\lambda_1(-\Delta + 2\theta_\mu; D, N - \mu, N) > 0. \tag{3.10}$$

2. The map $\mu \in (\mu_1, +\infty) \mapsto \theta_\mu \in X_2$ *is regular, increasing and*

$$\theta_\mu = m_1\psi_1(\mu - \mu_1) + O((\mu - \mu_1)^2) \quad \text{as } \mu \downarrow \mu_1, \tag{3.11}$$

being

$$m_1 = \frac{\displaystyle\int_{\Gamma_2} \psi_1^2}{\displaystyle\int_{\Omega} \psi_1^3}, \tag{3.12}$$

and ψ_1 is a principal positive eigenfunction associated to $\mu = \mu_1$, that is

$$-\Delta\psi_1 = 0 \quad \text{in } \Omega, \qquad B_2(\mu_1)\psi_1 = 0 \quad \text{on } \partial\Omega. \tag{3.13}$$

3. *We have that $\theta_\mu \to z$ in $C^{2,\gamma}(\Omega)$ as $\mu \to +\infty$ where z is the minimal solution of the problem*

$$\begin{cases} -\Delta z = -z^2 & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma_1, \\ z = +\infty & \text{on } \Gamma_2, \\ \dfrac{\partial z}{\partial n} = 0 & \text{on } \Gamma_3. \end{cases} \tag{3.14}$$

Proof. 1. Let v a positive solution of (3.8). Then

$$0 = \lambda_1(-\Delta + v; D, N - \mu, N) > \lambda_1(-\Delta; D, N - \mu, N),$$

and so $\mu > \mu_1$.

Assume now that $\mu > \mu_1$, or equivalently, $\lambda_1(-\Delta; D, N - \mu, N) < 0$. Consider $\psi_\mu > 0$ the positive eigenfunction associated to $\lambda_1(-\Delta; D, N - \mu, N)$ with $\|\psi_\mu\|_\infty = 1$. Then, it is not hard to show that

$$\underline{v} := \varepsilon\psi_\mu$$

with $\varepsilon > 0$ is a subsolution of (3.8) for $\varepsilon = -\lambda_1(-\Delta; D, N - \mu, N)$.

The construction of a supersolution is more involved. Define for $\delta > 0$ and small the sets

$$B_\delta := \{x \in \mathbb{R}^N : dist(x, \Gamma_1) \le \delta\} \quad \text{and} \quad \Omega_\delta := \Omega \cup B_\delta.$$

Now, consider the eigenvalue problem

$$\begin{cases} -\Delta\phi = \lambda\phi & \text{in } \Omega_\delta, \\ \phi = 0 & \text{on } \partial B_\delta \setminus \overline{\Omega}, \\ \dfrac{\partial\phi}{\partial n} - \mu\phi = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial\phi}{\partial n} = 0 & \text{on } \Gamma_3, \end{cases}$$

and take its principal eigenvalue, denoted by λ_1^δ, and its associated positive eigenfunction φ_δ. Observe that $\min_{\overline{\Omega}} \varphi_\delta > 0$. Then, again it is not hard to show that

$$\overline{v} := M\varphi_\delta$$

is a supersolution of (3.8) if

$$M \ge \frac{-\lambda_1^\delta}{\min_{\overline{\Omega}} \varphi_\delta}.$$

Now, we can take M large such that $\overline{v} > \underline{v}$ and apply the sub-supersolution method to conclude the existence of a positive solution of (3.8) such that $\underline{v} \leq v \leq \overline{v}$.

The uniqueness follows by a standard argument, observe that $v \mapsto -v^2/v$ is decreasing and so we can apply the general result of [4], see [30] for nonlinear boundary conditions. We would like to point that although the main uniqueness result of [4] is stated for the homogeneous Dirichlet boundary conditions, it is also valid for the mixed condition $B_2(\mu)v = 0$. So, we can conclude that

$$\underline{v} \leq \theta_\mu \leq \overline{v}.$$

Thanks to the above bound and the non-existence of solution for $\mu = \mu_1$, we conclude that

$$\lim_{\mu \to \mu_1} \|\theta_\mu\|_\infty = 0.$$

In order to prove (3.10) it is enough (see for instance Lemma 2.2 in [16]) to find a positive supersolution, that is, a positive function $\overline{v}$ such that

$$(-\Delta + 2\theta_\mu)\overline{v} \geq 0 \quad \text{in } \Omega, \qquad B_2(\mu)\overline{v} \geq 0 \quad \text{on } \partial\Omega,$$

and at least one of the inequalities is strict. We take as supersolution $\overline{v} = \theta_\mu$, then $B_2(\mu)\theta_\mu = 0$ on $\partial\Omega$ and

$$-\Delta\theta_\mu + 2\theta_\mu^2 = \theta_\mu^2 > 0 \quad \text{in } \Omega,$$

and so we conclude (3.10).

2. The proof of that $\mu \mapsto \theta_\mu$ is increasing is standard. Now, we show its regular character. For that, we use a continuation method. Define the regular map $\mathcal{F} : \mathbb{R} \times X_2 \mapsto C^\gamma(\overline{\Omega}) \times C^{1,\gamma}(\Gamma_2)$ defined by

$$\mathcal{F}(\mu, v) := (-\Delta v + v^2, \frac{\partial v}{\partial n} - \mu v).$$

It is clear that the solutions of (3.8) can be viewed as the zeros of the mapping $\mathcal{F}$. Consider a solution (μ_0, v_0) of (3.8) with $\mu_0 > \mu_1$. Then

$$D_v\mathcal{F}(\mu_0, v_0)w = (-\Delta w + 2v_0 w, \frac{\partial w}{\partial n} - \mu_0 w).$$

Now, thanks to (3.10) it is clear that $D_v\mathcal{F}(\mu_0, v_0)$ is an isomorphism, and so using the implicit function theorem we conclude the regularity of the map $\mu \mapsto \theta_\mu$ for $\mu > \mu_1$.

Now, we will show the expression (3.11). We have that $\mathcal{F}(\mu_1, 0) = 0$ and that $D_v\mathcal{F}(\mu, 0)w = (-\Delta w, \frac{\partial w}{\partial n} - \mu w)$. Hence, $Ker(D_v\mathcal{F}(\mu_1, 0)) = span\{\psi_1\}$ and $dim(Ker(D_v\mathcal{F}(\mu_1, 0))) = 1$, being ψ_1 the positive eigenfunction associated to μ_1.

On the other hand,

$$D_{\mu v}\mathcal{F}(\mu_1, 0)w = (0, -w),$$

and so $D_{\mu v}\mathcal{F}(\mu_1, 0)\psi_1 \notin R(D_v\mathcal{F}(\mu_1, 0))$. Indeed, if $D_{\mu v}\mathcal{F}(\mu_1, 0)\psi_1 \in R(D_v\mathcal{F}(\mu_1, 0))$ there exists $y \in X_2$ such that

$$-\Delta y = 0 \quad \text{in } \Omega, \quad \frac{\partial y}{\partial n} - \mu_1 y = -\psi_1 \quad \text{on } \Gamma_2,$$

and so multiplying by ψ_1 and integrating we arrive at

$$\int_{\Gamma_2} \psi_1^2 = 0,$$

an absurdum. Finally, it can shown that $D_v\mathcal{F}(\mu_1, 0)$ is a Fredholm operator of index 0, and consequently the co-dimension of $R(D_v\mathcal{F}(\mu_1, 0))$ is 1.

So the Crandall-Rabinowitz theorem applies, see [9], and for a complement W of $span\{\psi_1\}$, there are two C^1 functions $\mu(s) : (-\varepsilon_0, \varepsilon_0) \mapsto \mathbb{R}$ and $w(s) : (-\varepsilon_0, \varepsilon_0) \mapsto W$ for some $\varepsilon_0 > 0$, with $\mu(0) = \mu_1$, $w(0) = 0$ and

$$\mathcal{F}(\mu(s), s\psi_1 + sw(s)) = 0 \quad \text{for } s \in (-\varepsilon_0, \varepsilon_0).$$

Moreover, if $\mathcal{F}(\mu, v) = 0$ and (μ, v) is close to $(\mu_1, 0)$, then either $v = 0$ or for some $s \neq 0, s \in (-\varepsilon_0, \varepsilon_0)$, $(\mu, v) = (\mu(s), s\psi_1 + sw(s))$.

Writing $\mu(s) = \mu_1 + s\mu_2 + O(s^2)$, introducing the expressions of $\mu(s)$ and $s\psi_1 + sw(s)$ into (3.8) in the variables μ and v respectively, and after some calculations, we arrive at

$$\int_{\Gamma_2} (\mu_2 + O(s))(\psi_1 + w(s))\psi_1 = \int_{\Omega} (\psi_1 + w(s))^2\psi_1,$$

and so

$$\mu_2 = \frac{\displaystyle\int_{\Omega} \psi_1^3}{\displaystyle\int_{\Gamma_2} \psi_1^2}.$$

From $\mu(s)$, it suffices to calculate s as function of μ and conclude (3.11).

3. First, we can apply the general result Theorem 3.3 in [26] and conclude that for a open set $D \subset \overline{D} \subset \Omega$ there exists a constant M such that for any regular solution v_μ of

$$-\Delta v = -v^2 \quad \text{in } \Omega,$$

the following estimate holds

$$\|v_\mu\|_{C(\overline{D})} \leq M.$$

So, since the map $\mu \mapsto \theta_\mu$ is increasing we can define the pointwise limit

$$z(x) := \lim_{\mu \to +\infty} \theta_\mu(x) \quad x \in \Omega.$$

Thanks to the L^p elliptic estimates, this limit is in $C^{2,\gamma}(\Omega)$ and $z \in C^{2,\gamma}(\Omega)$. Now, it remains to prove that z is in fact solution of (3.14), that is, that $\lim_{dist(x,\Gamma_2)\to 0} z(x) = \infty$.

Now, since the proof of this paragraph is practically similar to Theorem 4 in [19], see also [28], we only sketch it. First, we can show that

$$z(x) = \lim_{m \to +\infty} v_m(x),$$

where v_m is the unique solution of

$$\begin{cases} -\Delta v = -v^2 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial v}{\partial n} = m & \text{on } \Gamma_2, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Gamma_3. \end{cases}$$

Take $\delta > 0$ small, and consider the set $\Omega_\delta := \{x \in \Omega : 0 < dist(x, \Gamma_2) < \delta\}$. Then, take now

$$\underline{w} := A(dist(x, \Gamma_2) + \tau)^{-\kappa}$$

for $A > 0$, $\tau > 0$ and $\kappa > 0$ to be chosen. It is not hard to show that $\underline{w} - k$ for some k is a subsolution of

$$\begin{cases} -\Delta v = -v^2 & \text{in } \Omega_\delta, \\ \dfrac{\partial v}{\partial n} = m & \text{on } \Gamma_2, \\ v = v_m & \text{on } \{x \in \Omega : dist(x, \Gamma_2) = \delta\}. \end{cases}$$

And then,

$$\underline{w} - k \le v_m \quad \text{in } \Omega_\delta,$$

and so taking $m \to \infty$ and $\tau \to 0$ we get,

$$A dist(x, \Gamma_2)^{-\kappa} - k \le z,$$

and so $z \to \infty$ as $dist(x, \Gamma_2) \to 0$. □

Remark 3.3. *The existence, uniqueness and paragraph 3 of the above result have been previously studied in [19] in the case $\partial\Omega = \Gamma_2$, see also [5] for mixed boundary problem with the parameter μ in the equation instead of at the boundary.*

Problems related to (3.14) have been extensively studied in the last years, see for example [28], [18], [12] and references therein.

With a similar reasoning to the above result we can study the general equation

$$\begin{cases} -\Delta v = -v^2 - r(x)v & \text{in } \Omega, \\ B_2(\mu)v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.15}$$

where $r \in C^\gamma(\overline{\Omega})$ is a positive function.

Proposition 3.4. *There exists a positive solution of (3.15) if, and only if,*

$$\lambda_1(-\Delta + r; D, N - \mu, N) < 0. \tag{3.16}$$

In case of existence of solution, this is the unique positive one, denoted by V_μ, and it is l. a. s., that is

$$\lambda_1(-\Delta + 2V_\mu + r; D, N - \mu, N) > 0. \tag{3.17}$$

Also, along the paper we will need to study the following equation

$$\begin{cases} -div(a(x)\nabla w) = b(x)w(\lambda - c(x)w) & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial w}{\partial n} + d(x)w = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \Gamma_3, \end{cases} \tag{3.18}$$

where $a \in C^1(\overline{\Omega})$, $b, c \in C^\gamma(\overline{\Omega})$ and $d \in C^{1,\gamma}(\Gamma_2)$ all of them positive. Although the following result is in fact true under more general conditions on the data, and perhaps it is more or less known, we include a proof for the reader's convenience and for the useful estimates obtained.

Proposition 3.5. *There exists a positive solution of* (3.18) *if, and only if,*

$$\lambda > \Gamma(a; b, D, N + d, N),$$

where $\Gamma(a; b, D, N + d, N)$ *is defined in (2.5). Moreover, the solution is unique, we call it* w *and the following estimate holds*

$$\frac{\lambda - \Gamma(a; b, D, N + d, N)}{\|\phi_1\|_\infty c_M}\phi_1 \le w \le \frac{\lambda}{c_L}, \tag{3.19}$$

where ϕ_1 *is a positive eigenfunction associated to* $\Gamma(a; b, D, N + d, N)$*.*

Proof. Assume that w is a positive solution of (3.18). Consider ϕ_1 the positive eigenfunction associated to $\Gamma(a; b, D, N + d, N)$. Multiplying (3.18) by ϕ_1 and integrating by parts we get that $\lambda > \Gamma(a; b, D, N + d, N)$.

The uniqueness of positive solution follows again by classical results, observe that $w \mapsto b(x)w(\lambda - c(x)w)/w$ is decreasing.

For the existence we use again the sub-supersolution method. Indeed, we can pick up $\overline{w} := K > 0$, for K constant. Then, $\overline{w}$ is a supersolution of (3.18) if

$$K = \frac{\lambda}{c_L}.$$

As subsolution we consider $\underline{w} := \varepsilon\phi_1$ for $\varepsilon > 0$. Then, $\underline{w}$ is subsolution of (3.18) provided of

$$\varepsilon \le \frac{\lambda - \Gamma(a; b, D, N + d, N)}{\|\phi_1\|_\infty c_M}.$$

This proves the estimate (3.19) and concludes the proof. □

4 Study of the stability of the semi-trivial solutions

In this section we study the stability of the two semi-trivial states. Let us to extend the definition of ϑ_λ and θ_μ. We write $\vartheta_\lambda \equiv 0$ as $\lambda \le \lambda_1$ and $\theta_\mu \equiv 0$ as $\mu \le \mu_1$.

Let us introduce now some maps. By Lemma 2.1 we have that for each $\lambda > \lambda_1$ there exists a unique value $\mu = F(\lambda)$ such that $\lambda_1(-\Delta + c\vartheta_\lambda; D, N - \mu, N) = 0$. Of course, if $\mu > F(\lambda)$ (resp. $\mu < F(\lambda)$) we have that $\lambda_1(-\Delta + c\vartheta_\lambda; D, N - \mu, N) < 0$ (resp. $\lambda_1(-\Delta + c\vartheta_\lambda; D, N - \mu, N) > 0$.)

On the other hand, for $\mu > \mu_1$ we consider the eigenvalue problem

$$\begin{cases} -\Delta\Phi + \alpha div(\Phi\nabla\theta_\mu) = \lambda\Phi & \text{in } \Omega, \\ B_1\Phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Denote by $\Lambda(\alpha, \mu)$ the principal eigenvalue of (4.1). This eigenvalue plays a crucial role in studying our problem, so a study in detail will be carried out later.

We extend the definitions of $F(\lambda)$ and $\Lambda(\alpha, \mu)$ in the following sense: $F(\lambda) = \mu_1$ if $\lambda \le \lambda_1$ and $\Lambda(\alpha, \mu) = \lambda_1$ for $\mu \le \mu_1$.

Proposition 4.1. *1. The trivial solution of (1.1) is l. a. s. if* $\lambda < \lambda_1$ *and* $\mu < \mu_1$ *and unstable if* $\lambda > \lambda_1$ *or* $\mu > \mu_1$*.*

2. *Assume that $\lambda > \lambda_1$. The semi-trivial solution $(\vartheta_\lambda, 0)$ is l. a. s. if $\mu < F(\lambda)$ and unstable if $\mu > F(\lambda)$.*

3. *Assume that $\mu > \mu_1$. The semi-trivial solution $(0, \theta_\mu)$ is l. a. s. if $\lambda < \Lambda(\alpha, \mu)$ and unstable if $\lambda > \Lambda(\alpha, \mu)$.*

Proof. First we prove only the second paragraph of the result, the third one follows similarly. Observe that the stability of $(\vartheta_\lambda, 0)$ is given by the real parts of the eigenvalues for which the following problem admits a solution $(\xi, \eta) \in X \setminus \{(0,0)\}$

$$\begin{cases} -\Delta\xi + \alpha div(\vartheta_\lambda \nabla\eta) - \lambda\xi + 2\vartheta_\lambda\xi = \sigma\xi & \text{in } \Omega, \\ -\Delta\eta + c\vartheta_\lambda\eta = \sigma\eta & \text{in } \Omega, \\ B_1\xi = B_2(\mu)\eta = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Assume that $\eta \equiv 0$, then for some $j \geq 1$ and by (3.7)

$$\sigma = \lambda_j(-\Delta + 2\vartheta_\lambda - \lambda; D, N, N) \geq \lambda_1(-\Delta + 2\vartheta_\lambda - \lambda; D, N, N) > 0.$$

Suppose now that $\eta \not\equiv 0$, then from the second equation of (4.2) we get

$$\sigma = \lambda_j(-\Delta + c\vartheta_\lambda; D, N - \mu, N) \geq \lambda_1(-\Delta + c\vartheta_\lambda; D, N - \mu, N) > 0$$

because $\mu < F(\lambda)$.

Assume now that $\mu > F(\lambda)$. Then,

$$\sigma_1 := \lambda_1(-\Delta + c\vartheta_\lambda; D, N - \mu, N) < 0.$$

Denote by η a positive eigenfunction associated to σ_1, that is

$$-\Delta\eta + c\vartheta_\lambda\eta = \sigma_1\eta \quad \text{in } \Omega, \qquad B_2(\mu)\eta = 0 \quad \text{on } \partial\Omega.$$

Since $\sigma_1 < 0$, then

$$\lambda_1(-\Delta + 2\vartheta_\lambda - \lambda - \sigma_1; D, N, N) > 0,$$

and so there exists

$$\xi = (-\Delta + 2\vartheta_\lambda - \lambda - \sigma_1)^{-1}_{B_1}(-\alpha div(\vartheta_\lambda \nabla\eta)),$$

that is,

$$-\Delta\xi + (2\vartheta_\lambda - \lambda)\xi + \alpha div(\vartheta_\lambda \nabla\eta) = \sigma_1\xi \quad \text{in } \Omega, \quad B_1\xi = 0 \quad \text{on } \partial\Omega.$$

Then, $\sigma_1 < 0$ is an eigenvalue of (4.2) with the eigenfunction associated (ξ, η), so $(\vartheta_\lambda, 0)$ is unstable.

The stability of the trivial solution follows in a similar way. □

5 The case $\alpha = 0$: no chemotaxis

In this section, we study the case when the chemotaxis is not present, that is $\alpha = 0$ and so the system (1.1) is uncoupled. In the following result we collect the main features:

Proposition 5.1. *Assume that $\alpha = 0$ in (1.1).*

1. *The trivial solution exists for all $\lambda, \mu \in \mathbb{R}$. It is l. a. s. for $\lambda > \lambda_1$ and $\mu > \mu_1$ and unstable for $\lambda < \lambda_1$ or $\mu < \mu_1$.*

2. *A coexistence state exists if, and only if, $\lambda > \lambda_1$ and $\mu > F(\lambda)$. In this case, the solution is unique and l. a. s.*

3. *Assume that $\lambda \le \lambda_1$. Then if $\mu \le \mu_1$ the only solution of (1.1) is the trivial $(0,0)$. If $\mu > \mu_1$ there exists the semi-trivial solution $(0, \theta_\mu)$, which is l. a. s.*

4. *Assume $\lambda > \lambda_1$ and $\mu \le F(\lambda)$. There exists the semi-trivial solution $(\vartheta_\lambda, 0)$, which is unstable. Moreover, if $\mu > \mu_1$ there exists the semi-trivial solution $(0, \theta_\mu)$ that is l. a. s.*

Proof. Assume that $\lambda \le \lambda_1$, then by Proposition 3.1 it follows that $u \equiv 0$, and so $v \equiv \theta_\mu$. Now, suppose that $\lambda > \lambda_1$, then again by Proposition 3.1 we have that $u \equiv \vartheta_\lambda$. Going back to the v-equation, we can apply Proposition 3.4 with $r = c\vartheta_\lambda$, and so by (3.16) there exists solution if, and only if,

$$\lambda_1(-\Delta + c\vartheta_\lambda; D, N - \mu, N) < 0,$$

or equivalently, $\mu > F(\lambda)$.

The stability results follow in a similar way that Proposition 4.1, except the stability of the coexistence state, we call it $(\vartheta_\lambda, V_\mu)$. For that, we need study the real parts of the eigenvalues for which the following problem admits a solution $(\xi, \eta) \in X \setminus \{(0,0)\}$

$$\begin{cases} -\Delta\xi - \lambda\xi + 2\vartheta_\lambda\xi = \sigma\xi & \text{in } \Omega, \\ -\Delta\eta + c\vartheta_\lambda\eta + 2V_\mu\eta + cV_\mu\xi = \sigma\eta & \text{in } \Omega, \\ B_1\xi = B_2(\mu)\eta = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

If $\xi \equiv 0$, then by (3.17)

$$\sigma = \lambda_j(-\Delta + 2V_\mu + c\vartheta_\lambda; D, N - \mu, N) \ge \lambda_1(-\Delta + 2V + c\vartheta_\lambda; D, N - \mu, N) > 0.$$

If $\xi \not\equiv 0$, then by (3.7)

$$\sigma = \lambda_j(-\Delta + 2\vartheta_\lambda - \lambda; D, N, N) \ge \lambda_1(-\Delta + 2\vartheta_\lambda - \lambda; D, N, N) > 0.$$

This completes the proof. $\square$

6 The existence of coexistence states

Our first result provides necessary conditions on λ and μ in order to have positive solutions of (1.1).

Lemma 6.1. *If there exists a positive solution* (u,v) *of* (1.1) *then*

$$\lambda > 0 \quad and \quad \mu > \mu_1.$$

Proof. If (u,v) is a positive solution of (1.1), then $v \leq \theta_\mu$, and so $\mu > \mu_1$.

On the other hand, by the change of variable

$$u = e^{\alpha v} w, \tag{6.1}$$

the equation of u in (1.1) transforms into

$$\begin{cases} -div(e^{\alpha v}\nabla w) = e^{\alpha v} w(\lambda - e^{\alpha v} w) & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial w}{\partial n} + \alpha\mu v w = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \Gamma_3. \end{cases} \tag{6.2}$$

Now, applying Proposition 3.5 we conclude that $w = 0$ if $\lambda \leq 0$. □

In fact, we can sharp this result building a non-existence region of positive solutions in the (λ,μ)-plane.

Proposition 6.2. *Assume that* $\mu > \mu_1$. *If*

$$\lambda \leq G(\mu) := \Gamma(1; e^{\alpha\theta_\mu}, D, N, N)$$

then (1.1) does not possess positive solution.

Moreover, $G(\mu_1) = \lambda_1$, $\mu \mapsto G(\mu)$ *is a decreasing map and so there exists the following limit*

$$\lim_{\mu\to+\infty} G(\mu) := \lambda_*(\alpha) \geq 0.$$

Proof. If (u,v) is a positive solution of (1.1) then w defined in (6.1) is a positive solution of (6.2), and so by Proposition 3.5 we have

$$\lambda > \Gamma(e^{\alpha v}; e^{\alpha v}, D, N + \alpha\mu v, N) \geq \Gamma(1; e^{\alpha\theta_\mu}, D, N, N) = G(\mu) > 0.$$

The properties of the map G follow by the ones of the eigenvalue $\Gamma(1; e^{\alpha\theta_\mu}, D, N, N)$. □

Remark 6.3. *Lemma 6.1 and Proposition 6.2 provide us a non-existence region of positive solutions in the* (λ,μ)*-plane. Indeed, if*

$$(\lambda,\mu) \in B := \{(\lambda,\mu) : \mu \leq \mu_1 \text{ or } \lambda \leq G(\mu)\},$$

then (1.1) does not possess positive solutions. See Figure 2 where we have drawn the region B *in different cases.*

In the following result, we show a priori bounds in X of the solutions of (1.1).

Proposition 6.4. *Consider that* $(\lambda, \mu) \in \mathcal{K} \subset \mathbb{R}^2$ *compact. Then, there exists a constant* C *(independent of* λ *and* μ*) such that for any solution* (u, v) *of* (1.1) *we have*

$$\|(u, v)\|_X \le C.$$

Proof. Suppose $(\lambda, \mu) \in \mathcal{K} \subset \mathbb{R}^2$ compact and let (u, v) be a solution of (1.1). Then, $u = e^{\alpha v} w$ transforms the equation for u into (6.2). So, by (3.19)

$$w \le \frac{\lambda}{(e^{2\alpha v})_L} \le \lambda.$$

On the other hand, since $v \le \theta_\mu$, we obtain that

$$u = e^{\alpha v} w \le \lambda e^{\alpha \theta_\mu} \le C,$$

for some constant C not depending on λ or μ. Hence, u and v are bounded in $L^\infty(\Omega)$. Now, going back to the v-equation and using the L^p-estimates of Agmon, Douglis and Nirenberg [1], we have that for p large

$$\|v\|_{C^1(\overline{\Omega})} \le C\|v\|_{W^{2,p}(\Omega)} \le C\| - v^2 - cuv\|_p \le C.$$

But, the u-equation in (1.1) can be written as follows

$$-\Delta u + \alpha \nabla u \cdot \nabla v = \lambda u - u^2 - \alpha u(v^2 + cuv)$$

and thus, u is bounded in $W^{2,p}(\Omega)$ for all $p > 1$, and so in $C^1(\overline{\Omega})$. Now, again using the v-equation and the Schauder Theory in Hölder spaces (see [17]), v is bounded in X_2, and finally u in X_1 with constants independent of λ and μ. $\square$

The following result shows that fixed μ, (1.1) does not have positive solutions for λ large enough.

Proposition 6.5. *Fix* $\mu > \mu_1$*. Then, there exists* $\Lambda_0 > 0$ *(depending on* μ*) such that there does not exist positive solution of* (1.1) *for* $\lambda > \Lambda_0$*.*

In order to clarify the proof of this result, we include several lemmas which will be used later. The first one is a useful interpolation inequality, which follows from the boundedness of the embedding operator from $H^1(\Omega)$ into $L^2(\partial\Omega)$, see for instance Theorem 2.1 in [23].

Lemma 6.6. *For any* $\varepsilon > 0$*, there exists a positive constant* $C(\varepsilon)$ *such that*

$$\int_{\partial\Omega} f^2 \le \varepsilon \int_\Omega |\nabla f|^2 + C(\varepsilon) \int_\Omega f^2, \quad \textit{for any } f \in H^1(\Omega).$$

Since we are going to move the parameter λ, let us write it as subscript. Denoting by

$$a_\lambda := e^{\alpha v}$$

we have that $1 \le a_\lambda \le e^{\alpha\theta_\mu} \le C$, for a positive constant independent of λ, and so that

$$\lambda_1 = \Gamma(1; 1, D, N, N) \le \Gamma(\lambda) := \Gamma(a_\lambda; 1, D, N, N) \le \Gamma(C; 1, D, N, N) := \overline{\Gamma}. \tag{6.3}$$

Denote now by ϕ_λ a positive eigenfunction associated to $\Gamma(\lambda)$. We will need a bound of ϕ_λ independent of the parameter λ. In the following result we obtain a bound on its L^∞-norm in function of its L^2-norm. Although the result and its proof are standard, see for instance Theorem 4.1 in [32], we have followed the proof of Lemma 5 in [20] because we can estimate the dependence on λ.

Lemma 6.7. *There exists a positive constant C depending on $\overline{\Gamma}$ but not on λ such that*

$$\|\phi_\lambda\|_\infty \le C(\|\phi_\lambda\|_2 + 1). \tag{6.4}$$

Proof. Denote by $z = \phi_\lambda + 1$ and take $\beta \ge 1$, and

$$\varphi := z^\beta - 1.$$

It is clear that

$$\nabla\varphi = \beta z^{\beta-1}\nabla z.$$

Now, taking φ as test function in the equation of ϕ_λ, and taking into account that $\nabla\phi_\lambda = \nabla z$, we obtain

$$\beta\int_\Omega a_\lambda z^{\beta-1}|\nabla z|^2 = \Gamma(a_\lambda; 1, D, N, N)\int_\Omega \phi_\lambda\varphi \le \overline{\Gamma}\int_\Omega z^{\beta+1},$$

with $\overline{\Gamma}$ defined in (6.3). Moreover, observe that $1 \le a_\lambda$, and so

$$\beta\int_\Omega z^{\beta-1}|\nabla z|^2 \le \overline{\Gamma}\int_\Omega z^{\beta+1}.$$

Now, denoting by $\Psi := z^{\frac{\beta+1}{2}}$, we get

$$\beta z^{\beta-1}|\nabla z|^2 = \frac{4\beta}{(\beta+1)^2}|\nabla\Psi|^2.$$

Hence,

$$\int_\Omega |\nabla\Psi|^2 \le \frac{\overline{\Gamma}(\beta+1)^2}{4\beta}\int_\Omega z^{\beta+1}.$$

Using now that $H^1(\Omega) \hookrightarrow L^r(\Omega)$ for some $r > 2$, we conclude that

$$\left(\int_\Omega |\Psi|^r\right)^{2/r} \le \frac{\overline{\Gamma}(\beta+1)^2}{4\beta}\int_\Omega z^{\beta+1} + C\int_\Omega \Psi^2,$$

for some constant C independent of λ. Now, since $\Psi^2 = z^{\beta+1}$ and $(\beta+1)^2/(4\beta) \le \beta+1$, we get

$$\left(\int_\Omega z^{\frac{(\beta+1)}{2}r}\right)^{\frac{2}{r(\beta+1)}} \le (C(\overline{\Gamma})(\beta+1))^{\frac{1}{\beta+1}}\left(\int_\Omega z^{\beta+1}\right)^{\frac{1}{\beta+1}}.$$

Thus, if we call $p = \beta+1$ and $q = r/2 > 1$ we have

$$\|z\|_{qp} \le (Cp)^{1/p}\|z\|_p.$$

Now, taking $p = 2q^n$, for $n = 0, 1, 2, ...$ we have

$$\|z\|_{2q^n} \le (2C)^{1/2}(2qC)^{1/2q}\dots(2q^nC)^{1/2q^n}\|z\|_2,$$

letting $n \to \infty$ we obtain that for some constant C

$$\|z\|_\infty \le C\|z\|_2,$$

and so we conclude (6.4). $\square$

Now, we are ready to prove the result.

Proof of Proposition 6.5. We reason by contradiction. Assume that there exists a coexistence state for all $\lambda > 0$. We use again the change of variable (6.1), that is, $u = e^{\alpha v} w$, that transforms the equation of u in (1.1) into (6.2). Using that $1 \le e^{\alpha v} \le e^{\alpha \theta_\mu}$, we get that

$$-div(e^{\alpha v}\nabla w) \ge \lambda w - e^{2\alpha\theta_\mu} w^2 \quad \text{in } \Omega,$$

and

$$\frac{\partial w}{\partial n} + \alpha\mu v w \ge \frac{\partial w}{\partial n} \quad \text{on } \Gamma_2,$$

and so w is a supersolution of the equation

$$\begin{cases} -div(e^{\alpha v}\nabla w) = \lambda w - e^{2\alpha\theta_\mu} w^2 & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \Gamma_3. \end{cases} \tag{6.5}$$

By (3.19) we have that

$$\frac{\lambda - \Gamma(e^{\alpha v}; 1, D, N, N)}{r(\mu,\lambda)}\phi_\lambda \le w,$$

where $r(\mu,\lambda) := (e^{2\alpha\theta_\mu})_M \|\phi_\lambda\|_\infty$ and ϕ_λ is the positive eigenfunction associated to $\Gamma(\lambda)$ such that $\|\phi_\lambda\|_2 = 1$. Observe that by Lemma 6.4 we have that

$$\|\phi_\lambda\|_\infty \le C(\|\phi_\lambda\|_2 + 1) = C.$$

On the other hand, by the monotony of the eigenvalue $\Gamma(a; 1, D, N, N)$ with respect to a, we have

$$\Gamma(e^{\alpha v}; 1, D, N, N) \le \Gamma(e^{\alpha\theta_\mu}; 1, D, N, N) := s(\mu),$$

and so, denoting

$$\tau(\lambda) := \frac{\lambda - s(\mu)}{r(\mu,\lambda)},$$

we have

$$\tau(\lambda)\phi_\lambda \le w \le u. \tag{6.6}$$

Observe that

$$\tau(\lambda) \ge \frac{\lambda - s(\mu)}{C(e^{2\alpha\theta_\mu})_M} \to +\infty \quad \text{as } \lambda \to +\infty.$$

Now, fix $\mu > \mu_1$. Then, since v is a positive solution of the second equation of (1.1) and using (6.6), we get

$$0 = \lambda_1(-\Delta + v + cu; D, N - \mu, N) \ge \lambda_1(-\Delta + c\tau(\lambda)\phi_\lambda; D, N - \mu, N) := g(\lambda).$$

Observe that

$$g(\lambda) = \inf_{\varphi\in\mathcal{S},\varphi\neq 0} \frac{\displaystyle\int_\Omega |\nabla\varphi|^2 + c\tau(\lambda)\int_\Omega \phi_\lambda\varphi^2 - \mu\int_{\Gamma_2}\varphi^2}{\displaystyle\int_\Omega \varphi^2},$$

being

$$\mathcal{S} := \{u \in H^1(\Omega) : u = 0 \text{ in } \Gamma_1\}.$$

We claim that

$$\lim_{\lambda \to +\infty} g(\lambda) = +\infty,$$

whence we deduce that λ can not reach a value bigger than Λ_0. Suppose otherwise that $g(\lambda)$ is bounded. There exists a sequence $\varphi_\lambda \in \mathcal{S}$ such that $\|\varphi_\lambda\|_2 = 1$ and

$$\int_\Omega |\nabla \varphi_\lambda|^2 + c\tau(\lambda) \int_\Omega \phi_\lambda \varphi_\lambda^2 - \mu \int_{\Gamma_2} \varphi_\lambda^2 = g(\lambda) \int_\Omega \varphi_\lambda^2 = g(\lambda). \tag{6.7}$$

Using now Lemma 6.6 we get that

$$(1 - \mu\varepsilon) \int_\Omega |\nabla \varphi_\lambda|^2 + c\tau(\lambda) \int_\Omega \phi_\lambda \varphi_\lambda^2 \leq g(\lambda) + C(\varepsilon)\mu, \tag{6.8}$$

and hence if we take ε small enough, φ_λ is bounded in $H^1(\Omega)$, and so passing to a subsequence there exists $\varphi_0 \geq 0$, $\|\varphi_0\|_2 = 1$ and $\varphi_0 \neq 0$ such that

$$\varphi_\lambda \rightharpoonup \varphi_0 \quad \text{in } H^1(\Omega), \quad \varphi_\lambda \to \varphi_0 \quad \text{in } L^2(\Omega). \tag{6.9}$$

We study now ϕ_λ. By (6.3) it follows that there exists $\Gamma_0 > 0$ such that

$$\Gamma(a_\lambda; 1, D, N, N) \to \Gamma_0 \quad \text{as } \lambda \to +\infty.$$

We know that

$$-div(a_\lambda \nabla \phi_\lambda) = \Gamma(a_\lambda; 1, D, N, N)\phi_\lambda \quad \text{in } \Omega, \tag{6.10}$$

and so,

$$\int_\Omega |\nabla \phi_\lambda|^2 \leq \int_\Omega a_\lambda |\nabla \phi_\lambda|^2 = \Gamma(a_\lambda; 1, D, N, N) \int_\Omega \phi_\lambda^2 \leq C,$$

whence we deduce that ϕ_λ is bounded in $H^1(\Omega)$, and hence

$$\phi_\lambda \rightharpoonup \phi_0 \quad \text{in } H^1(\Omega), \quad \phi_\lambda \to \phi_0 \quad \text{in } L^2(\Omega). \tag{6.11}$$

We get that $\Gamma(a_\lambda)\phi_\lambda \to \Gamma_0\phi_0$ in $L^2(\Omega)$. Observe that $\phi_0 \geq 0$ and nontrivial because $\|\phi_0\|_2 = 1$.

Observe that the equation (6.10) is verified in $H^{-1}(\Omega)$, and so we can apply the homogenization technique (see for instance [10] and Theorem 2.1 in [22]), and conclude that there exists an uniformly elliptic symmetric matrix $A \in (L^\infty(\Omega))^{N\times N}$ such that following equation is verified in $H^{-1}(\Omega)$

$$-div(A\nabla\phi_0) = \Gamma_0\phi_0,$$

and so, since $\Gamma_0\phi_0 \geq 0$ and non-trivial, by the strong maximum principle $\phi_0 > 0$, see Theorem 8.19 in [17]. Then by (6.8) we get

$$\limsup_{\lambda \to +\infty} \int_\Omega \phi_\lambda \varphi_\lambda^2 = 0,$$

and so by (6.9) and (6.11) we get that

$$\int_\Omega \phi_0 \varphi_0^2 \leq 0,$$

an absurdum. This completes the proof. $\square$

We are ready to prove the main existence result:

Theorem 6.8. *Assume that $\mu > \mu_1$ and $\lambda > 0$. Then, if some of the following conditions are satisfied*

$$\lambda > \Lambda(\alpha, \mu) \quad and \quad \mu > F(\lambda)$$

or

$$\lambda < \Lambda(\alpha, \mu) \quad and \quad \mu < F(\lambda),$$

then, there exists at least a coexistence state of (1.1).

Proof. We are going to apply the bifurcation method. We fix $\mu > \mu_1$ and consider λ as bifurcation parameter. First, we apply the Crandall-Rabinowitz theorem in order to find the bifurcation point from the semi-trivial solution $(0, \theta_\mu)$. Consider the map $\mathcal{F} : \mathbb{R} \times X_1 \times \tilde{X_2} \mapsto C^\alpha(\overline{\Omega}) \times C^\alpha(\overline{\Omega})$ defined by

$$\mathcal{F}(\lambda, u, v) := (-\Delta u + \alpha div(u\nabla v) - \lambda u + u^2, -\Delta v + v^2 + cuv),$$

being $\tilde{X_2} := \{v \in X_2 : \partial v/\partial n - \mu v = 0 \quad \text{on } \Gamma_2\}$. It is clear that $\mathcal{F}$ is regular, that $\mathcal{F}(\lambda, 0, \theta_\mu) = 0$ and

$$D_{(u,v)}\mathcal{F}(\lambda_0, u_0, v_0)\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -\Delta\xi + \alpha div(\xi\nabla v_0) - \lambda_0\xi + 2u_0\xi + \alpha div(u_0\nabla\eta) \\ -\Delta\eta + 2v_0\eta + cu_0\eta + cv_0\xi \end{pmatrix}.$$

Hence, for $\lambda = \lambda_0 = \Lambda(\alpha, \mu)$ and $(u_0, v_0) = (0, \theta_\mu)$ we get that

$$Ker[D_{(u,v)}\mathcal{F}(\lambda_0, 0, \theta_\mu)] = span\{(\Phi_1, \Phi_2)\}$$

where Φ_1 is an eigenfunction associated to $\Lambda(\alpha, \mu)$ and

$$\Phi_2 := (-\Delta + 2\theta_\mu)^{-1}_{B_2(\mu)}(c\theta_\mu\Phi_1),$$

which is well defined by (3.10). Hence, $dim(Ker[D_{(u,v)}\mathcal{F}(\lambda_0, 0, \theta_\mu)]) = 1$.

On the other hand, observe that

$$D_{\lambda(u,v)}\mathcal{F}(\lambda_0, u_0, v_0)\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -\xi \\ 0 \end{pmatrix}.$$

We can show that $D_{\lambda(u,v)}\mathcal{F}(\lambda_0, 0, \theta_\mu)(\Phi_1, \Phi_2)^t \notin R(D_{(u,v)}\mathcal{F}(\lambda_0, 0, \theta_\mu))$. Indeed, suppose that there exists $(\xi, \eta) \in X$ such that $D_{(u,v)}\mathcal{F}(\lambda_0, 0, \theta_\mu)(\xi, \eta)^t = (-\Phi_1, 0)$, and so

$$-\Delta\xi + \alpha div(\xi\nabla\theta_\mu) - \lambda_0\xi = -\Phi_1 \quad \text{in } \Omega, \quad B_1\xi = 0 \quad \text{on } \partial\Omega.$$

Under the change of variable $\xi = e^{\alpha\theta_\mu}\varsigma$, the above equation is transformed into

$$\begin{cases} -div(e^{\alpha\theta_\mu}\nabla\varsigma) - \lambda_0 e^{\alpha\theta_\mu}\varsigma = -\Phi_1 & \text{in } \Omega, \\ \varsigma = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial\varsigma}{\partial n} + \alpha\mu\theta_\mu\varsigma = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial\varsigma}{\partial n} = 0 & \text{on } \Gamma_3. \end{cases} \tag{6.12}$$

In a similar way, under the change of variable $\Phi_1 = e^{\alpha\theta_\mu}\psi_1$, (4.1) transforms into

$$\begin{cases} -div(e^{\alpha\theta_\mu}\nabla\psi_1) = \lambda_0 e^{\alpha\theta_\mu}\psi_1 & \text{in } \Omega, \\ \psi_1 = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial\psi_1}{\partial n} + \alpha\mu\theta_\mu\psi_1 = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial\psi_1}{\partial n} = 0 & \text{on } \Gamma_3. \end{cases} \tag{6.13}$$

Now, multiplying (6.12) by ψ_1 and (6.13) by ς, and subtracting we get

$$0 = \int_\Omega \Phi_1\psi_1,$$

an absurdum. Again, it can be showed that $R(D_{(u,v)}\mathcal{F}(\lambda_0, 0, \theta_\mu))$ has co-dimension 1.

Hence, the point $(\lambda, u, v) = (\lambda_0, 0, \theta_\mu)$ is a bifurcation point from the semi-trivial solution $(0, \theta_\mu)$.

Now, we can apply Theorem 4.1 of [24] and conclude the existence of a continuum $\mathcal{C}^+ \subset \mathbb{R} \times X_1 \times \tilde{X}_2$ of positive solutions of (1.1) emanating from the point $(\lambda, u, v) = (\Lambda(\alpha, \mu), 0, \theta_\mu)$ such that:

i) $\mathcal{C}^+$ is unbounded in $\mathbb{R} \times X_1 \times \tilde{X}_2$; or

ii) there exists $\lambda_\infty \in \mathbb{R}$ and a solution $\vartheta_{\lambda_\infty}$ of (3.1) such that $(\lambda_\infty, \vartheta_{\lambda_\infty}, 0) \in cl(\mathcal{C}^+)$; or

iii) there exists $\overline{\lambda} \in \mathbb{R}$ such that $(\overline{\lambda}, 0, 0) \in cl(\mathcal{C}^+)$.

Alternative iii) is not possible. Indeed, if a sequence of positive solutions $(\lambda_n, u_n, v_n) \in cl(\mathcal{C}^+)$ such that $\lambda_n \to \overline{\lambda}$ and $(u_n, v_n) \to (0, 0)$ uniformly, then denoting by

$$V_n = \frac{v_n}{\|v_n\|_\infty},$$

and using the elliptic regularity, we have that $V_n \to V \geq 0$ and non-trivial in $C^2(\overline{\Omega})$ with

$$-\Delta V = 0 \quad \text{in } \Omega, \quad B_2(\mu)V = 0 \quad \text{on } \partial\Omega,$$

and so $\mu = \mu_1$, a contradiction.

On the other hand, fixed $\mu > \mu_1$, for λ negative or λ large, (1.1) does not possess positive solution by Lemma 6.1 and Proposition 6.5. Moreover, by Proposition 6.4 it follows that $\mathcal{C}^+$ is bounded in X uniformly on compact subintervals of λ. Hence, alternative i) does not occur.

Therefore, alternative ii) holds. When this alternative occurs, we can proceed as above and it follows that $\lambda_\infty > \lambda_1$ is such that

$$\lambda_1(-\Delta + c\vartheta_{\lambda_\infty}; D, N - \mu, N) = 0,$$

that is, $\mu = F(\lambda_\infty)$. So, we can conclude the existence of a coexistence state for

$$\lambda \in (\min\{(\Gamma(\alpha, \mu), \lambda_\infty)\}, \max\{(\Gamma(\alpha, \mu), \lambda_\infty)\}).$$

This completes the proof. $\square$

As consequence of this result, it is very important to study the behavior of the functions

$$\mu = F(\lambda) \quad \text{and} \quad \lambda = \Lambda(\alpha, \mu).$$

Recall that we have defined $F(\lambda) = \mu_1$ for $\lambda \le \lambda_1$ and $\Lambda(\alpha, \mu) = \lambda_1$ for $\mu \le \mu_1$.

Proposition 6.9. *Denote by φ_1 and ψ_1 positive principal eigenfunctions associated to λ_1 and μ_1 defined in (3.4) and (3.13), respectively.*

1. *The map $\lambda \in (\lambda_1, +\infty) \mapsto F(\lambda)$ is increasing, regular and satisfies*

$$\lim_{\lambda \to +\infty} F(\lambda) = +\infty. \tag{6.14}$$

Moreover,

$$F(\lambda) = \mu_1 + c n_1 l_1 (\lambda - \lambda_1) + O((\lambda - \lambda_1)^2), \quad \text{as } \lambda \downarrow \lambda_1, \tag{6.15}$$

where n_1 is defined in (3.6), and

$$l_1 = \frac{\displaystyle\int_\Omega \varphi_1 \psi_1^2}{\displaystyle\int_{\Gamma_2} \psi_1^2}.$$

2. *Fix $\mu > \mu_1$. Then $\alpha \in (0, +\infty) \mapsto \Lambda(\alpha, \mu)$ is increasing, regular and*

$$\lim_{\alpha \to 0} \Lambda(\alpha, \mu) = \lambda_1, \quad \text{and} \quad \lim_{\alpha \to +\infty} \Lambda(\alpha, \mu) = +\infty.$$

3. *Fix $\alpha > 0$. Then, $\mu \in (\mu_1, +\infty) \mapsto \Lambda(\alpha, \mu)$ is regular and satisfies*

$$\Lambda(\alpha, \mu) = \lambda_1 + \mu_1 m_1 \frac{\alpha}{2} k_1 (\mu - \mu_1) + O((\mu - \mu_1)^2), \quad \text{as } \mu \downarrow \mu_1, \tag{6.16}$$

where m_1 is defined in (3.12) and

$$k_1 = \frac{\displaystyle\int_{\Gamma_2} \varphi_1^2 \psi_1}{\displaystyle\int_\Omega \varphi_1^2}.$$

Finally,

$$\lim_{\mu \to +\infty} \Lambda(\alpha, \mu) = \lambda_1\left(-\Delta + \frac{\alpha^2}{4}|\nabla z|^2 + \frac{\alpha}{2} z^2; D, D, N\right), \tag{6.17}$$

where z is the minimal solution of (3.14).

Proof. 1. Recall that $\mu = F(\lambda)$ if, and only if, $\lambda_1(-\Delta + c\vartheta_\lambda; D, N - F(\lambda), N) = 0$.

Since $\lambda \mapsto \vartheta_\lambda$ is increasing, we get that $\lambda \mapsto F(\lambda)$ is also increasing. To prove (6.14) we argue by contradiction. Assume that $F(\lambda)$ is bounded for λ large, $F(\lambda) \le C$, then

$$0 = \lambda_1(-\Delta + c\vartheta_\lambda; D, N - F(\lambda), N) \ge \lambda_1(-\Delta + c\vartheta_\lambda; D, N - C, N).$$

On the other hand, by (3.3)

$$(\lambda-\lambda_1)\varphi_1 \leq \vartheta_\lambda,$$

being φ_1 the positive eigenfunction associated to λ_1 with $\|\varphi_1\|_\infty = 1$. Hence,

$$0 \geq \lambda_1(-\Delta + c\vartheta_\lambda; D, N-C, N) \geq \lambda_1(-\Delta + c(\lambda-\lambda_1)\varphi_1; D, N-C, N) \to \infty$$

as $\lambda \to +\infty$. The fact that this last eigenvalue diverges to $+\infty$ as $\lambda \to +\infty$ follows with a similar argument to the used in Proposition 6.5, see also Theorem 6.4 in [25].

Now, take ψ_λ the eigenfunction associated to $\lambda_1(-\Delta + c\vartheta_\lambda; D, N-F(\lambda), N) = 0$, that is

$$\begin{cases} -\Delta\psi_\lambda + c\vartheta_\lambda\psi_\lambda = 0 & \text{in } \Omega, \\ \psi_\lambda = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial\psi_\lambda}{\partial n} = F(\lambda)\psi_\lambda & \text{on } \Gamma_2, \\ \dfrac{\partial\psi_\lambda}{\partial n} = 0 & \text{on } \Gamma_3. \end{cases} \tag{6.18}$$

Since the map $\lambda \mapsto \vartheta_\lambda$ is regular, $F(\lambda)$ and ψ_λ are also regular in λ, see [21], [3] and Example 3.5 in [7]. Hence, using (3.5) we can write

$$\begin{aligned} \psi_\lambda &= \psi_1 + \psi_2(\lambda-\lambda_1) + O((\lambda-\lambda_1)^2), \\ \vartheta_\lambda &= n_1\varphi_1(\lambda-\lambda_1) + O((\lambda-\lambda_1)^2), \\ F(\lambda) &= \mu_1 + \mu_2(\lambda-\lambda_1) + O((\lambda-\lambda_1)^2). \end{aligned}$$

We would like to compute μ_2. Introducing these expressions into the equation (6.18), the terms of order 0 drive to show that ψ_1 is a positive eigenfunction associated to μ_1. The terms of order $(\lambda-\lambda_1)$ satisfy the following equation:

$$\begin{cases} -\Delta\psi_2 + cn_1\varphi_1\psi_1 = 0 & \text{in } \Omega, \\ \psi_2 = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial\psi_2}{\partial n} = \mu_1\psi_2 + \mu_2\psi_1 & \text{on } \Gamma_2, \\ \dfrac{\partial\psi_2}{\partial n} = 0 & \text{on } \Gamma_3. \end{cases} \tag{6.19}$$

Multiplying by ψ_1 and integrating by parts, we get

$$\mu_2 = cn_1 \frac{\displaystyle\int_\Omega \varphi_1\psi_1^2}{\displaystyle\int_{\Gamma_2} \psi_1^2}.$$

This proves (6.15).

2. We make a change of variable yet used in a slight different context in [31] and [3].

Indeed, under the change of variables $\Phi = e^{(\alpha/2)\theta_\mu}\psi$ in (4.1) we obtain

$$\begin{cases} -\Delta\psi + (\dfrac{\alpha^2}{4}|\nabla\theta_\mu|^2 + \dfrac{\alpha}{2}\Delta\theta_\mu)\psi = \lambda\psi & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial\psi}{\partial n} + \dfrac{\alpha}{2}\mu\theta_\mu\psi = 0 & \text{on } \Gamma_2, \\ \dfrac{\partial\psi}{\partial n} = 0 & \text{on } \Gamma_3, \end{cases} \tag{6.20}$$

and so

$$\Lambda(\alpha,\mu) = \lambda_1(-\Delta + \frac{\alpha^2}{4}|\nabla\theta_\mu|^2 + \frac{\alpha}{2}\theta_\mu^2; D, N + \frac{\alpha}{2}\mu\theta_\mu, N).$$

This implies that the map is increasing in α, $\Lambda_1(\alpha,\mu) \to +\infty$ as $\alpha \to +\infty$ and $\Lambda_1(\alpha,\mu) \to \lambda_1$ as $\alpha \to 0$.

3. First, observe that $\Lambda(\alpha,\mu_1) = \lambda_1$. Let Φ_μ be the principal eigenfunction associated to $\Lambda(\alpha,\mu)$. Using now (3.11) we can write

$$\begin{aligned} \Phi_\mu &= \Phi_0 + \Phi_1(\mu-\mu_1) + O((\mu-\mu_1)^2), \\ \theta_\mu &= m_1\psi_1(\mu-\mu_1) + O((\mu-\mu_1)^2), \\ \Lambda(\alpha,\mu) &= \lambda_1 + \lambda_2(\mu-\mu_1) + O((\mu-\mu_1)^2). \end{aligned}$$

Again, we can easily check that Φ_0 is a eigenfunction associated to λ_1, that is $\Phi_0 = \varphi_1$, and that Φ_1 verifies

$$\begin{cases} -\Delta\Phi_1 + \alpha m_1 div(\varphi_1\nabla\psi_1) = \lambda_1\Phi_1 + \lambda_2\varphi_1 & \text{in } \Omega, \\ B_1\Phi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.21}$$

Multiplying by φ_1 and integrating, we get that

$$\lambda_2 \int_\Omega \varphi_1^2 = \alpha m_1 \left(\mu_1 \int_{\Gamma_2} \psi_1\varphi_1^2 - \frac{1}{2}\int_\Omega \nabla\psi_1 \cdot \nabla(\varphi_1^2) \right).$$

Finally, multiplying the equation of ψ_1 by φ_1^2 we have

$$\int_\Omega \nabla\psi_1 \cdot \nabla(\varphi_1^2) = \mu_1 \int_{\Gamma_2} \varphi_1^2\psi_1,$$

and hence

$$\lambda_2 = \mu_1 m_1 \frac{\alpha}{2} \frac{\displaystyle\int_{\Gamma_2} \varphi_1^2\psi_1}{\displaystyle\int_\Omega \varphi_1^2} > 0.$$

This proves (6.16).

The proof of (6.17) is more involved. For that, we use the equation (6.20). Denote

$$g_\mu(x) := \frac{\alpha^2}{4}|\nabla\theta_\mu|^2 + \frac{\alpha}{2}\Delta\theta_\mu = \frac{\alpha^2}{4}|\nabla\theta_\mu|^2 + \frac{\alpha}{2}\theta_\mu^2.$$

Take $\Omega_0 \Subset \Omega$, by Theorem 3.2 we know that $\theta_\mu \to z$ in $C^{2,\alpha}(\overline{\Omega}_0)$, where z is the solution of (3.14), and so $g_\mu(x) \le G(z)$ in Ω_0 for some function G. Then,

$$\Lambda(\alpha,\mu) = \lambda_1(-\Delta + g_\mu(x); D, N + \alpha\mu\theta_\mu, N) \le \lambda_1(-\Delta + g_\mu(x); D, D, N) \le \lambda_1^{\Omega_0}(-\Delta + G(z)),$$

where $\lambda_1^{\Omega_0}(-\Delta + G(z))$ represents the first eigenvalue of the operator $-\Delta + G(z)$ in Ω_0 and homogeneous Dirichlet boundary conditions, see for example Propositions 3.1 and 3.2 in [6].

Then, $\Lambda(\alpha,\mu)$ is bounded for all μ and so there exists Λ^* such that for a sub-sequence μ_n we have that

$$\Lambda(\alpha,\mu_n) \to \Lambda^* \quad \text{as } n \to +\infty.$$

Now, consider ψ_n the positive eigenfunction associated to $\Lambda(\alpha,\mu_n)$ such that $\|\psi_n\|_2 = 1$. Then,

$$\int_\Omega |\nabla\psi_n|^2 + \int_\Omega g_{\mu_n}(x)\psi_n^2 + \alpha\mu_n \int_{\Gamma_2} \theta_{\mu_n}\psi_n^2 = \Lambda(\alpha,\mu_n)\int_\Omega \psi_n^2 = \Lambda(\alpha,\mu_n), \tag{6.22}$$

and then ψ_n is bounded in $H^1(\Omega)$. So, we can conclude that there exists $\psi_0 \in H^1(\Omega)$ such that

$$\psi_n \rightharpoonup \psi_0 \quad \text{in } H^1(\Omega), \quad \psi_n \to \psi_0 \quad \text{in } L^2(\Omega),$$

with $\psi_0 \ge 0$ and non-trivial. On the other hand, for $\mu_n \ge \mu^* > \mu_1$ for μ^* fixed, we have

$$\alpha\mu_n \int_{\Gamma_2} \theta_{\mu^*}\psi_n^2 \le \alpha\mu_n \int_{\Gamma_2} \theta_{\mu_n}\psi_n^2 \le C$$

this last inequality by (6.22). Hence, as $\mu_n \to \infty$

$$\int_{\Gamma_2} \theta_{\mu^*}\psi_0^2 = 0$$

whence $\psi_0 = 0$ on Γ_2. Moreover, we can show that ψ_0 is a weak solution of

$$-\Delta\psi_0 + g_\alpha(z)\psi_0 = \Lambda^*\psi_0$$

being

$$g_\alpha(z) = \frac{\alpha^2}{4}|\nabla z|^2 + \frac{\alpha}{2}z^2.$$

Hence by Lemma 2.2 we get that $\Lambda^* = \lambda_1(-\Delta + g_\alpha(z); D, D, N)$. This completes the proof. □

In Figure 2 we have represented different cases of the regions of non-existence and existence of coexistence states of (1.1). The region denoted by B is the non-existence region (recall Remark 6.3). We have maintained this notation in the case of no chemotaxis, Figure 2 a).

With respect to the region of existence of coexistence state, this is delimited by the curves $\mu = F(\lambda)$ and $\lambda = \lambda_1$ when $\alpha = 0$, by $\mu = F(\lambda)$ and $\lambda = \Lambda(\alpha,\mu)$ when $\alpha > 0$ according to Proposition 5.1 and Theorem 6.8, respectively. Observe that we are not able to show if $\Lambda(\alpha,\mu)$ is monotone in μ. In any case we know the existence of its limit as $\mu \to +\infty$ and we can show different situations depending on the size of α.

In Figure 2 a) we have represented the case $\alpha = 0$, see Proposition 5.1.

In Figure 2 b) we have plotted the case when α is small, for example for

$$\alpha < \frac{2}{c\mu_1 n_1 m_1 k_1 l_1},$$

the curve $\mu = F(\lambda)$ is below that $\lambda = \Lambda(\alpha, \mu)$ in a neighbourhood of the point (λ_1, μ_1) by Proposition 6.9. If we assume that α is small enough, we have that $\mu = F(\lambda)$ is below that $\lambda = \Lambda(\alpha, \mu)$ in all the plane. In this case we are in Figure 2 b), and the existence region is denoted by A; in this region both semi-trivial solutions are unstable.

In Figures 2 c) and d) we have represented the existence region for α large, in the first case $\Lambda(\alpha, \mu)$ is not increasing in μ and in the second one is increasing.

Now, we have divided the existence region in $A \cup C$, being A where the semi-trivial solutions are unstable and C when they are stable. Of course, we have represented only the case in which the curves intersect one time, but several intersections could occur.

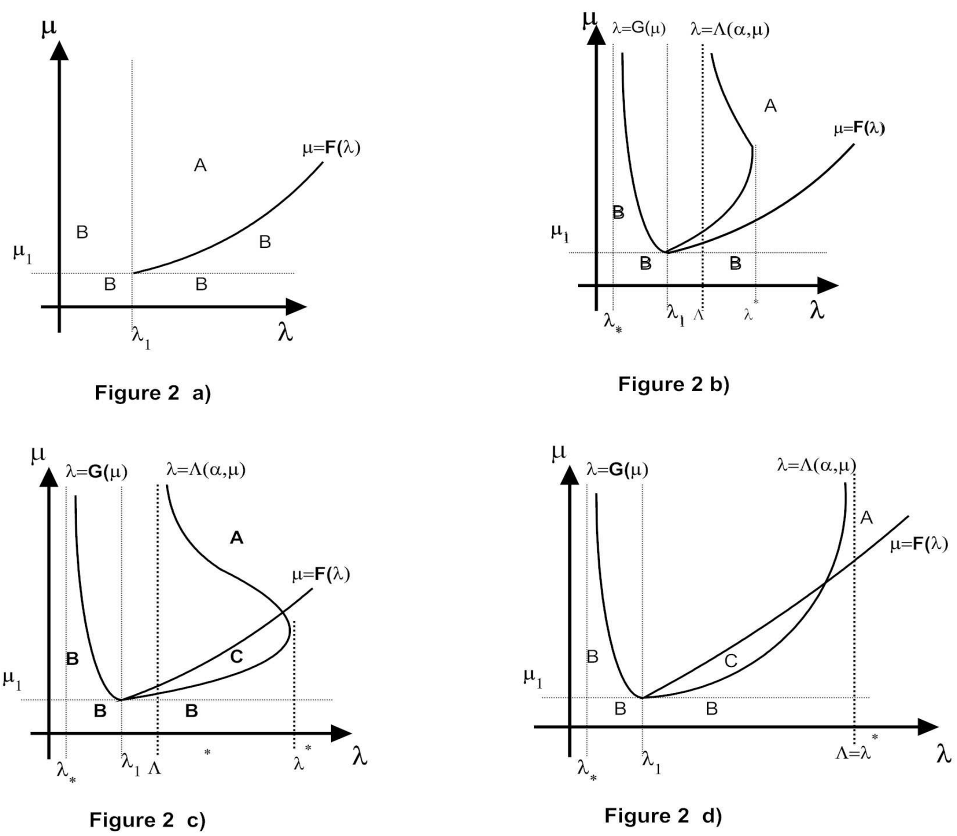

Figure 2: Regions of existence and non-existence of positive solutions of (1.1).

7 Conclusions

We have presented a model arising from angiogenesis where the CEs grow following a logistic law and move toward the TAF, appearing so a chemotaxis term. With respect to the TAF, they are consumed by the CEs and it appears a flux of TAF towards inside of the domain. The results obtained in this paper can be interpreted in different ways. We are focusing our attention in the stability semi-trivial solution $(0, \theta_\mu)$, that is, the solution where CEs disappear and so the angiogenesis does not occur.

First, we divide our conclusions depending if the chemotaxis is present.

Case 1: No chemotaxis: Assume that $\alpha = 0$ (see Figure 2 a)). By Proposition 5.1, if $\lambda \leq \lambda_1$ then $u \equiv 0$ and the semi-trivial solution $(0, \theta_\mu)$ is stable. So, if the growth rate of the CEs is small, they disappear independently of the TAF generated by the tumor.

However, if $\lambda > \lambda_1$ for all $\mu > F(\lambda)$ a stable coexistence state exists and so the angiogenesis occurs. Hence, if the growth rate of the CEs is large and the tumor segregates enough TAF, then angiogenesis occurs.

Case 2: With chemotaxis: Now, we introduce the chemotaxis, $\alpha > 0$. We need some notations (see Figures 2 b)-d)). We recall from Proposition 6.2 that if $\lambda \leq G(\mu)$, then $u \equiv 0$ and that $\lim_{\mu\to\infty} G(\mu) = \lambda_*(\alpha) \geq 0$.

Moreover, fixed $\lambda \in (\lambda_*(\alpha), \lambda_1)$ denote by μ_λ the number such that $\lambda = G(\mu_\lambda)$.

Also, we need to remember (6.17) and so define

$$\lim_{\mu\to+\infty} \Lambda(\alpha, \mu) = \lambda_1(-\Delta + \frac{\alpha^2}{4}|\nabla z|^2 + \frac{\alpha}{2}z^2; D, D, N) := \Lambda(\alpha).$$

Also denote by

$$\lambda^*(\alpha) := \sup_{\mu\geq\mu_1} \Lambda(\alpha, \mu),$$

Since $\alpha \mapsto \Lambda(\alpha, \mu)$ is increasing, it is evident that $\lambda_*(\alpha) < \lambda_1 < \Lambda(\alpha) \leq \lambda^*(\alpha)$.

Finally, fixed $\lambda > \lambda_1$ denote by $\mu^\lambda(\alpha) = \sup\{\mu : \Lambda(\alpha, \mu) = \lambda\}$.

Now, the behaviour of the system depends on the size of λ. We distinguish several cases:

- If $\lambda \leq \lambda_*(\alpha)$: then $u \equiv 0$ and $(0, \theta_\mu)$ is stable, angiogenesis does not occur.
- If $\lambda \in (\lambda_*(\alpha), \lambda_1)$: then if $\mu \leq \mu_\lambda$ we have that $u \equiv 0$, and if $\mu > \mu_\lambda$ there could exist a coexistence state, but in any case $(0, \theta_\mu)$ is stable.
- If $\lambda \in (\lambda_1, \Lambda(\alpha))$: in this case for $\mu > \mu^\lambda$ the solution $(0, \theta_\mu)$ is stable.
- If $\lambda \in (\Lambda(\alpha), \lambda^*(\alpha))$: see in this case Figures 2 b) and c). There exists a value $\mu_1(\lambda) < \mu^\lambda$ such that $\lambda = \Lambda(\alpha, \mu_1(\lambda))$. In the first case (Figure 2 b)) $\mu_1(\lambda) > F(\lambda)$ while in the second one (Figure 2 c)) $\mu_1(\lambda) < F(\lambda)$. In both cases, there exists at least coexistence state if

  $$\mu \in (\min\{\mu_1(\lambda), F(\lambda)\}, \max\{\mu_1(\lambda), F(\lambda)\}) \cup (\mu^\lambda, +\infty).$$

  If $\mu \in (\max\{\mu_1(\lambda), F(\lambda)\}, \mu^\lambda)$ then the semi-trivial solution $(0, \theta_\mu)$ is l. a. s. and for $\mu > \mu^\lambda$ large is unstable.
- If $\lambda > \lambda^*(\alpha)$: in this case $(0, \theta_\mu)$ is unstable, and there exists a coexistence state for $\mu > F(\lambda)$.

Roughly speaking, we have three kinds of behaviour:

1. When λ (the growth rate of the CEs) is small, then CEs tend to disappear independently of the value of μ.

2. When λ is bigger but not so much, then for μ large again CEs disappear. This could seem a little strange, because we could think that a bigger generation of concentration of TAF benefits the CEs, but when μ is large, there is a lot of TAF, and so there is a "saturation of movement" of CEs that produces that they bump into each other and so they disappear.

3. Finally, if λ is large, then both populations coexist for all μ large enough.

There is also an important change depending on the size of chemotaxis parameter. When α is small, small chemotaxis, (see Figure 2 b)) for $\mu \in (F(\lambda), \mu_1(\lambda)) \cup (\mu^\lambda, +\infty)$ there exists a coexistence state, and in this range both semi-trivial solutions are unstable and so the coexistence state is generically stable, however when α is large, big chemotaxis, (see Figures 2 c) and d)) in the range $\mu \in (\mu_1(\lambda), F(\lambda))$ both semi-trivial solutions are l. a. s., and so the coexistence state is generically unstable.

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