Bifurcations in non-autonomous scalar equations

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Abstract

In a previous paper we introduced various definitions of stability and instability for non-autonomous differential equations, and applied these to investigate the bifurcations in some simple models. In this paper we present a more systematic theory of local bifurcations in scalar non-autonomous equations.

Keywords: Non-autonomous differential equations, bifurcation theory, pullback attracting sets.

1 Introduction

In a previous paper (Langa, Robinson, & Suárez [15]) we introduced various definitions of stability and instability that seemed to be potentially useful in discussing the dynamics of the solutions of non-autonomous differential equations. In particular we applied these definitions to various simple model problems that exhibited non-autonomous versions of standard autonomous bifurcations: an explicitly solvable pitchfork bifurcation problem, a saddle-node type bifurcation, and a general n-dimensional 'loss of stability'.

In this paper we develop a more general theory, concentrating on the well-known 'local bifurcations' from the autonomous theory, and finding conditions for similar bifurcations in the scalar non-autonomous equation

$$\dot{x} = f(x, t, \lambda),$$

where λ is a parameter. By imposing conditions on the Taylor coefficients in the expansion of f near $x = \lambda = 0$ (which reduce to the standard conditions in the autonomous case) we are able to prove various general theorems guaranteeing transcritical, pitchfork, and saddle node bifurcations. Although we require a strong 'balance hypothesis' on the terms in the Taylor expansion, we believe that these results are a further step towards a general non-autonomous theory of bifurcations. We do not present any concrete examples here, instead concentrating on the development of an abstract theory which we believe should be applicable to a wide variety of particular models.

Some particular examples have been analysed in various settings: using the framework of skew product flows Johnson [8] and Johnson and Yi [9] have considered a generalised notion of a Hopf bifurcation; Shen and Yi [19] treat almost periodic scalar differential equations (but leave bifurcation phenomena largely untouched); more recently Kloeden [12] has analysed transcritical and pitchfork bifurcations in an almost periodic equation; Johnson, Kloeden, & Pavani [10] have considered a non-autonomous 'two step bifurcation'; and Kloeden & Siegmund [14] give a nice discussion of the general problem in the context of skew product flows.

In this paper we do not adopt the skew product approach and the restrictions on the generality of f that it would entail, preferring to use the language of processes.

2 Non-autonomous equations as processes

For the solution of any non-autonomous equation

$$\dot{x} = f(x,t) \qquad x(s) = x_0 \qquad \text{with} \qquad x \in \mathbb{R}^m \tag{2.1}$$

the initial time (s) is as important as the final time (t). In order to treat these equations as dynamical systems we consider a family of solution operators $\{S(t,s)\}_{t \geq s}$ (termed a "process", see Dafermos [6] or Sell [18]) that depend on both the final and initial times. We can then denote the solution of (2.1) at time t by $S(t,s)x_0$. If f is sufficiently smooth (which it will be in all that follows) then it is clear that $S(t,s) : \mathbb{R}^m \to \mathbb{R}^m$ must satisfy

a) $S(t,t)$ is the identity for all $t \in \mathbb{R}$,

b) $S(t,\tau)S(\tau,s) = S(t,s)$ for all t, τ, and $s \in \mathbb{R}$, and

c) $S(t,s)x_0$ is continuous in t, s, and x_0.

There may in fact be solutions of (2.1) that do not exist for all time, and some restrictions to the possible values of s and t may be necessary, giving rise to only a 'local process'. Although we pass over them here, we will deal with such technicalities where necessary in what follows.

Since in this paper we will only treat *scalar* equations with unique solutions both forwards and backwards in time, the resulting process will be order-preserving, i.e.

$$x_s > y_s \qquad \Rightarrow \qquad S(t,s)x_s > S(t,s)y_s \qquad \text{for all} \qquad t,s \in \mathbb{R}$$

(allowing $S(t,s)x_s$ or $S(t,s)y_s$ to be $\pm\infty$ if necessary allows us to take values of t and s from all of $\mathbb{R}$).

3 Stability & instability in non-autonomous systems

We now recall some of the definitions from Langa et al. [15] which we will use in our bifurcation analysis. The simple notion of a complete trajectory will be central:

Definition 1 *The continuous map $x : \mathbb{R} \to \mathbb{R}^m$ is a* complete trajectory *if*

$$S(t,s)x(s) = x(t) \qquad \textit{for all} \qquad t,s \in \mathbb{R}.$$

We will investigate the appearance and disappearance of complete trajectories that are 'stable' or 'unstable' in certain senses that appear to be appropriate for non-autonomous systems. Note that complete trajectories are merely particular examples of *invariant sets* in non-autonomous systems:

Definition 2 *A time-varying family of sets* $\{\Sigma(t)\}_{t\in\mathbb{R}}$ *is* invariant *(we say "*$\Sigma(\cdot)$ *is invariant") if*

$$S(t,s)\Sigma(s) = \Sigma(t) \qquad \textit{for all} \qquad t,s \in \mathbb{R}.$$

In what follows we make constant use of the Hausdorff semidistance between two sets A and B, $\text{dist}[A,B]$, which is defined as

$$\text{dist}[A,B] = \sup_{a\in A}\inf_{b\in B} d(a,b):$$

note that this only measures how far A is from B ($\text{dist}[A,B]=0$ only implies that $A\subseteq B$). We also use the notation $N(X,\epsilon)$ to denote the closed ϵ-neighbourhood of a set X:

$$N(X,\epsilon) = \{y:\ y = x+z,\ x\in X,\ z\in\mathbb{R}^m \text{ with } |z|\le\epsilon\}.$$

3.1 Notions of attraction

First we define formally the familiar notion of a set that is attracting forwards in time, with a specified domain of attraction D. For any choice of D we say that $\Sigma(\cdot)\subset D$ if $\Sigma(t)\subset D$ for every $t\in\mathbb{R}$.

Definition 3 *An invariant set* $\Sigma(\cdot)$ *is* forwards attracting within D *if* $\Sigma(\cdot)\subset D$ *and for each* $s\in\mathbb{R}$

$$\lim_{t\to\infty}\text{dist}[S(t,s)K,\Sigma(t)] = 0$$

for all compact subsets[1] K *of* D.

In a non-autonomous system the notion of being 'locally forwards attracting' is a little more subtle; we allow the neighbourhood of Σ that is attracted to depend on the initial time. It is clear that if $\Sigma(\cdot)$ is forwards attracting within D then it is also locally forwards attracting within D.

[1] Note that the definition implies attraction of every initial condition in K at a uniform rate. Our definition in Langa et al. [15] only required convergence for each fixed initial condition. Contrary to the statement in the footnote in that paper, the two definitions are most certainly *not* equivalent, even for finite-dimensional systems.

Definition 4 *An invariant set* $\Sigma(\cdot)$ *is* locally forwards attracting within D *if* $\Sigma(\cdot) \subset D$ *and for each* $s \in \mathbb{R}$ *there exists a* $\delta(s)$ *such that*

$$\lim_{t \to \infty} \text{dist}[S(t,s)K, \Sigma(t)] = 0$$

for all compact $K \subset N(\Sigma(s), \delta(s)) \cap D$.

We now introduce the notion of pullback attraction

Definition 5 *An invariant set* $\Sigma(\cdot)$ *is* pullback attracting within D *if* $\Sigma(\cdot) \subset D$ *and for every* $t \in \mathbb{R}$ *and every compact set* $K \subset D$,

$$\lim_{s \to -\infty} \text{dist}[S(t,s)K, \Sigma(t)] = 0.$$

$\Sigma(\cdot)$ *is* globally pullback attracting *if we can take* $D = \mathbb{R}^m$.

For a set $\Sigma(\cdot)$ to be locally pullback attracting, the neighbourhood of $\Sigma(\cdot)$ that is attracted can depend only on the final time. Note that the definition allows a different collection of compact sets $K(\cdot)$ to be attracted to $\Sigma(t)$ for each fixed $t \in \mathbb{R}$.

Definition 6 *We say that* $\Sigma(\cdot)$ *is* locally pullback attracting within D *if* $\Sigma(\cdot) \subset D$ *and for every* $t \in \mathbb{R}$ *there exists a* $\delta(t) > 0$ *such that if* $K(\cdot) \subset D$ *is compact and*

$$\lim_{s \to -\infty} \text{dist}[K(s), \Sigma(s)] < \delta(t)$$

then

$$\lim_{s \to -\infty} \text{dist}[S(t,s)K(s), \Sigma(t)] = 0. \tag{3.1}$$

If D is bounded it is once again clear that any set that is pullback attracting within D is locally pullback attracting within D. However, it is an uncomfortable consequence of our definitions that a set can be globally pullback attracting but not locally pullback attracting if D is unbounded. Nevertheless, this cannot occur if the set is 'bounded in the past', as shown by the following lemma.

Lemma 1 *If an invariant set* $\Sigma(\cdot)$ *is pullback attracting within* D *and bounded 'in the past', i.e.*

$$\bigcup_{t<T} \Sigma(t)$$

is bounded for some T, *then* $\Sigma(\cdot)$ *is locally pullback attracting.*

Proof. We show that $\Sigma(\cdot)$ is locally pullback attracting for any choice of constant δ (this was called 'uniformly pullback attracting' in Langa et al. [15]). If

$$\lim_{s\to-\infty} \operatorname{dist}[K(s),\Sigma(s)] < \delta$$

then for some τ, which we choose to be less than T, we must have $\operatorname{dist}[K(s),\Sigma(s)] < 2\delta$ for all $s < \tau$. Since $\Sigma(s)$ is bounded for $s < T$, all such $K(s)$ are contained in a bounded set X_δ.

Since Σ is globally pullback attracting, this bounded set is (pullback) attracted to Σ: there exists a σ such that

$$\operatorname{dist}[S(t,s)X_\delta,\Sigma(t)] < \epsilon \qquad \text{for all} \qquad s \leq \sigma.$$

Since $K(s) \subset X_\delta$ for all $s < T$, it follows that

$$\operatorname{dist}[S(t,s)K(s),\Sigma(t)] < \epsilon \qquad \text{for all} \qquad s \leq \sigma,$$

and so Σ is locally pullback attracting. □

3.2 Stability

We now give a definition of 'stability' in the pullback sense.

Definition 7 $\Sigma(\cdot)$ *is* pullback Lyapunov stable *if for every $t \in \mathbb{R}$ and $\epsilon > 0$ there exists a $\delta(t) > 0$ such that for any $s < t$, $x_s \in N(\Sigma(s),\delta(t))$ implies that $S(t,s)x_s \in N(\Sigma(t),\epsilon)$.*

The following result, analogous to the fact that attraction implies stability for stationary points of scalar autonomous systems, means that in what follows we need not be concerned with Lyapunov stability properties of complete trajectories, but only their attraction properties.

Lemma 2 *Let $x^*(\cdot)$ be a complete trajectory in a non-autonomous scalar ODE that is locally pullback attracting; then this trajectory is also pullback Lyapunov stable.*

Proof. Fix $t \in \mathbb{R}$. Given an $\epsilon > 0$, we can guarantee that if $x_\pm(s) = x^*(s) \pm \frac{1}{2}\delta(t)$ then

$$\lim_{s\to-\infty} |S(t,s)x_\pm(s) - x^*(t)| = 0,$$

and so in particular there exists a σ such that

$$|S(t,s)x_\pm(s) - x^*(t)| < \epsilon \qquad \text{for all} \qquad s \leq \sigma.$$

Since the system is order preserving

$$|x_s - x^*(s)| < \frac{\delta(t)}{2} \qquad \Rightarrow \qquad |S(t,s)x_s - x^*(t)| < \epsilon \qquad \text{for all} \qquad s \leq \sigma.$$

Now we can use the continuous dependence on initial conditions for $s \in [\sigma, t]$, along with the invariance of $x^*(\cdot)$, to guarantee that for $\delta_\sigma < \delta(t)$ and sufficiently small

$$|x_s - x^*(s)| < \delta_\sigma \qquad \Rightarrow \qquad |S(t,s)x_s - x^*(t)| < \epsilon \qquad \text{for all} \qquad \sigma \leq s \leq t.$$

Thus $x^*(\cdot)$ is pullback Lyapunov stable. □

We note here that Kloeden [11] has shown that one can generalise the classical notion of a Lyapunov function to cover many non-autonomous systems in such a way that there is a Lyapunov function associated with any pullback attracting set. In particular his results imply the existence of a Lyapunov function for a bounded locally pullback attracting trajectory for the equation $\dot{x} = f(x,t)$ provided that $f(x,t)$ is locally Lipschitz in x.

3.3 Notions of instability

In Langa et al. [15] we introduced two notions of instability. One is simply the converse of Lyapunov stability, while the other, stronger, property appears to be more useful.

Definition 8 *We say that $\Sigma(\cdot)$ is* pullback unstable *if it is not pullback Lyapunov stable, i.e. if there exists a $t \in \mathbb{R}$ and an $\epsilon > 0$ such that, for each $\delta > 0$, there exists an $s < t$ and an $x_0 \in N(\Sigma(s), \delta)$ such that*

$$\text{dist}[S(t,s)x_0, \Sigma(t)] > \epsilon.$$

We say that $\Sigma(\cdot)$ is 'asymptotically unstable' if its unstable set $U_\Sigma(\cdot)$, defined below (cf. Crauel [4]), is non-trivial (i.e. if $U_\Sigma(t) \neq \Sigma(t)$).

Definition 9 *If $\Sigma(\cdot)$ is an invariant set then the unstable set of Σ, $U_\Sigma(\cdot)$, is defined as*

$$U_\Sigma(s) = \{x_0 : \lim_{t \to -\infty} \text{dist}\,[S(t,s)x_0, \Sigma(t)] = 0\}.$$

We say that $\Sigma(\cdot)$ is asymptotically unstable *if for some t we have*

$$U_\Sigma(t) \neq \Sigma(t). \tag{3.2}$$

The power of this definition comes from the following simple result (see Langa et al. [15] for the proof).

Proposition 3 *If $\Sigma(\cdot)$ is asymptotically unstable then it is also pullback unstable and cannot be locally pullback attracting.*

Most notions of instability are related to the behaviour of solutions $x(t)$ as $t \to -\infty$; the notion of 'asymptotic instability' defined above is essentially a time-reversed notion of 'forwards attraction'. It should therefore be unsurprising that it is possible to define an alternative notion of instability based on a time-reversed version of pullback attraction:

Definition 10 *An invariant set $\Sigma(\cdot)$ is* (locally) pullback repelling within D *if it is (locally) pullback attracting within D for the time-reversed system, i.e. if $\Sigma(\cdot) \subset D$ and for any compact set $K \subset D$ and for each $t \in \mathbb{R}$,*

$$\lim_{s \to +\infty} \text{dist}[S(t,s)K, \Sigma(t)] = 0.$$

3.4 An aside: linear stability in non-autonomous systems

We mention here that we make little use of linear notions of stability in this paper. There appear to be major problems with deducing anything from such 'infinitesimal' behaviour without further constraints. As an example, consider the equation

$$\dot{x} = x - \frac{\mathrm{e}^{-t}}{1+t^2} x^2,$$

whose solution can be given explicitly as

$$x(t,s;x_s) = \frac{\mathrm{e}^t}{\mathrm{e}^s x_s^{-1} + \tan^{-1}(t) - \tan^{-1}(s)}.$$

It is clear that if x_s is fixed then as $s \to -\infty$

$$x(t,s;x_s) \to x^*(t) = \frac{\mathrm{e}^t}{\tan^{-1}(t) + \pi/2}.$$

The trajectory $x^*(t)$ is globally pullback attracting, and also, since it is bounded as $t \to -\infty$, locally pullback attracting (Lemma 1). Since we are treating a scalar equation, the trajectory is also pullback Lyapunov stable (Lemma 2). However, suppose that we linearise about $x^*(t)$, and obtain

$$\begin{aligned} \dot{X} &= \left[1 - 2\frac{\mathrm{e}^{-t}}{1+t^2} x^*(t)\right] X \\ &= \left[1 - \frac{2}{(1+t^2)(\tan^{-1}(t) + \pi/2)}\right] X \end{aligned}$$

Therefore

$$\begin{aligned} X(t,s;X_0) &= \exp\left(\int_s^t 1 - \frac{2}{(1+r^2)(\tan^{-1}(r)+\pi/2)}\,\mathrm{d}r\right) X_0 \\ &= \exp\left((t-s) - 2\ln[\tan^{-1}(r)+\pi/2]_{r=s}^{t}\right) X_0 \\ &= \mathrm{e}^{t-s}\left(\frac{\tan^{-1}(s)+\pi/2}{\tan^{-1}(t)+\pi/2}\right)^2 X_0. \end{aligned}$$

Now, as $s \to -\infty$ we have $|X(t,s;X_0)| \to \infty$, so that $x^*(t)$ is pullback linearly *unstable.*

4 Pullback attractors

The use of the pullback notion in the above definitions was inspired by the theory of pullback attractors (Cheban *et al.*, [2]; Crauel, Debussche, & Flandoli [5]; Kloeden & Schmalfuß, [13]; Schmalfuß, [17]; Chepyzhov & Vishik, [3]). Although such attractors are not central to our approach here, they will be a useful tool.

Definition 11 *An invariant set* $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ *is said to be the* pullback attractor *of the process* S *within* D *if it is*

a) a compact subset of D *for each* $t \in \mathbb{R}$*,*

b) pullback attracting within D *(in the sense of Definition 5), and*

c) minimal in the sense that if $\{C(t)\}_{t\in\mathbb{R}}$ *is another family of closed sets that are pullback attracting within* D *then* $\mathcal{A}(t) \subset C(t)$ *for all* $t \in \mathbb{R}$*.*

The condition required to guarantee the existence of such a pullback attractor is simple (see Crauel *et al.*, [5]; Schmalfuß, [17]). The following theorem also provides some information of the structure of the attractor for scalar systems (for a result valid for more general order-preserving systems, see Langa & Suárez, [16]).

Theorem 4 *Assume that there exists a family of compact pullback absorbing sets, i.e. a family* $\{K(t)\}_{t\in\mathbb{R}}$ *of nonempty compact sets such that for each* t_0 *and each compact set* $B \subset D$ *there exists a* $T = T(t_0, B)$ *such that*

$$S(t_0,s)B \subset K(t_0) \qquad \textit{for all} \qquad s \le T.$$

Then there is a pullback attractor $\mathcal{A}(t)$ within D, which is a connected set for each $t \in \mathbb{R}$. If $S(t,s)$ arises from a scalar ODE then

$$\mathcal{A}(t) = [a_-(t), a_+(t)],$$

and $a_\pm(t)$ are complete trajectories.

Proof. The proof of existence of an attractor is standard, as is its connectedness (see Crauel et al. [5], for example) so we only prove the final part of the theorem here. First, it is clear that since $\mathcal{A}(t)$ is a compact connected set for each t then it must be an interval $[a_-(t), a_+(t)]$; it only remains to show that $a_\pm(t)$ are complete trajectories, i.e. that

$$S(t,s)a_+(s) = a_+(t)$$

(and similarly for $a_-(\cdot)$). Since $\mathcal{A}(t)$ is invariant, we must have

$$a_-(t) \le S(t,s)a_+(s) \le a_+(t).$$

Suppose that $S(t,s)a_+(s) < a_+(t)$; then applying $S(s,t)$ (which is order-preserving) to both sides we obtain $a_+(s) < S(s,t)a_+(t)$. Since $\mathcal{A}(t)$ is invariant, it follows that $S(s,t)a_+(t) \in \mathcal{A}(s)$, and so $a_+(s) < S(s,t)a_+(t) \le a_+(s)$, a contradiction. So $S(t,s)a_+(s) = a_+(t)$ and $a_+(\cdot)$ is a complete trajectory as claimed. A similar argument shows that $a_-(\cdot)$ is also a complete trajectory. □

5 Non-autonomous transcritical bifurcation

The standard autonomous example of an equation exhibiting a transcritical bifurcation is

$$\dot{x} = \lambda x - x^2.$$

For $\lambda < 0$ the origin is locally stable and there is an unstable negative fixed point at $x = \lambda < 0$; when $\lambda > 0$ the stability is swapped, with the origin becoming unstable and the fixed point at $x = \lambda > 0$ becoming stable.

Our analysis of the general non-autonomous problem will be heavily based on the explicitly solvable model

$$\dot{x} = \lambda f(t)x - g(t)x^2 \qquad x(s) = x_s \tag{5.1}$$

which we treat in Section 5.1. We then move on to the more general situation, with our assumptions motivated by the explicit model. We delay a formal definition of a 'transcritical bifurcation' in a non-autonomous system until after our more informal discussion of (5.1).

5.1 An explicitly solvable model

First we treat the model equation

$$\dot{x} = \lambda f(t)x - g(t)x^2 \qquad x(s) = x_s, \tag{5.2}$$

which has the explicit solution

$$x(t,s;x_s) = \frac{e^{\lambda F(t)}}{x_s^{-1}e^{\lambda F(s)} + \int_s^t e^{\lambda F(r)} g(r)\,dr}, \tag{5.3}$$

where F is any anti-derivative of f. Despite the simplicity of the model, and the fact that it can be solved explicitly, we will need to impose a number of conditions to ensure the transcritical behaviour we require.

First, we assume that f and g are 'essentially positive',

$$\int_{-\infty}^t f(s)\,ds = \int_{-\infty}^t g(s)\,ds = +\infty, \tag{5.4}$$

which in particular implies that $\lim_{s\to-\infty} F(s) = -\infty$. Under this condition if we fix t and let $s \to -\infty$ in (5.3) then for $\lambda > 0$ and any $x_s \in \mathbb{R}$ we have

$$\lim_{s\to-\infty} x(t,s;x_s) = x_\lambda(t) := \frac{e^{\lambda F(t)}}{\int_{-\infty}^t e^{\lambda F(r)} g(r)\,dr}. \tag{5.5}$$

We naturally impose the positivity of this candidate pullback attracting trajectory, and assume that it is uniformly bounded above: $0 < x_\lambda(t) \le M_\lambda$ for all $t \in \mathbb{R}$. Note that the lower bound is equivalent to the assumption that $\int_{-\infty}^t e^{\lambda F(r)} g(r)\,dr > 0$. (This is consistent with, but does not follow from, the assumption of the essential positivity of g.)

Although it appears at first that $x_\lambda(\cdot)$ is pullback attracting, we must also be able to guarantee that for every $t \in \mathbb{R}$ and for $s \le \sigma_t$ (for some σ_t) the solution $x(\tau,s;x_s)$ exists for all $s \le \tau \le t$. Indeed, while the limit in (5.5) is independent of x_s, it is clear that for every fixed $x_s < 0$ if s sufficiently large and negative then $x(\tau,s;x_s)$ will blow up for some $\tau \ge s$ (while $x_s^{-1}e^{\lambda F(s)}$ is negative and tends to zero as $s \to -\infty$, the integral term in the denominator of (5.3) is positive and bounded below). When $x_s > 0$, to ensure that the solution exists on the interval $[s,t]$ we need

$$x_s^{-1}e^{\lambda F(s)} + \int_s^\tau e^{\lambda F(r)} g(r)\,dr > 0 \qquad \text{for all} \qquad \tau \in [s,t]. \tag{5.6}$$

While this holds if we allow x_s to depend on time and require $x_s < x_\lambda(s)$ (which implies that $x_\lambda(\cdot)$ is pullback attracting 'from below') the essential positivity of g alone is not sufficient to guarantee the existence of solutions that start 'above' $x_\lambda(\cdot)$.

Requiring that for some $\alpha_t > 0$ any solution with $x_s < (1+\alpha_t)x_\lambda(s)$ exists on $[s,t]$ is equivalent (by rearrangement of (5.6)) to the requirement that

$$\int_{-\infty}^{\tau} \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r > \frac{\alpha_t}{1+\alpha_t}\int_{-\infty}^{s} \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r \tag{5.7}$$

for all $s \leq \tau \leq t$. The most natural way to ensure this seems to be to require that g is asymptotically positive as $t \to -\infty$, i.e. $g(t) \geq \gamma^- > 0$ for all $t \leq T^-$, for some $T^- \in \mathbb{R}$. One can then take $s \leq T^-$ and it then suffices to show that (5.7) holds for τ in the bounded interval $[T^-, t]$, which can easily be done by choosing $\alpha_t > 0$ appropriately.

In order to ensure that $x_\lambda(\cdot)$ is locally pullback attracting we require in addition that x_λ is bounded uniformly away from zero: $x_\lambda(t) \geq m_\lambda > 0$ for all $t \in \mathbb{R}$. It then follows that we can apply Definition 6 with $\delta(t) = \alpha_t m_\lambda$.

When $\lambda < 0$ the essential positivity of f and the asymptotic positivity of g combine to ensure that

$$\left| x_s^{-1}\mathrm{e}^{\lambda F(s)} + \int_s^t \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r \right| \to \infty$$

as $s \to -\infty$, which implies that $\lim_{s\to-\infty} x(t,s;x_s) = 0$. However, we again have to ensure that the solution $x(\tau,s;x_s)$ exists for all $\tau \in [s,t]$. Considering the case $x_s < 0$ this requires

$$|x_s|^{-1}\mathrm{e}^{\lambda F(s)} > \int_s^{\tau} \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r.$$

This should hold for all s sufficiently large and negative, and so in particular we must have

$$\liminf_{s\to-\infty} \frac{\mathrm{e}^{\lambda F(s)}}{\int_s^{\tau} \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r} > 0$$

(note that the left-hand side of this expression does not depend on τ). We show in the proof of the following result that this is in fact sufficient to obtain local pullback attraction to the origin.

Proposition 5 *Consider the equation*

$$\dot{x} = \lambda f(t)x - g(t)x^2. \tag{5.8}$$

Suppose that f is essentially positive,

$$\int_{-\infty}^{t} f(s)\,\mathrm{d}s = +\infty \qquad \text{for all} \qquad t \in \mathbb{R}, \tag{5.9}$$

g is asymptotically positive, i.e. there exists a T^- such that

$$g(t) \geq \gamma^- > 0 \qquad \text{for all} \qquad t \leq T^-,$$

and that there exists a $\lambda_0 > 0$ *such that the 'balance conditions'*

$$0 < m_\lambda \leq x_\lambda(t) = \frac{\mathrm{e}^{\lambda F(t)}}{\int_{-\infty}^{t} \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r} \leq M_\lambda \qquad \text{for all} \qquad t \in \mathbb{R},\ 0 < \lambda < \lambda_0, \tag{5.10}$$

and

$$\liminf_{s \to -\infty} \frac{\mathrm{e}^{\lambda F(s)}}{\int_{s}^{t} \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r} \geq m_\lambda > 0 \qquad \text{for all} \qquad -\lambda_0 < \lambda < 0 \tag{5.11}$$

hold, where F *is any anti-derivative of* f. *Then for* $-\lambda_0 < \lambda < 0$ *the zero solution is locally pullback attracting in* $\mathbb{R}$*; for* $\lambda = 0$ *the origin is asymptotically unstable but still locally pullback attracting in* $\mathbb{R}^+$*; and for* $0 < \lambda < \lambda_0$ *the origin is asymptotically unstable and the trajectory* $x_\lambda(t)$ *is locally pullback attracting. In addition for each* t *we have* $x_\lambda(t) \to 0$ *as* $\lambda \to 0$.

Furthermore if there exists a T^+ *such that*

$$g(t) \geq \gamma^+ > 0 \qquad \text{for all} \qquad t \geq T^+, \tag{5.12}$$

and

$$\int_{t}^{\infty} f(s)\,\mathrm{d}s = +\infty, \tag{5.13}$$

then for $-\lambda_0 < \lambda < 0$ *the origin is locally forwards attracting, and for* $0 < \lambda < \lambda_0$ *the trajectory* $x_\lambda(\cdot)$ *is locally forwards attracting. Assuming in addition that*

$$0 < m_\lambda \leq x_\lambda(t) = \frac{\mathrm{e}^{\lambda F(t)}}{\int_{t}^{\infty} \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r} \leq M_\lambda \qquad \text{for all} \qquad t \in \mathbb{R},\ \lambda < 0 \tag{5.14}$$

then for $-\lambda_0 < \lambda < 0$ *the trajectory* $x_\lambda(t)$ *is both asymptotically unstable and locally pullback repelling. Once again, for each* $t \in \mathbb{R}$ *we have* $x_\lambda(t) \to 0$ *as* $\lambda \to 0$.

Proof. *When* $\lambda < 0$. As remarked above we have $x(t, s; x_s) \to 0$ for any $x_s \neq 0$, but we must also guarantee that the solution $x(\tau, s; x_s)$ exists for all $\tau \in (s, t)$.

For $x_s > 0$ we need to ensure that

$$x_s^{-1} \mathrm{e}^{\lambda F(s)} + \int_{s}^{\tau} \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r > 0 \qquad \text{for all} \qquad s \leq \tau \leq t. \tag{5.15}$$

Since there exists a T^- such that $g(r) \geq \gamma > 0$ for all $r \leq T^-$, (5.15) is assured provided that

$$x_s^{-1} \mathrm{e}^{\lambda F(s)} + \int_{T^-}^{\tau} \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r > 0 \qquad \text{for all} \qquad T^- \leq \tau \leq t.$$

Since $\lambda < 0$ and $F(s) \to -\infty$ as $s \to -\infty$, for s small enough it is certainly true that $\mathrm{e}^{\lambda F(s)}$ is bounded below on $(-\infty, T^-]$. Thus (5.15) follows provided that we choose

$$x_s < \frac{\inf_{s \le T^-} \mathrm{e}^{\lambda F(s)}}{\sup_{\tau \in [T^-, t]} \left| \int_{T^-}^{\tau} \mathrm{e}^{\lambda F(r)} g(r) \, \mathrm{d}r \right|}$$

(note that the right-hand side depends only on t).

For $x_s < 0$ the argument is a little more involved, and requires the balance condition (5.11). First we note that the asymptotic positivity of g implies that there exists a T_t such that

$$\int_{\tau}^{t} \mathrm{e}^{\lambda F(r)} g(r) \, \mathrm{d}r > 0 \qquad \text{for all} \qquad \tau \le T_t,$$

Given this T_t, it follows from (5.11) and the fact that $\mathrm{e}^{\lambda F(s)} \to \infty$ as $s \to -\infty$ that there exists a σ_t such that

$$\frac{\mathrm{e}^{\lambda F(s)}}{\int_s^t \mathrm{e}^{\lambda F(r)} g(r) \, \mathrm{d}r} \ge \frac{m_\lambda}{2} \tag{5.16}$$

and

$$\frac{\mathrm{e}^{\lambda F(s)}}{\int_s^{T_t} \mathrm{e}^{\lambda F(r)} g(r) \, \mathrm{d}r + \inf_{\tau \in [T_t, t]} \int_{T_t}^{\tau} \mathrm{e}^{\lambda F(r)} g(r) \, \mathrm{d}r} \ge \frac{m_\lambda}{2} \tag{5.17}$$

for all $s \le \sigma_t$: in that follows we will take $s \le \sigma_t$. We now require that the denominator in (5.3) is negative, i.e. that

$$x_s^{-1} \mathrm{e}^{\lambda F(s)} + \underbrace{\int_s^{\tau} \mathrm{e}^{\lambda F(r)} g(r) \, \mathrm{d}r}_{I(s,\tau)} < 0 \qquad \text{for all} \qquad s \le \tau \le t. \tag{5.18}$$

We consider three cases. (i) If $I(s, \tau) < 0$ then clearly (5.18) is satisfied. (ii) If $I(s, \tau) > 0$ and $\tau \le T_t$ then $I(\tau, t) > 0$ and

$$x_s^{-1} \mathrm{e}^{\lambda F(s)} + I(s, \tau) < x_s^{-1} \mathrm{e}^{\lambda F(s)} + I(s, \tau) + I(\tau, t) = x_s^{-1} \mathrm{e}^{\lambda F(s)} + I(s, t).$$

For $x_s^{-1} \mathrm{e}^{\lambda F(s)} + I(s, t)$ to be negative we require

$$|x_s| < \frac{\mathrm{e}^{\lambda F(s)}}{\int_s^t \mathrm{e}^{\lambda F(r)} g(r) \, \mathrm{d}r},$$

but the right-hand side of this expression is bounded below by $m_\lambda/2$ using (5.16). (iii) If $I(s, \tau) > 0$ and $T_t < \tau \le t$ then we require

$$|x_s| < \frac{\mathrm{e}^{\lambda F(s)}}{\int_s^{T_t} \mathrm{e}^{\lambda F(r)} g(r) \, \mathrm{d}r + \int_{T_t}^{\tau} \mathrm{e}^{\lambda F(r)} g(r) \, \mathrm{d}r},$$

and once again the right-hand side is bounded below by $m_\lambda/2$, this time using (5.17).

Thus for each fixed t there exists a σ_t such that if $s \leq \sigma_t$ and $|x_s|$ is sufficiently small the solution exists on $[s,t]$ and hence the origin is locally pullback attracting.

When $\lambda = 0$. When $\lambda = 0$ the explicit solution is

$$x(t) = \frac{1}{x_s^{-1} + \int_s^t g(r)\,\mathrm{d}r}, \tag{5.19}$$

and for $x_s > 0$ it follows from the asymptotic positivity of g and a simplified version of the above argument that the origin is locally pullback attracting in $\mathbb{R}_+$; and that for $x_s < 0$ but sufficiently small (depending on s), $x(t,s;x_s) \to 0$ as $t \to -\infty$, and so the origin is asymptotically unstable.

When $\lambda > 0$. This case was treated before the formal statement of the proposition. Only the asymptotic instability of the origin and the convergence of x_λ to zero remain.

We deal first with the asymptotic instability of the origin. Since $x(t) \equiv 0$ and $x_\lambda(\cdot)$ are solutions and the equation is order-preserving, any solution with $0 < x_s < x_\lambda(s)$ exists for all $t \leq s$. Since $0 < \int_{-\infty}^s \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r < +\infty$ and $\mathrm{e}^{\lambda F(t)} \to 0$ as $t \to -\infty$ it follows that for such a solution $x(t,s;x_s) \to 0$ as $t \to -\infty$.

To show that $x_\lambda(t) \to 0$ as $\lambda \to 0$, fix t and $\epsilon > 0$. Choose T such that

$$\int_T^t g(r)\,\mathrm{d}r > 2\mathrm{e}^{\lambda F(t)}/\epsilon$$

(which is possible since g is asymptotically positive). Then

$$\int_{-\infty}^t \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r = \int_{-\infty}^T \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r + \int_T^t \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r > \int_T^t \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r.$$

Now, choose λ sufficiently small that

$$\sup_{r\in[T,t]} |\mathrm{e}^{\lambda F(r)} - 1| < \frac{\mathrm{e}^{\lambda F(t)}}{\epsilon \int_T^t |g(r)|\,\mathrm{d}r},$$

and then

$$\int_{-\infty}^t \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r > \mathrm{e}^{\lambda F(t)}/\epsilon,$$

which implies that $x_\lambda(t) < \epsilon$.

Including the extra 'forwards' conditions in (5.13) and (5.12), when $\lambda < 0$ the origin is locally forwards attracting when x_s is sufficiently small, since (5.12) guarantees that

$$\inf_{t\geq s} \int_s^t \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r > -\infty.$$

When $\lambda = 0$ the origin becomes locally forwards attracting. When $\lambda > 0$ the trajectory $x_\lambda(\cdot)$ is now locally forwards attracting: to show this we can rearrange the explicit solution into the alternative form

$$\left(\frac{1}{x(t)} - \frac{1}{x_\lambda(t)}\right) = \mathrm{e}^{\lambda(F(s)-F(t))}\left(\frac{1}{x_s} - \frac{1}{x_\lambda(s)}\right). \tag{5.20}$$

Therefore

$$|x(t) - x_\lambda(t)| = \frac{x_\lambda(t)x(t)}{\mathrm{e}^{\lambda F(t)}}\,\frac{\mathrm{e}^{\lambda F(s)}}{x_\lambda(s)x_s}\,|x_\lambda(s) - x_s|. \tag{5.21}$$

The balance condition in (5.10) implies that *any* solution with $x_s > 0$ is bounded as $t \to +\infty$. To see this, consider

$$\begin{aligned}\frac{\mathrm{e}^{\lambda F(t)}}{x_s^{-1}\mathrm{e}^{\lambda F(s)} + \int_s^t \mathrm{e}^{\lambda F(r)}g(r)\,\mathrm{d}r} &\le M_\lambda \frac{\int_{-\infty}^t \mathrm{e}^{\lambda F(r)}g(r)\,\mathrm{d}r}{x_s^{-1}\mathrm{e}^{\lambda F(s)} + \int_s^t \mathrm{e}^{\lambda F(r)}g(r)\,\mathrm{d}r}\\ &= M_\lambda \frac{\int_{-\infty}^s \mathrm{e}^{\lambda F(r)}g(r)\,\mathrm{d}r + \int_s^t \mathrm{e}^{\lambda F(r)}g(r)\,\mathrm{d}r}{x_s^{-1}\mathrm{e}^{\lambda F(s)} + \int_s^t \mathrm{e}^{\lambda F(r)}g(r)\,\mathrm{d}r}.\end{aligned}$$

Condition (5.12) guarantees that the second terms in the numerator and denominator are positive for t sufficiently large, and so

$$\limsup_{t\to\infty} x(t) \le M_\lambda \max\left(1, \frac{x_s}{x_\lambda(s)}\right).$$

It therefore follows from (5.21) that $x_\lambda(\cdot)$ is forwards attracting while solutions exist.

To show that solutions do not blow up for $x_s < (1+\alpha_s)x_\lambda(s)$, observe that

$$\begin{aligned}x_s^{-1}\mathrm{e}^{\lambda F(s)} + \int_s^t \mathrm{e}^{\lambda F(r)}g(r)\,\mathrm{d}r &> \frac{1}{1+\alpha_s}\int_{-\infty}^s \mathrm{e}^{\lambda F(r)}g(r)\,\mathrm{d}r + \int_s^t \mathrm{e}^{\lambda F(r)}g(r)\,\mathrm{d}r\\ &= \int_{-\infty}^t \mathrm{e}^{\lambda F(r)}g(r)\,\mathrm{d}r - \frac{\alpha_s}{1+\alpha_s}\int_{-\infty}^s \mathrm{e}^{\lambda F(r)}g(r)\,\mathrm{d}r.\end{aligned}$$

Using the asymptotic positivity of g this expression is positive for α_s sufficiently small. This implies that $x_\lambda(\cdot)$ is locally forwards attracting.

Under the final condition the results follow by making the transformations

$$\lambda \mapsto -\lambda, \qquad x \mapsto -x, \qquad \text{and} \qquad t \mapsto -t.$$

□

We note here that an alternative to requiring stronger conditions at infinity (such as the asymptotic positivity of g) might be to make assumptions on integrals of f and g that are uniform in time, e.g.

$$\int_t^{t+T} g(s)\,\mathrm{d}s \ge \gamma > 0 \qquad \text{and} \qquad \int_t^{t+T} |g(s)|\,\mathrm{d}s \le \Gamma < +\infty \qquad \text{for all} \qquad t \in \mathbb{R}.$$

Since (see Bohr [1]) almost periodic functions $\varphi(\cdot)$ have time averages that converge uniformly,

$$\sup_{t\in\mathbb{R}} \left| \int_t^{t+T} \varphi(s)\,\mathrm{d}s - \bar{\varphi} \right| \to 0$$

as $T \to \infty$ (here $\bar{\varphi}$ is the time average of φ), such conditions would naturally include this important class of specific examples.

5.2 Conditions for localised bifurcating solutions

We now give stronger, but perhaps more natural, conditions on $f(t)$ and $g(t)$ that ensure that the balance conditions (5.10) and (5.11) hold.

Lemma 6 *Suppose that*

$$\liminf_{t\to-\infty} g(t) > 0 \tag{5.22}$$

and that

$$0 < m = \liminf_{t\to-\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to-\infty} \frac{f(t)}{g(t)} = M < +\infty. \tag{5.23}$$

Then for $\lambda > 0$

$$\lambda m \le \liminf_{t\to-\infty} x_\lambda(t) \le \limsup_{t\to-\infty} x_\lambda(t) \le \lambda M, \tag{5.24}$$

while for $\lambda < 0$ *we have*

$$\liminf_{s\to-\infty} \frac{\mathrm{e}^{\lambda F(s)}}{\int_s^t \mathrm{e}^{\lambda F(r)} g(r)\,\mathrm{d}r} \ge -m\lambda, \tag{5.25}$$

where F *is an antiderivative of* f.

Proof. For any $K > M$ there exists a T such that for all $t \le T$ we have $g(t) > 0$ and

$$\frac{f(t)}{g(t)} \le K.$$

For such t it follows that

$$\begin{aligned}
\int_{-\infty}^t \mathrm{e}^{\lambda F(s)} g(s)\,\mathrm{d}s &\ge \frac{1}{K} \int_{-\infty}^t \mathrm{e}^{\lambda F(s)} f(s)\,\mathrm{d}s \\
&\ge \frac{1}{K} \left[\frac{\mathrm{e}^{\lambda F(s)}}{\lambda} \right]_{s=-\infty}^t \\
&= \frac{1}{\lambda K}\,\mathrm{e}^{\lambda F(t)},
\end{aligned}$$

since $F(t) \to -\infty$ as $t \to -\infty$ by (5.22) and (5.23). Therefore

$$x_\lambda(t) = \frac{\mathrm{e}^{\lambda F(t)}}{\int_{-\infty}^{t} \mathrm{e}^{\lambda F(s)} g(s)\,\mathrm{d}s} \le K\lambda \qquad \text{for all} \qquad t \le T,$$

and hence

$$\limsup_{t\to-\infty} x_\lambda(t) \le M\lambda.$$

For the lower bound the proof is similar, but now using the fact that for any $k < m$ there exists a T such that

$$\frac{f(t)}{g(t)} > k \qquad \text{for all} \qquad t \le T.$$

The proof of (5.25) follows the same lines. □

5.3 The general case

We will now consider the general equation $\dot{x} = G(t, x, \lambda)$, and prove a bifurcation theorem based on assumptions on the Taylor coefficients of G. Since we will impose conditions on these coefficients similar to those in Lemma 6, we will be able to show that the system undergoes a transcritical bifurcation that is a little more akin to its autonomous counterpart than that in Proposition 5.

We now give our formal definition of a 'transcritical bifurcation' in a non-autonomous system. Note that we insist in the definition that the non-zero trajectory is in some sense 'localised' near the origin, and that the required behaviour depends only on the system in the past (pullback attraction and asymptotic instability). In our results we will be able to deduce further details of the behaviour of solutions by making additional assumptions on the system in the future.

Definition 12 *The system $\dot{x} = f(x, t, \lambda)$ undergoes a* local transcritical bifurcation *at $x = 0$, $\lambda = 0$ if there exists a $\lambda_0 > 0$ and an $\epsilon > 0$ such that*

(i) for all $-\lambda_0 < \lambda < 0$ the zero solution is locally pullback attracting within $(-\epsilon, 0]$ and pullback attracting within $[0, \epsilon)$; and there is another negative complete trajectory $x_\lambda(t)$ within $(-\epsilon, 0)$ that is asymptotically unstable and satisfies

$$x_\lambda(t) \to 0 \qquad \text{as} \qquad \lambda \to 0; \tag{5.26}$$

(ii) for $\lambda = 0$ the zero solution is asymptotically unstable but still pullback attracting within $[0, \epsilon)$; and

(iii) for $0 < \lambda < \lambda_0$ the zero solution is asymptotically unstable, and there is another positive complete trajectory $x_\lambda(t)$ within $(0, \epsilon)$ that satisfies

$$x_\lambda(t) \to 0 \qquad \textit{as} \qquad \lambda \to 0 \tag{5.27}$$

and is pullback attracting within $(0, \epsilon)$.

While we only require pointwise convergence in (5.26) and (5.27), we will in fact obtain uniform convergence in Theorem 7, which treats the equation $\dot{x} = G(t, x, \lambda)$ whose right-hand side has the Taylor expansion

$$\begin{aligned} G(t, x, \lambda) \quad = \quad & G + G_x x + G_\lambda \lambda + \tfrac{1}{2} G_{xx} x^2 + G_{x\lambda} x\lambda + \tfrac{1}{2} G_{\lambda\lambda} \lambda^2 \\ & + \tfrac{1}{6} G_{xxx} x^3 + \tfrac{1}{3} G_{xx\lambda} x^2 \lambda + \tfrac{1}{3} G_{x\lambda\lambda} x \lambda^2 + \tfrac{1}{6} G_{\lambda\lambda\lambda} \lambda^3 + \ldots \end{aligned}$$

(all expressions involving G and its derivatives on the right-hand side are evaluated at $(t, 0, 0)$). We assume that $G(t, 0, \lambda) = 0$ for all t and λ, and furthermore that $G_x(t, 0, 0) = 0$. This implies that $\partial^k G / \partial \lambda^k (t, 0, 0) = 0$ for all t and $k \in \mathbb{Z}_+$.

We therefore have

$$G(t, x, \lambda) = \lambda \left[G_{x\lambda} + \tfrac{1}{3} G_{x\lambda\lambda} \lambda + \ldots \right] x + \left[\tfrac{1}{2} G_{xx} + \tfrac{1}{6} G_{xxx} x + \tfrac{1}{3} G_{xx\lambda} \lambda + \ldots \right] x^2$$

and this motivates the following theorem.

Theorem 7 *Consider*

$$\dot{x} = G(t, x, \lambda),$$

and assume that

$$G(t, 0, \lambda) = 0 \quad \textit{for all} \quad \lambda \in \mathbb{R} \qquad \textit{and} \qquad G_x(t, 0, 0) = 0.$$

Set $f(t) = G_{x\lambda}(t, 0, 0)$ and $g(t) = -\frac{1}{2} G_{xx}(t, 0, 0)$, and rewrite the equation as

$$\dot{x} = \lambda [f(t) + \lambda \phi(t, \lambda)] x - [g(t) + \gamma(t, x, \lambda)] x^2,$$

where

$$\phi(t, 0) = \tfrac{1}{3} G_{x\lambda\lambda}(t, 0, 0) \qquad \text{and} \qquad \gamma(t, 0, 0) = 0. \tag{5.28}$$

Assume that

$$\liminf_{t \to \pm\infty} g(t) > 0, \tag{5.29}$$

that

$$0 < m = \liminf_{t\to\pm\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\pm\infty} \frac{f(t)}{g(t)} = M < +\infty, \tag{5.30}$$

and that

$$|\phi(t,\lambda)| \le h(t), \quad |\gamma_\lambda(t,x,\lambda)| \le h(t), \quad \textit{and} \quad |\gamma_x(t,x,\lambda)| \le h(t), \tag{5.31}$$

where

$$\limsup_{t\to\pm\infty} \frac{h(t)}{g(t)} \le K.$$

Then there is a local transcritical bifurcation as λ passes through zero. Furthermore when $\lambda < 0$ the 'unstable' trajectory is pullback repelling in $(-\epsilon, 0)$; when $\lambda = 0$ the origin is locally forwards attracting in $\mathbb{R}_+$; and when $\lambda > 0$ the pullback attracting trajectory $x_\lambda(\cdot)$ is forwards attracting in $(0,\epsilon)$.

Note that the standard conditions for a transcritical bifurcation in the autonomous equation $\dot{x} = f(x,\lambda)$ are (see Glendinning, [7]):

$$f(0,\lambda) = 0, \qquad f_x(0,0) = 0, \qquad f_{x\lambda}(0,0) > 0, \qquad \text{and} \qquad f_{xx}(0,0) < 0.$$

If $G(t,x,\lambda) = f(x,\lambda)$ then we recover these conditions in our theorem.

Proof. We assume throughout that $|\lambda| \le \epsilon$, where ϵ will be chosen 'sufficiently small'. Note that it follows from (5.28) and (5.31) that

$$|\gamma(t,x,\lambda)| \le h(t)[|x| + |\lambda|]. \tag{5.32}$$

The origin is locally pullback attracting in $(-\epsilon,\epsilon)$ for $\lambda < 0$. While $0 < x(t,s;x_s) \le \epsilon$ we have $0 \le x(t,s;x_s) \le v(t,s;x_s)$ where $v(t)$ solves

$$\dot{v} = \lambda[f(t) + \epsilon h(t)]v - [g(t) - 2\epsilon h(t)]v^2 \qquad \text{with} \qquad v(s) = x_s.$$

There exists a T such that if $s \le t \le T$ then we can neglect the second term; changing the definition of T if necessary, we can use the bound $|h(t)| \le K'f(t)/m$ (for some $K' > K$) to deduce that

$$\dot{v} \le \lambda(1 - (\epsilon K'/m))f(t)v,$$

from which it follows that

$$v(t) \le \mathrm{e}^{(1-(\epsilon K'/m))\lambda(F(t)-F(t_0))}v(t_0).$$

Once more decreasing T if necessary, so that $f(t) > 0$ for all $t \leq T$, it follows that for $s \leq t \leq T$ we have $v(t,s;x_s) \leq \epsilon$ provided that $0 < x_s \leq \epsilon$ and hence, since the comparison $x(t,s;x_s) \leq v(t,s;x_s)$ remains valid, it follows that

$$\lim_{s\to-\infty} S(t,s)x_s = 0 \qquad \text{for all} \qquad t \leq T.$$

Since $S(\tau,t)$ is continuous and zero is invariant we have

$$\lim_{s\to-\infty} S(\tau,s)x_s = S(\tau,t)\left[\lim_{s\to-\infty} S(t,s)x_s\right] = S(\tau,t)0 = 0 \qquad \text{for all} \qquad \tau \in \mathbb{R},$$

and the origin is pullback attracting within $[0,\epsilon)$.

While $-\epsilon \leq x(t,s;x_s) \leq 0$ we have $u(t,s;x_s) \leq x(t,s;x_s) \leq 0$ where $u(t)$ solves

$$\dot{u} = \lambda[f(t) - \epsilon h(t)]u - [g(t) + 2\epsilon h(t)]u^2 \qquad \text{with} \qquad u(s) = x_s;$$

therefore while $u \geq -\epsilon$

$$\dot{u} \geq \lambda f(t)[(1 - \epsilon K/m)u] - (1 + 2\epsilon K)u^2/m.$$

For T chosen such that $f(t) > 0$ for all $t \leq T$, and for $0 \geq x_s \geq -\lambda(m - \epsilon K)/m(1 + 2\epsilon K)$ it follows that

$$\lim_{s\to-\infty} S(t,s)x_s = 0 \qquad \text{for all} \qquad t \leq T,$$

and arguing as above the origin is locally pullback attracting within $(-\epsilon, 0]$.

When $\lambda = 0$. While $|x| \leq \epsilon$ we have

$$\dot{x} \leq -[g(t) - 2\epsilon h(t)]x^2,$$

which immediately gives the pullback attraction of the zero solution within $[0,\epsilon)$, and the asymptotic instability of zero, since for $x_s < 0$ we have $x(t,s;x_s) \to 0$ as $t \to -\infty$.

There is a positive trajectory that is pullback attracting in $[0,\infty)$ *when* $\lambda > 0$. While $|x(t,s;x_s)|, |\lambda| < \epsilon$ we have

$$u(t,s;x_s) \leq x(t,s;x_s) \leq v(t,s;x_s), \tag{5.33}$$

where $u(t,s;x_s)$ and $v(t,s;x_s)$ are the solutions of

$$\dot{u} = \lambda \underbrace{[f(t) - \epsilon h(t)]}_{f_-(t)} u - \underbrace{[g(t) + 2\epsilon h(t)]}_{g_+(t)} u^2 \qquad \text{with} \qquad u(s) = x_s$$

and

$$\dot{v} = \lambda \underbrace{[f(t) + \epsilon h(t)]}_{f_+(t)} v - \underbrace{[g(t) - 2\epsilon h(t)]}_{g_-(t)} v^2 \qquad \text{with} \qquad v(s) = x_s. \tag{5.34}$$

In particular, we have an explicit form for the solution of (5.34), namely

$$v(t) = \frac{\mathrm{e}^{\lambda F_+(t)}}{x_s^{-1}\mathrm{e}^{\lambda F_+(s)} + \int_s^t \mathrm{e}^{\lambda F_+(r)} g_-(r)\,\mathrm{d}r}.$$

Using the balance condition (5.23) it follows that for λ and x_s sufficiently small, $v(t) \leq \epsilon$ for all $t \leq 0$. In this case the comparison (5.33) remains valid for all such t.

Due to the two-sided balance and the balance between h and g it follows that we can define the upper and lower solutions

$$x_+(t) = \frac{\mathrm{e}^{\lambda F_+(t)}}{\int_{-\infty}^t \mathrm{e}^{\lambda F_+(r)} g_-(r)\,\mathrm{d}r}$$

and

$$x_-(t) = \frac{\mathrm{e}^{\lambda F_-(t)}}{\int_{-\infty}^t \mathrm{e}^{\lambda F_-(r)} g_+(r)\,\mathrm{d}r},$$

the pullback attractors of the upper and lower equations. We then have

$$x_-(t) \leq \liminf_{s\to-\infty} x(t,s;x_s) \leq \limsup_{s\to-\infty} x(t,s;x_s) \leq x_+(t).$$

Therefore there exists a pullback attractor $\mathcal{A}(t)$ within the phase space consisting of the interval $(0,\epsilon)$. Since the system is order-preserving, there are two solutions $x_1(t)$ and $x_2(t)$ such that $\mathcal{A}(t) = [x_1(t), x_2(t)]$, and so we have $x_-(t) \leq x_j(t) \leq x_+(t)$ for $j = 1, 2$.

If we set $z(t) = x_1(t) - x_2(t)$ then

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &\leq \lambda[f(t) + \epsilon h(t)]z - g(t)(x_1 + x_2)z - [\gamma(t, x_1, \lambda)x_1^2 - \gamma(t, x_2, \lambda)x_2^2] \\ &\leq \lambda f_+(t)z - g(t)(x_1 + x_2)z - \gamma(t, x_1, \lambda)(x_1^2 - x_2^2) \\ &\qquad +[\gamma(t, x_1, \lambda) - \gamma(t, x_2, \lambda)]x_2^2 \\ &\leq \lambda f_+(t)z - g(t)(x_1 + x_2)z + 2\epsilon h(t)[x_1 + x_2]z + \epsilon x_1 h(t)z \\ &\leq [\lambda f_+(t) - (2g(t) - 5\epsilon h(t))x_-(t)]z. \end{aligned}$$

Since

$$2g(t) - 5\epsilon h(t) \geq \frac{2 - 5\epsilon K}{1 + \epsilon K} g_+(t)$$

this gives

$$\frac{\mathrm{d}z}{\mathrm{d}t} \leq \left[\lambda f_+(t) - \frac{2 - 5\epsilon K}{1 + 2\epsilon K} \frac{g_+(t)\mathrm{e}^{\lambda F_-(t)}}{\int_{-\infty}^t \mathrm{e}^{\lambda F_-(r)} g_+(r)\,\mathrm{d}r}\right] z.$$

We have

$$z(t) \le z(t_0)\mathrm{e}^{I(t,t_0)},$$

where

$$\begin{aligned} I(t,t_0) &:= \int_{t_0}^{t} \lambda f_+(s) - \frac{2-5\epsilon K}{1+2\epsilon K}\frac{g_+(s)\mathrm{e}^{\lambda F_-(s)}}{\int_{-\infty}^{t}\mathrm{e}^{\lambda F_-(r)}g_+(r)\,\mathrm{d}r}\,\mathrm{d}s \\ &= \lambda(F_+(t)-F_+(t_0)) - \frac{2-5\epsilon K}{1+2\epsilon K}\left[\ln\int_{-\infty}^{s}\mathrm{e}^{\lambda F_-(r)}g_+(r)\,\mathrm{d}r\right]_{s=t_0}^{t}. \end{aligned}$$

Now,

$$\frac{f_-}{M} \le g_+ \le \frac{1+2\epsilon K}{m-\epsilon K}f_-,$$

and so

$$\begin{aligned} \left[\ln\int_{-\infty}^{s}\mathrm{e}^{\lambda F_-(r)}g_+(r)\,\mathrm{d}r\right]_{s=t_0}^{t} &\ge \ln\left(\frac{1}{\lambda M}\mathrm{e}^{\lambda F_-(t)}\right) - \ln\left(\frac{1+2\epsilon K}{\lambda(m-\epsilon K)}\mathrm{e}^{\lambda F_-(t_0)}\right) \\ &= \lambda(F_-(t)-F_-(t_0)) + \ln\frac{m-\epsilon K}{M(1+2\epsilon K)}. \end{aligned}$$

Therefore

$$I \le \lambda(F_+(t)-F_+(t_0)) - \frac{2-5\epsilon K}{1+2\epsilon K}\,\lambda(F_-(t)-F_-(t_0)) + C_\epsilon,$$

where

$$C_\epsilon = -\frac{2-5\epsilon K}{1+2\epsilon K}\,\ln\frac{m-\epsilon K}{M(1+2\epsilon K)} > 0.$$

Since

$$f_- \ge \frac{1-(\epsilon K/m)}{1+(\epsilon K/m)}f_+$$

we also have

$$F_-(t)-F_-(t_0) \ge \frac{1-(\epsilon K/m)}{1+(\epsilon K/m)}[F_+(t)-F_+(t_0)],$$

and so

$$I \le \lambda(F_+(t)-F_+(t_0))\left[1-2\,\frac{(1-\frac{5}{2}\epsilon K)(1-\epsilon K/m)}{(1+2\epsilon K)(1+\epsilon K/m)}\right] + C_\epsilon.$$

It follows that for ϵ sufficiently small we can guarantee that $z(t)=0$, and hence that there is a single pullback attracting positive trajectory $x^*(\cdot)$.

Now note that the above argument is in fact valid for any two trajectories $x_1(\cdot)$ and $x_2(\cdot)$ that are bounded below by $x_-(t)$. Now also note that any trajectory $x(t,s;x_s)$ with $x_s>0$ has $x(t,s;x_s) > \frac{3}{4}x_-(t)$ for t large enough (cf. argument following (5.20) in the proof of Proposition 5); this is also enough to apply the above argument, and so $x^*(\cdot)$ is attracting in $(0,\epsilon)$ as $t\to+\infty$.

The origin is unstable 'downwards' when $\lambda > 0$. We have $0 \geq x(t) \geq u(t)$ where $u(t)$ solves

$$\dot{u} = \lambda[f(t) - \epsilon h(t)]u.$$

As $t \to -\infty$ we therefore have $u(t) \to 0$, and so we have $x(t) \to 0$ too.

The unstable trajectory when $\lambda < 0$. The transformation $x \mapsto -x$, $t \mapsto -t$, gives the existence of a candidate for the negative unstable trajectory; its instability follows from the fact that $x^*(\cdot)$ is attracting 'from above' as $t \to +\infty$. □

6 Non-autonomous 'simple pitchfork' bifurcation

The canonical autonomous example of an equation exhibiting a pitchfork bifurcation is

$$\dot{y} = \mu y - y^3. \tag{6.1}$$

For $\mu < 0$ the only fixed point is the origin, which is stable; while for $\mu > 0$ the origin is unstable and there are two new fixed points at $\pm\sqrt{\mu}$ which are stable.

We now give a formal definition of what we understand by a 'pitchfork bifurcation' for a non-autonomous system. Note that as before all the behaviour in the definition only relies on the properties of the equation 'in the past'.

Definition 13 *The system $\dot{x} = f(x, t, \lambda)$ undergoes a* localised pitchfork bifurcation *at $x = 0$, $\lambda = 0$ if there exists a $\lambda_0 > 0$ and an $\epsilon > 0$ such that*

(i) for all $-\lambda_0 < \lambda \leq 0$ the zero solution is pullback attracting within $(-\epsilon, \epsilon)$;

(ii) when $0 < \lambda < \lambda_0$ the zero solution is asymptotically unstable, and there exist bounded trajectories $x_\lambda^+(t)$ and $x_\lambda^-(t)$ that are pullback attracting in $(0, \epsilon)$ and $(-\epsilon, 0)$ respectively, and satisfy

$$x_\lambda^\pm(t) \to 0 \qquad \text{as} \qquad \lambda \downarrow 0$$

uniformly on compact subsets of $\mathbb{R}$.

Since equation (6.1) is invariant under the transformation $y \mapsto -y$ it is convenient to consider the new variable $x = 2y^2$, which satisfies the equation

$$\dot{x} = 2\mu x - x^2.$$

With a rescaled bifurcation parameter $\lambda = 2\mu$, we have

$$\dot{x} = \lambda x - x^2,$$

where we can restrict attention to $x \geq 0$.

For our general non-autonomous example we retain the simplifying factor of reflectional symmetry to ease our treatment, but as in the autonomous case this requirement could be weakened. With an original equation

$$\dot{y} = H(y, t, \lambda)$$

that is invariant under the transformation $y \mapsto -y$ we set $x = y^2$ and consider instead

$$\dot{x} = G(x, t, \lambda) = 2yH(y, t, \lambda).$$

The existence of a non-autonomous pitchfork bifurcation under appropriate conditions is now a simple consequence of Theorem 7:

Theorem 8 *Let the conditions of Theorem 7 hold for the transformed equation* $\dot{x} = G(x, t, \lambda)$*, except that all limit conditions are only required as* $t \to -\infty$*. Then there is a local pitchfork bifurcation as* λ *passes through zero for* $\dot{y} = H(y, t, \lambda)$*.*

7 The non-autonomous saddle node bifurcation

The canonical example of an autonomous equation in which a saddle-node bifurcation occurs is

$$\dot{x} = \lambda - x^2. \tag{7.1}$$

For $\lambda < 0$ every trajectory tends to $-\infty$ (in a finite time), while for $\lambda > 0$ there are two fixed points: a stable point at $x = \sqrt{\lambda}$ and an unstable point at $x = -\sqrt{\lambda}$.

In the non-autonomous case we make the following definition, consistent with our practice of requiring only behaviour that depends on the past.

Definition 14 *The equation* $\dot{x} = f(x, t, \lambda)$ *undergoes a* local saddle node bifurcation *at* $x = 0$, $\lambda = 0$ *provided that there exists a* $\lambda_0 > 0$*, an* $\epsilon > 0$*, and a* δ *with* $0 < \delta < \epsilon$ *such that*

(i) for $-\lambda_0 < \lambda \leq 0$ *there are no complete trajectories lying within* $(-\epsilon, \epsilon)$*;*

(ii) for $0 < \lambda < \lambda_0$ there exists a complete trajectory $x_\lambda^+(\cdot)$ that is pullback attracting within $(-\delta, \epsilon)$ and another complete trajectory $x_\lambda^-(\cdot)$ that lies within $(-\epsilon, \epsilon)$ and is asymptotically unstable. Furthermore

$$\lim_{\lambda \to 0} x_\lambda^\pm(t) \to 0$$

uniformly on compact subintervals of $\mathbb{R}$.

Note that a more natural definition might require the 'unstable' complete trajectory $x_\lambda^-(\cdot)$ to be pullback repelling within $(-\epsilon, \delta)$, rather than asymptotically unstable.

7.1 The simple case

First we treat the simplest non-autonomous version of (7.1).

Theorem 9 *Consider the equation*

$$\dot{x} = \lambda f(t) - g(t)x^2 \tag{7.2}$$

where f is 'essentially positive'

$$\int_{-\infty}^{t} f(s)\,\mathrm{d}s = \int_{t}^{\infty} f(s)\,\mathrm{d}s = +\infty, \tag{7.3}$$

and the balance conditions

$$\liminf_{t \to \pm\infty} g(t) > 0 \qquad \text{and} \qquad 0 < m \le \lim_{t \to \pm\infty} \frac{f(t)}{g(t)} \le M$$

hold. Then for $\lambda \le 0$ there are no nonzero bounded complete trajectories: when $\lambda < 0$ for any fixed x_s there is a σ such that, for $s \le \sigma$, $x(t, s; x_s) \to -\infty$ as $t \to t^*(s) < \infty$, and similarly for any fixed t we have $x(t, s; x_s) \to -\infty$ as $s \to s^*(t) > -\infty$. For $\lambda = 0$ the zero solution is locally forwards and locally pullback attracting within $[0, \infty)$, while for negative initial conditions we have the same behaviour as for $\lambda < 0$.

For $\lambda > 0$ there are two trajectories $\pm x^*(t)$, such that $x^*(t)$ is both forwards and pullback attracting,

$$\lim_{s \to -\infty} S(t, s)x_0 = x^*(t) \qquad \text{for all} \qquad x_0 > -\sqrt{\lambda m}$$

and

$$\lim_{t \to +\infty} \mathrm{dist}[S(t, s)x_0, x^*(t)] = 0 \qquad \text{for all} \qquad x_0 > -\sqrt{\lambda m},$$

and $-x^*(t)$ is asymptotically unstable and pullback repelling,

$$\lim_{s\to+\infty} S(t,s)x_0 = -x^*(t) \qquad \text{for all} \qquad x_0 < \sqrt{\lambda m}$$

and

$$\lim_{t\to-\infty} \operatorname{dist}[S(t,s)x_0, -x^*(t)] = 0 \qquad \text{for all} \qquad x_0 < \sqrt{\lambda m}.$$

Proof. First we consider $\lambda < 0$, and assume initially that $x_s < 0$. Since there exists a T such that for $t \le T$ the functions f and g are positive, we have

$$\dot{x} \le -g(t)x^2$$

for all such t. It follows that

$$x(t,s;x_s) \le \frac{1}{x_s^{-1} + \int_s^t g(r)\,\mathrm{d}r} \qquad \text{for all} \qquad s \le t \le T,$$

and hence, since g is essentially positive, that there exists an $s^*(t) > -\infty$ such that

$$\lim_{s\to s^*(t)} x(t,s;x_s) = -\infty.$$

Similarly if x_s is fixed there exists a $\sigma(t)$ such that if $s \le \sigma(t)$ we have

$$\lim_{t\to t^*(s)} x(t,s;x_s) = -\infty$$

for some $t^* < +\infty$.

If $x_s > 0$, observe that we can argue from the above results applied for $x_s = -1$: there exists a σ_1 such that if $s \le \sigma_1$ then $x(t,s;-1) \to -\infty$ as $t \to t^*(s) < \infty$. Now, since for $t \le T$ we have $\dot{x} \le \lambda f(t) < 0$, and so

$$x(t,s;x_s) < x_s + \lambda \int_s^t f(r)\,\mathrm{d}r. \tag{7.4}$$

The essential positivity of f now implies that there exists a σ_2 such that if $s \le \sigma_2$ then $x(t,s;x_s) \le -1$ for some $t \le \sigma_1$; it follows that for some $t^*(s)$ we have $x(t,s;x_s) \to -\infty$ as $t \to t^*(s)$.

It is also the case that for each fixed t we have $x(t,s;-1) \to -\infty$ as $s \to s_1(t) > -\infty$. Using (7.4) once again there exists an $s_2(t)$ such that $x(t,s;x_s) \to -1$ as $s \to s_2$: it follows that $x(t,s;x_s) \to -\infty$ as $s \to s_1(s_2)$.

When $\lambda = 0$ the local attractivity of the origin follows from the explicit solution

$$x(t,s;x_s) = \frac{1}{x_s^{-1} + \int_s^t g(r)\,\mathrm{d}r},$$

while the behaviour for $x_s < 0$ is a consequence of the argument used above for $\lambda < 0$.

When $\lambda > 0$ we have

$$\dot{x} \le g(t)[M\lambda - x^2]$$

and

$$\dot{x} \ge g(t)[m\lambda - x^2].$$

It follows that if $x_0 > -\sqrt{m\lambda}$ then

$$\sqrt{m\lambda} \le \lim_{s \to -\infty} x(t, s; x_0) \le \sqrt{M\lambda}.$$

Considering the difference of two solutions of (7.2), $z = x_1 - x_2$, we have

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -g(t)[x_1 + x_2]z.$$

Since g is essentially positive, and $x_1, x_2 \ge \sqrt{m\lambda}$ it follows that $x_1(t) = x_2(t)$. This gives a positive solution $x^*(t)$ that attracts (pullback and forwards) all trajectories with $x_0 > -\sqrt{\lambda m}$.

Without the assumption on what happens as $t \to +\infty$, we can only note that for $x_0 < -\sqrt{\lambda M}$ the solution tends to $-\infty$ (pullback and forwards). There is some indeterminate band of conditions between $-\sqrt{\lambda M}$ and $-\sqrt{\lambda m}$.

Since the conditions on f and g are symmetric in t we can consider the time-reversed problem. The same argument now shows that there is a negative solution $y^*(t)$ that attracts all trajectories with $x_0 < \sqrt{m\lambda}$ both backwards in time and is 'pullback repelling',

$$\lim_{t \to \infty} x(s, t; x_0) = y^*(s). \tag{7.5}$$

We want to show that in fact that if $x_s > y^*(s)$ then

$$\lim_{t \to \infty} [x(t, s; x_s) - x^*(t)] = 0.$$

We now that this convergence holds if $x_s > -\sqrt{\lambda m}$. So now consider an initial condition $x_s > y^*(s)$. We know that (7.5) holds in particular for $x_0 = 0$; i.e.

$$\lim_{t \to \infty} x(t, s; 0) = y^*(s).$$

In particular, for t large enough we must have

$$x(t, s; 0) < x_s.$$

Since the equation is order-preserving, it follows that

$$x(s, t; x_s) > 0;$$

from time t this solution is therefore (since it is greater than $\sqrt{-m\lambda}$) attracted to $x^*(t)$.

Reversing the argument shows that $y^*(t)$ attracts any initial condition less than $x^*(t)$ as $t \to -\infty$, and the result follows. $\square$

7.2 General saddle node

We now consider

$$\dot{x} = G(t, x, \lambda),$$

where the right-hand side has Taylor expansion (where expressions on the right-hand side involving G are evaluated at $(t, 0, 0)$)

$$\begin{aligned} G(t, x, \lambda) &= G + G_x x + G_\lambda \lambda + \tfrac{1}{2}G_{xx}x^2 + G_{x\lambda}x\lambda + \tfrac{1}{2}G_{\lambda\lambda}\lambda^2 \\ &\quad + \tfrac{1}{6}G_{xxx}x^3 + \tfrac{1}{3}G_{xx\lambda}x^2\lambda + \tfrac{1}{3}G_{x\lambda\lambda}x\lambda^2 + \tfrac{1}{6}G_{\lambda\lambda\lambda}\lambda^3 + \ldots \end{aligned}$$

We assume that $G(t, 0, 0) = G_x(t, 0, 0) = 0$, and so have

$$\begin{aligned} G(t, x, \lambda) &= \lambda[G_\lambda + G_{x\lambda}x + \tfrac{1}{2}G_{\lambda\lambda}\lambda + \tfrac{1}{3}G_{xx\lambda}x^2 + \tfrac{1}{3}G_{x\lambda\lambda}x\lambda + \tfrac{1}{6}G_{\lambda\lambda\lambda}\lambda^2 + \ldots] \\ &\quad + [\tfrac{1}{2}G_{xx} + \tfrac{1}{6}G_{xxx}x + \ldots]x^2. \end{aligned}$$

This motivates the following theorem.

Theorem 10 *Consider*

$$\dot{x} = G(t, x, \lambda),$$

and assume that

$$G(t, 0, 0) = G_x(t, 0, 0) = 0.$$

Set $f(t) = G_\lambda(t, 0, 0)$ *and* $g(t) = -\frac{1}{2}G_{xx}(t, 0, 0)$*, and rewrite the equation as*

$$\dot{x} = \lambda[f(t) + \phi(t, x, \lambda)] - x^2[g(t) + \psi(t, x)],$$

where $\psi(t, 0) = 0$*. Assume that*

$$\liminf_{t\to\pm\infty} g(t) > 0, \tag{7.6}$$

that

$$0 < m = \liminf_{t\to\pm\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\pm\infty} \frac{f(t)}{g(t)} = M < +\infty, \tag{7.7}$$

and that

$$|\phi(t,x,\lambda)| \leq h(t)[|x|+|\lambda|] \qquad \text{with} \qquad |\phi_x(t,x,\lambda)| \leq h(t), \tag{7.8}$$

and finally

$$|\psi_x(t,x)| \leq h(t),$$

where

$$\limsup_{t\to\pm\infty} \frac{h(t)}{g(t)} \leq K.$$

Then there is a local saddle node bifurcation as λ passes through zero. Furthermore when $\lambda > 0$ the pullback attracting trajectory $x_\lambda(\cdot)$ is forwards attracting in $(0,\epsilon)$, and the unstable trajectory is pullback repelling within $(-\epsilon,\delta)$.

The standard conditions for a saddle-node bifurcation in the autonomous equation $\dot{x} = f(x,\lambda)$ are (see Glendinning, [7]):

$$f(0,0)=0, \qquad f_x(0,0)=0, \qquad f_{x\lambda}(0,0)>0, \qquad \text{and} \qquad f_{xx}(0,0)<0.$$

Once again we recover these conditions above if we set $G(t,x,\lambda) = f(x,\lambda)$.

Proof. First note that the two assumptions on the x derivatives of ϕ and ψ imply the Lipschitz bounds

$$|\phi(t,x_1,\lambda)-\phi(t,x_2,\lambda)| \leq h(t)|x_1-x_2| \qquad \text{and} \qquad |\psi(t,x_1)-\psi(t,x_2)| \leq h(t)|x_1-x_2|.$$

For $\lambda < 0$ and ϵ sufficiently small we have

$$\dot{x} \leq \lambda[f(t)-\epsilon h(t)] \leq \lambda f(t)[1+(K\epsilon/m)]$$

for $t \leq -T$ or $t \geq T$. It follows as in the proof of Theorem 9 that there are no complete nonzero trajectories that lie entirely within $(-\epsilon,\epsilon)$.

When $\lambda = \delta^2$ we have, for all $|x| \leq \epsilon$,

$$\dot{x} \leq g(t)[\delta^2(M+\epsilon K) - x^2(1-\epsilon K)]$$

and

$$\dot{x} \geq g(t)[\delta^2(m-\epsilon K) - x^2(1+\epsilon K)].$$

With the choice

$$\delta = \epsilon\sqrt{\frac{1-\epsilon K}{M+\epsilon K}}$$

it follows that any trajectory with

$$x_- := -\delta\sqrt{\frac{m-\epsilon K}{1+\epsilon K}} < x_s \le \epsilon$$

has $|x(t,s;x_s)| \le \epsilon$ for all $t \ge s$, and hence that

$$\delta\sqrt{\frac{m-\epsilon K}{1+\epsilon K}} \le \lim_{s\to-\infty} x(t,s;x_s) \le \epsilon.$$

Thus the pullback attractor in $(x_-, \epsilon]$ consists of the interval $[x_1(t), x_2(t)]$. Considering the difference $z = x_1 - x_2$ this satisfies

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \lambda[\phi(t,x_1,\lambda) - \phi(t,x_2,\lambda)] - (x_1+x_2)g(t)z - x_1^2\psi(t,x_1) + x_2^2\psi(t,x_2) \\
&\le \delta^2 h(t)z - (x_1+x_2)g(t)z + (x_2^2 - x_1^2)\psi(t,x_1) + [\psi(t,x_2) - \psi(t,x_1)]x_1^2 \\
&\le \delta^2 h(t)z - (x_1+x_2)g(t)z + [\epsilon+\delta^2]h(t)(x_1+x_2)z + h(t)z\epsilon^2 \\
&\le C[\epsilon^2 h(t) - \epsilon g(t)]z \\
&\le -C(1-K\epsilon)\epsilon g(t)z
\end{aligned}$$

as $\epsilon \to 0$. It follows that for ϵ chosen sufficiently small, $z(t) = 0$.

Using the same argument for the time-reversed systems gives a saddle-node bifurcation. □

8 The balance hypothesis: examples

In this final section we give some examples demonstrating that without some kind of 'balance' between successive terms in the Taylor series we cannot expect the type of bifurcation results above. Note that while all these examples are asymptotically autonomous (as $t \to \infty$), the behaviour of the non-autonomous equation is different from that of its autonomous limit.

Our simplest example is

$$\dot{x} = \lambda x - \mathrm{e}^{-t}x^2 \qquad \text{with} \qquad x(s) = x_s \ge 0,$$

where the exponential term produces very strong dissipativity as $t \to -\infty$. From the explicit solution

$$x(t,s;x_s) = \frac{\mathrm{e}^{\lambda t}}{x_s^{-1}\mathrm{e}^{\lambda s} + (\lambda-1)^{-1}(\mathrm{e}^{(\lambda-1)t} - \mathrm{e}^{(\lambda-1)s})}$$

it is clear that while for $\lambda < 0$ the origin is pullback attracting in $\mathbb{R}_+$, this is also the case when $0 < \lambda < 1$. Thus the 'one-sided pitchfork' type bifurcation that we might expect is suppressed. [Note, however, that the complete (but unbounded) trajectory $x^*(t) = (\lambda - 1)e^t$ is forwards attracting for all $\lambda > 0$.]

In the previous example we made one of the terms of the Taylor expansion that plays a prime rôle in the bifurcation blow up as $t \to -\infty$. However, we can also shift this behaviour to the higher-order terms and run into similar problems. For the equation

$$\dot{x} = \lambda x - x^2 - e^{-t}x^3 \qquad \text{with} \qquad x(s) = x_s \geq 0$$

it is clear that for $\lambda < 0$ the origin is globally pullback (and forwards) attracting; while for $\lambda > 0$ we have

$$\lambda x - x^2 - e^{-t}x^3 \leq \lambda x - e^{-t}x^3,$$

so that the continued pullback attraction of the zero solution follows the previous example after setting $y = x^2$ and $\mu = 2\lambda$ (see Section 6).

A similar example, but one in which the higher-order terms produce instability (rather than enhance the stability), is

$$\dot{x} = \lambda x - 2x^2 + e^{-t}x^3.$$

Given an initial condition x_s, whatever the value of λ we can choose T sufficiently large and negative that

$$\lambda x - 2x^2 + e^{-t}x^3 \geq \tfrac{1}{2}e^{-t}x^3 \qquad \text{for all} \qquad t \leq T.$$

It follows that for any x_s,

$$\lim_{s \to -\infty} x(t, s; x_s) = +\infty,$$

and there is never a pullback attracting trajectory.

9 Conclusion

We have tried to develop a general theory for bifurcations in non-autonomous scalar systems, in particular giving a set of possible definitions for transcritical, pitchfork, and saddle-node bifurcations that depend only on properties of the system in the past.

There are, of course, many ways in which these results could be improved. The main problem is the restrictive nature of some of the conditions that we have required on the terms in our Taylor expansion. As we remarked at the end of Section 5.1, it should be possible to prove similar results replacing assumptions such as the asymptotic positivity of terms in the equation by time integrated (or perhaps time-averaged) conditions such as

$$\int_t^{t+T} g(s)\,\mathrm{d}s \geq \gamma > 0 \qquad \text{for all} \qquad t \in \mathbb{R}.$$

There are higher-dimensional bifurcation results for certain systems, in particular in the almost periodic case (see Kloeden [12], for example). We hope to extend the results here to general higher dimensional systems, by considering the scalar systems obtained by restricting attention to an appropriate centre manifold, as is done in the autonomous case.

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