Programa de Doctorado "Matemáticas"

PHD DISSERTATION

On some families of links and new approaches to link homologies

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A mis padres.

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Resumen

Esta tesis se enmarca dentro de la Teoría de Nudos. En los Capítulos 1 y 2 se usan herramientas clásicas para estudiar las propiedades de ciertas familias de enlaces y las relaciones de inclusión entre ellas. En los Capítulos 3 y 4 se estudian nuevas aproximaciones a invariantes homológicos de enlaces introducidos en este siglo.

En 1983 Louis Kauffman conjeturó que las familias de enlaces alternativos y pseudoalternantes eran iguales. En el Capítulo 1 se prueba esta conjetura para el caso de enlaces cuyo primer número de Betti es menor que 3. También se dan dos contraejemplos que muestran que, en general, la conjetura no es cierta.

El polinomio de Conway de los enlaces positivos es positivo. En el Capítulo 2 se extiende esta propiedad a la familia de enlaces fuertemente cuasipositivos con índice de trenza 3. Así mismo, se prueba que esta propiedad no puede ser extendida a los enlaces fuertemente cuasipositivos con índice de trenza mayor que 5.

El Capítulo 3 se centra en los intentos de dar una definición de la homología Knot Floer en términos de los estados FKT de Kauffman. Concretamente, se muestra la no invariancia de una propuesta de Y. Rong bajo el movimiento de Reidemeister II.

En el Capítulo 4 se da una definición geométrica de la cohomología extrema de Khovanov, en términos del grafo de Lando asociado al diagrama de un enlace. Este nuevo punto de vista permite construir una familia de nudos H-gruesos cuya cohomología extrema de Khovanov cuenta con tantos grupos no triviales como se desee.

Los conceptos y resultados básicos asociados a cada capítulo se encuentran recogidos en sus respectivas introducciones. Así mismo, al final de cada uno de ellos se incluyen cuestiones abiertas a las que da lugar el trabajo desarrollado.

Abstract

This thesis falls within the scope of Knot Theory. In Chapters 1 and 2 we use classical tools for studying the properties and relations among some families of links. In Chapters 3 and 4 we discuss new approaches to homological link invariants introduced in this century.

In 1983 Louis Kauffman conjectured that the families of alternative and pseudoalternating links coincide. In Chapter 1 we prove that this conjecture holds for the special case of links whose first Betti number is at most 2. We also disprove the conjecture for the general case by showing two counterexamples.

Positive links have positive Conway polynomial. In Chapter 2 we extend this property to the family of strongly quasipositive links whose braid index equals 3. Moreover, we prove that this result cannot be extended when considering strongly quasipositive links whose braid index is greater than 5.

Chapter 3 focuses in the attemps of giving a definition of Knot Floer homology in terms of Kauffman FKT-states. Namely, we show that the definition given by Y. Rong is not invariant under the Reidemeister II move.

In Chapter 4 we give a geometric realization of the extreme Khovanov cohomology in terms of the Lando graph associated to a link diagram. This new approach allows us to give a family of H-thick knots with any number of nontrivial extreme Khovanov cohomology modules.

Each chapter has its own introduction containing its associated motivation and background. Moreover, at the end of each of them we have included open questions related to the results exposed in the chapter.

Contents

Introduction

The main problem in Knot Theory is determining if two given links are equivalent. In 1927 Reidemeister [44] showed that two diagrams represent equivalent links if and only if they are related by a sequence of what are known as Reidemeister moves. However, studying the equivalence problem is still a hard problem. With this purpose link invariants (functions whose value depends only on the equivalence class of the link) were introduced, and attending to these invariants and to other aspects, links are grouped into families. Determining the inclusion of an arbitrary link in a particular family is sometimes a difficult problem.

The starting point of this work has been comparing different families of links which extend the well known families of alternating and positive links. More precisely, we have studied those families involved in a conjecture by Louis Kauffman [27] (alternative, homogeneous and pseudoalternating links) and the family of strongly quasipositive links, introduced by Lee Rudolph [46]. A nice property for a link belonging to one of these families is that its genus can be easily computed.

Going back to link invariants, it is a fact that although most of them are used mainly to distinguish different links, many are related to their geometric or topological properties. This is the case of Khovanov [28] and Knot Floer homologies ([40] and [43]), two of the most effective link invariants introduced in the last two decades which categorify polynomial invariants, giving more information about the link. Khovanov homology, whose graded Euler characteristic is the Jones polynomial, distinguish some knots having the same Jones polynomial. Knot Floer homology, which categorifies the Alexander polynomial, determines the genus of a knot and its fiberedness (while the Alexander polynomial gives only bounds on the genus and obstructions about the fiberedness of a link).

In this work we also deal with these homological invariants. Namely, we review Khovanov and Knot Floer homologies and discuss a new approach to each of them.

This work is divided into two parts: in the first one, covering Chapters 1 and 2, we use classical tools for studying the relation among the above mentioned families of links, and we also give a necessary condition for a subset of links to be strongly quasipositive. In the second part, including Chapters 3 and 4, taking as starting point the fact that Khovanov and Knot Floer homologies categorify Alexander and Jones polynomials respectively, we study new approaches to these homological invariants in terms of classical tools like Lando graphs or Kauffman

states.

Let us detail the contents of each chapter, making a brief description of the main results.

In 1983 Louis Kauffman [27] introduced the family of alternative links as an extension of the class of alternating links preserving some of their properties. They are defined in terms of the sign of the edges in their Seifert diagrams. Previously, in 1976 E.J. Mayland and K. Murasugi [36] introduced the class of pseudoalternating links as those bounding a special kind of surface called generalized flat surface. It is easy to check that alternative links are pseudoalternating. In Chapter 1 we deal with these two families of links and the intermediate family of homogeneous links, introduced by Peter Cromwell in 1989 [19].

More precisely, we study a conjecture posed by Kauffman in [27] which states that the classes of alternative and pseudoalternating links are identical (Conjecture 1.1). This conjecture has been an open problem for 30 years. We disprove it in Section 1.3 by finding a two components link and a knot whose first Betti numbers equal 3 and 4 respectively, which are pseudoalternating and non-homogeneous, hence non-alternative (Theorems 1.11 and 1.13). However, Conjecture 1.1 holds for the special case of links whose first Betti number is at most 2 and for four components links whose genus equals 0, as we show in Sections 1.4 and 1.5. We also give a complete classification of homogeneous knots of genus 1 in Theorem 1.22.

Chapter 2 deals with strongly quasipositive links, introduced by Lee Rudolph as those links bounding a special kind of surface called quasipositive surface [46]. This geometric definition can be translated into algebraic language by saying that strongly quasipositive links are those links which can be seen as closures of positive braids in terms of band (Birman-Ko-Lee) generators [12]. Positive links are those links having a diagram with no negative crossings. Rudolph showed that positive links are strongly quasipositive [48], but the converse is not true, even when we restrict to the class of links whose braid index equals 3, as we shown in Proposition 2.11.

By using resolution trees, Cromwell proved that positive links have positive Conway polynomial (that is, with all its coefficients being non-negative) [19]. In Chapter 2 we study if this property can be extended to the class of strongly quasipositive links. By defining a special kind of resolutions trees, in Theorem 2.8 we show that this extension works when considering links whose braid index equals 3, that is, we prove that strongly quasipositive links with braid index 3 have positive Conway polynomials. In Section 2.5 we show that this result cannot be extended to a higher number of strands by providing a strongly quasipositive braid (with 6 strands) whose closure has non-positive Conway polynomial.

Chapter 3 is devoted to present our contribution to the aim of finding a combinatorial definition of Knot Floer homology in terms of FKT-states. These states were introduced by Kauffman in his book *Formal Knot Theory* in his description

of the Alexander polynomial. Knot Floer homology was developed independently Ozsváth and Szabó [40] and Rasmussen [43] as a categorification of Alexander polynomial. Based on the fact that the Alexander polynomial of a given link can be computed from the FKT-states associated to a diagram representing the link, some authors have tried to give a definition of Knot Floer homology in terms of Kauffman states. Their attempts have not been successful. In Section 3.3 we focus on a model proposed by Y. Rong [45] and we show that it is not appropriate by finding an example where the model is not invariant under Reidemeister moves (Example 3.12). Moreover, in Section 3.2 we show that the definition of Alexander polynomial given by Kauffman in [27] coincides with the one given by Ozsváth and Szabó in [41].

In Chapter 4 we deal with another homological invariant, Khovanov cohomology, introduced by M. Khovanov in [28] and redefined by O. Viro in terms of enhanced states [57]. In [17] the Lando graph of a given link diagram was introduced. Morton and Bae proved that the hypothetical extreme coefficient of the Jones polynomial coming from a link diagram is equal to the independence number of its Lando graph [7]. Hence, on one hand the Jones polynomial can be seen as the bigraded Euler characteristic of the Khovanov cohomology, and on the other hand the formula for the independence number certainly suggests the formula of an Euler characteristic. In Section 4.4 we combine both ideas, and as a result we get a new approach to extreme Khovanov cohomology in terms of Lando graph. More precisely, in Theorem 4.4 we prove that the extreme Khovanov cohomology of a link diagram equals the cohomology of the independence complex of its Lando graph. In fact, we prove that the extreme Khovanov cochain complex is a copy, shifted by some degree, of the cochain complex associated to the independent sets of vertices of the Lando graph.

Using this new approach, in Section 4.5 we find a relation between the homology of a certain and specific simplicial complex and the extreme Khovanov cohomology of a specific link diagram (Theorem 4.11). Then in Theorem 4.13 we show a link diagram with exactly two non trivial extreme Khovanov cohomology groups. This link is a basic piece in Section 4.6, where we present a family of knots with an arbitrary number of non-trivial extreme Khovanov cohomology modules, which are indeed examples of H-thick knots.

Most part of contents in Chapters 1 and 2 are contained in papers [51] and [52] by myself. Section 3.2 is inspired in a joint work with Louis Kauffman [26] and Chapter 4 contains some results appearing in [23], a joint work with Juan González-Meneses and Pedro M. G. Manchón.

Chapter 1

Kauffman's conjecture

1.1 Introduction

In this chapter we study a conjecture posed by Louis Kauffman in 1983 in his book "Formal Knot Theory" [27] involving alternative and pseudoalternating links.

In this book Kauffman introduced alternative links as an extension of the class of alternating links (remember that an oriented link is alternating if it can be represented by an alternating diagram, that is, a diagram where the crossings alternate under, over, under, over... as one travels along each component); this extension preserves some nice properties from alternating links. For instance, the canonical surface constructed from an alternative diagram representing a link is a minimal genus spanning surface for the link [27].

Previously, E.J. Mayland and K. Murasugi introduced the class of pseudoalternating links as those bounding a special kind of surface called generalized flat surface [36]. These links, which also extend the family of alternating ones, are interesting by themselves, as they have not only topological but algebraic properties (Proposition 1.7 is such an example).

It is not hard to prove that alternative links are pseudoalternating (in fact, combining Propositions 1.5 and 1.8 gives a direct proof of this result). Kauffman conjectured that the converse also holds:

Conjecture 1.1. *[27] The classes of alternative and pseudoalternating links are identical.*

Although this conjecture was stated by Kauffman, Mayland and Murasugi posed a similar question in [36]. As their paper was written 7 years before the notion of alternative diagram was introduced, they spoke about $*$ - products and wrote, literally, *"(...) we do not know whether every pseudoalternating link is a $*$ - product"*.

In addition, in 1989 Peter Cromwell introduced homogeneous links, an intermediate class between alternative and pseudoalternating ones [19]. In that

paper he referred explicitly to Conjecture 1.1, noting that its veracity would imply the equality between the three classes of links (alternative, homogeneous and pseudoalternating). In his own words, *"Kauffman conjectures that the class of alternative links equals the class of pseudoalternating ones, implying that all three classes are equivalent. This is not obvious"*.

One key point regarding the difference between these three concepts is the fact that alternative and homogeneous links are defined in terms of diagrams, (a link is alternative [homogeneous] if it can be represented by an alternative [homogeneous] diagram), but the definition of pseudoalternating links is more geometric, as they are defined as those links bounding a surface having some properties. There are some examples in the literature of authors using alternative and homogeneous as synonyms [54].

In Section 1.4 we prove that Conjecture 1.1 holds for those links whose first Betti number is at most 2 (this includes the class of knots of genus one). However, it is not true in general when this value increases, as we show in Section 1.3 by finding two counterexamples: a link and a knot whose first Betti numbers equal 3 and 4, respectively. In the way we work with the intermediate family of homogeneous links.
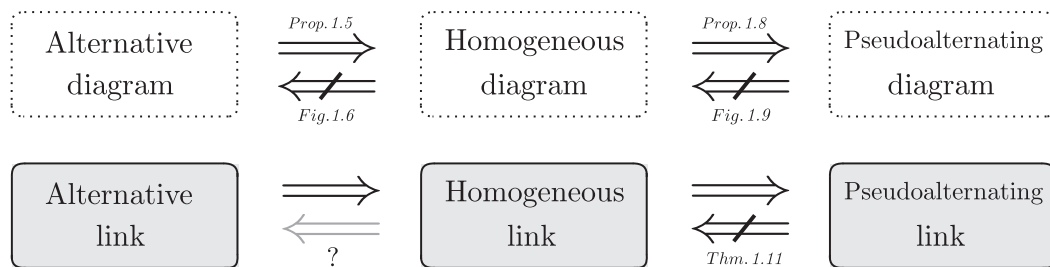

Figure 1.1: The implications between the families of alternative, homogeneous and pseudoalternating families.

1.2 Definitions

In this section we provide the definitions of the three families of links mentioned in Section 1.1. We remark that the alternative, homogeneous and pseudoalternating characters of a link are orientation dependant, so from now on all links will be oriented (and non-split).

1.2.1 Alternative links

Given an oriented diagram D of a link L, it is possible to smooth every crossing coherently with the orientation of the diagram, (that is, in the only way preserving orientation, as in Figure 1.3). Doing this for all crossings in D, we obtain a set of topological circles called *Seifert circles*. Following Kauffman, the *spaces* of the
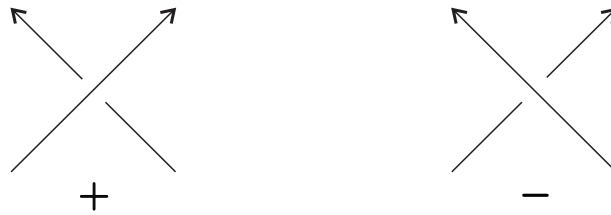
Figure 1.2: The sign convention is shown. It will be used throughout this work.

Figure 1.3: The smoothing used in Seifert's algorithm.

diagram D are the connected components of the complement of its Seifert circles in S^2 (one can also think on their complement in the plane), not to be confused with the regions of the knot diagram. We can then draw an edge joining two Seifert circles at the place where there was a crossing in D, and label the edge with the sign of the corresponding crossing according to Figure 1.2. We will refer to the resulting set of topological circles and labeled edges as the *Seifert diagram* of D, because of the analogy of this process to Seifert's algorithm for constructing an orientable surface spanning a link [50].

Definition 1.2. [27] *An oriented diagram D is alternative if all edges in any given space of D have the same sign. An oriented link is alternative if it admits an alternative diagram.*

Notice that there are non-alternative diagrams representing alternative links, as can be seen in Figure 1.6.

Positive and negative links are alternative (a positive/negative diagram leads to a Seifert diagram with all edges having the same sign). Alternating links are alternative. There are nevertheless alternative links which are not alternating: for instance, every positive (hence alternative) non-alternating link, like the knot 8_{19}. Although the following result is not written in terms of links but of diagrams, it clarifies the relation between both families:

Proposition 1.3. [27, Lemma 9.2] *A link diagram is alternating if and only if it is alternative and the sign of the edges in its Seifert diagram changes alternatively when passing through adjacent spaces.*

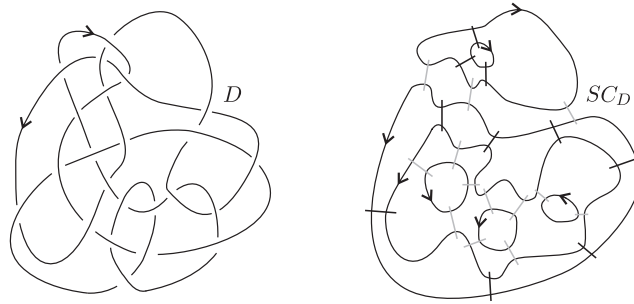Figure 1.4 illustrates the previous Proposition.

Figure 1.4: An alternating diagram D representing a two components link and its associated Seifert diagram SC_D. A dark edge has a positive label; a light edge a negative one. As D is alternating, it is an alternative diagram and the sign of the edges in SC_D alternates when passing through adjacent spaces.
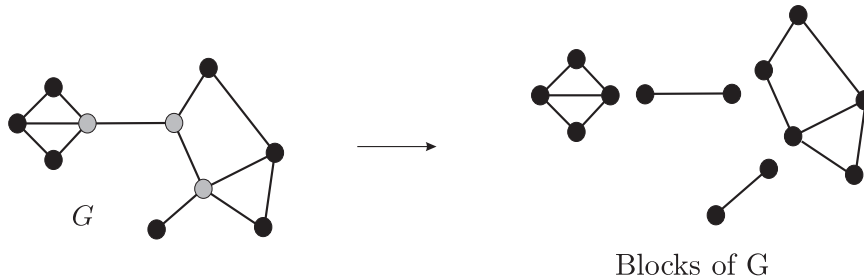

Figure 1.5: A connected graph and its associated blocks. The cut vertices are represented by grey dots.

1.2.2   Homogeneous links

We consider now the family of homogeneous links, introduced by Peter Cromwell in 1989 [19]. From the Seifert diagram associated to D, we can construct a graph G_D as follows: associate a vertex to each Seifert circle and draw an edge connecting two vertices in G_D for each edge joining the associated circles in the Seifert diagram; each edge must be labeled with the sign $+$ or $-$ of its associated crossing in D. The signed graph G_D is called the *Seifert graph* associated to D. G_D can be obtained from the Seifert diagram of D by collapsing each circle to a vertex. See Figure 1.6. Note that, as Seifert circles are oriented, Seifert diagrams have no edges with both endpoints in the same circle, hence Seifert graphs do not contain loops.

Given a connected graph G with no loops, a vertex v is a *cut vertex* if removing it disconnects the graph, that is, if $G\backslash\{v\}$ is disconnected. A block of G is a maximal subgraph of G with no cut vertices. Blocks of the graph G can be thought of in the following way: remove all the cut vertices of G; each remaining connected component together with its adjacent cut vertices is a block of G. In Figure 1.5 we show a connected graph and its associated blocks.

Definition 1.4. [19] *A Seifert graph is homogeneous if all the edges of a block*
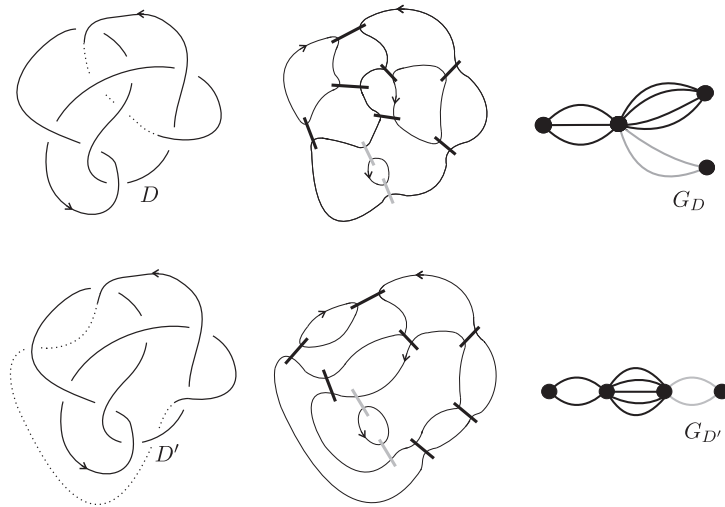
Figure 1.6: Two diagrams D and D' of the knot 9_{43}, their associated Seifert diagrams and Seifert graphs. A dark edge has a positive label; a light edge a negative one. D shows that 9_{43} is homogeneous, as the diagram in the figure is so; D is non-alternative, but since D' is so, 9_{43} is alternative.

have the same sign, for all blocks in the graph. An oriented diagram D is homogeneous if its associated Seifert graph G_D is homogeneous. An oriented link is homogeneous if it admits a homogeneous diagram.

Note that the original diagram D can be recovered from its Seifert diagram, as the sign and position of the crossings in the diagram are preserved (see Figure 1.6). However, as the relative position of the circles and the order of the edges are not encoded in the Seifert graph, D cannot be recovered from G_D. In this sense, one can say that there is a loss of information when considering Seifert graphs instead of Seifert diagrams.

Proposition 1.5. *Alternative links are homogeneous.*

Proof. The proof lies in the fact that alternative diagrams are homogeneous. The key point is to notice that the circles "touching" positive and negative edges in the Seifert diagram become cut vertices in the Seifert graph (hence, there are no positive and negative edges in the same block). $\qquad\square$

It is clear that the converse is not true in terms of diagrams, that is, homogeneous diagrams are not necessarily alternative (diagram D in Figure 1.6 is such an example).

Before introducing pseudoalternating links, let us give an answer to a question posed by Peter Cromwell in [19]. Motivated by the fact that a non-alternating diagram of a prime alternating link cannot have minimal crossing number [38], he wondered if there exists any homogenous link with a non-homogeneous diagram of minimal crossing number. The answer is that there exists such a link. As
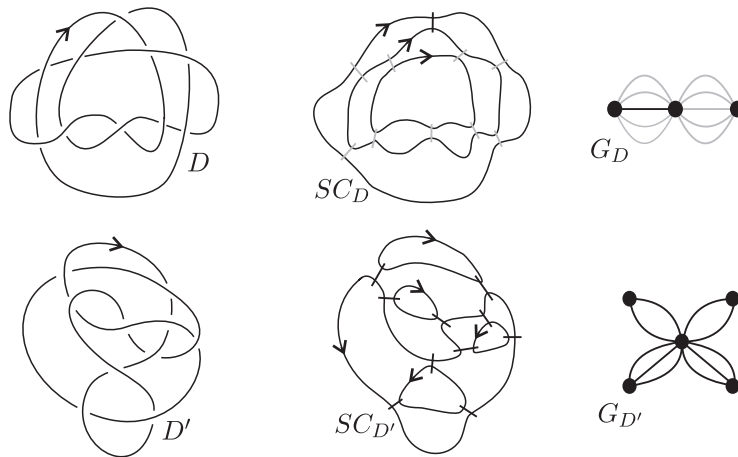
Figure 1.7: Two equivalent diagrams representing Perko pair (D and D' are the classical diagrams for knots 10_{161} and 10_{162} in Rolfsen's table, respectively). Both of them have minimal crossing number (10), however D' is homogeneous and D is not.

an example, consider the Perko pair given by the equivalent diagrams of knots $10_{161} \equiv 10_{162}$ [9]. The first one is minimal and non-homogeneous, and the second one is positive hence homogeneous (see Figure 1.7). [Notice that in some updated versions knot 10_{162} has been removed and subsequent knots have been renumbered, so that knot 10_{163} is called 10_{162}, and so on.]

1.2.3   Pseudoalternating links

Remember that starting from an oriented diagram D of a link L one can obtain its associated canonical surface S_D by applying Seifert's algorithm [50]. The graph G_D can also be thought as the spine graph of the corresponding canonical surface.

Primitive flat surfaces [36] are those canonical surfaces arising from positive or negative diagrams whose Seifert diagrams have no nested circles. A *generalized flat surface* is, roughly speaking, an orientable surface obtained by gluing a finite number of primitive flat surfaces along some of their discs, without overlapping bands. (Figure 1.8 helps to understand the idea.)

More precisely, given two primitive flat surfaces S_1 and S_2, choose a disc of each one, d_1 and d_2. Now, identify both discs in such a way that there exists a sphere $S^2 \subset S^3$ separating S^3 into two non empty 3-balls B_1 and B_2 such that $S_i \subset B_i$ and $S^2 \cap S_i = d_i$, for $i = 1, 2$. Bands starting at d_1 and d_2 are not allowed to overlap when identifying d_1 and d_2. This special kind of Stallings plumbing (or Murasugi sum) will be noted by $*$ (see Figure 1.8). Generalized flat surfaces are obtained as a finite iteration of this process, plumbing a primitive flat surface at each step.

The *first Betti number* of a surface S, $\beta(S)$, is the rank of its first homology group. For a primitive flat surface this is just the number of holes in the surface, or equivalently the number of connected components of the complement of its
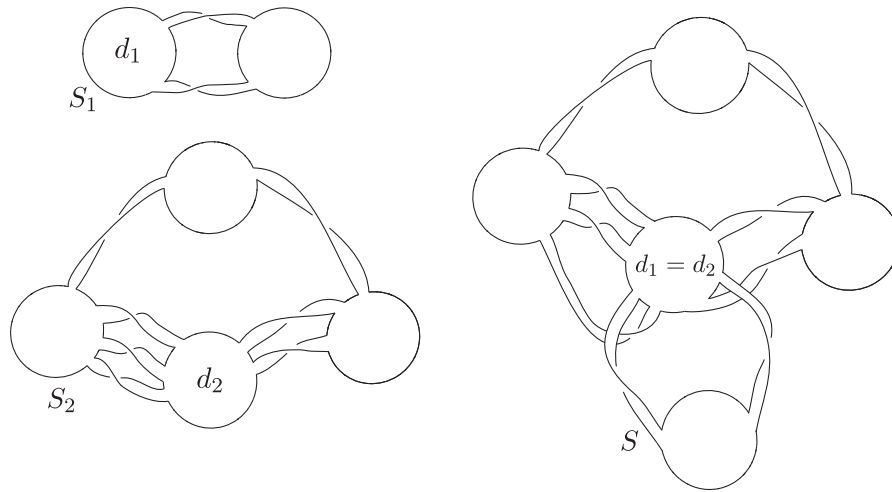
Figure 1.8: S_1 and S_2 are primitive flat surfaces, with $\beta(S_1) = 1$ and $\beta(S_2) = 4$. By an identification of discs d_1 and d_2 one obtains the generalized flat surface $S = S_1 * S_2$, having $\beta(S) = 5$. The link spanned by S is a pseudoalternating link.

spine graph in the plane minus 1. As the Euler characteristic of one of these surfaces can be computed as its number of discs d_S minus its number of bands b_S, it follows that $\beta(S) = b_S - d_S + 1$. Notice that the first Betti number is additive under this special kind of plumbing: each time one plumbs two surfaces both plumbing discs are identified, so the resulting spine graph can be thought as gluing the two original graphs along a vertex.

As generalized flat surfaces are orientable, the following definition makes sense:

Definition 1.6. [36] *An oriented link is said to be pseudoalternating if it is the boundary of a generalized flat surface, with the natural inherited orientation.*

Proposition 1.7. [36] *Let L be a pseudoalternating link with associated generalized flat surface S_L. Then S_L has maximal Euler characteristic and it is algebraically unknotted. Namely, $\pi_1(S^3 - S) = F_{2g+\mu-1}$, a free group of rank $2g + \mu - 1$, where g is the genus of S_L and μ the number of components of L.*

The fact that generalized flat surfaces have maximal Euler characteristic (or equivalently, minimal genus) among all the connected Seifert surfaces that span a given pseudoalternating link (Proposition 1.7) implies that the first Betti number of a pseudoalternating link L, $\beta(L) = \min\{\text{rank}(H_1(S)) \mid \partial S = L\}$, is given by any generalized flat surface spanning it. It also follows from a well-known result by Gabai [22, Corollary 6.7].

Proposition 1.8. *Homogeneous links are pseudoaltenating.*

Proof. Let D be a homogeneous diagram of a link and S_D its associated canonical surface. Let $G_1, \ldots, G_n$ be the blocks of the associated homogeneous Seifert graph
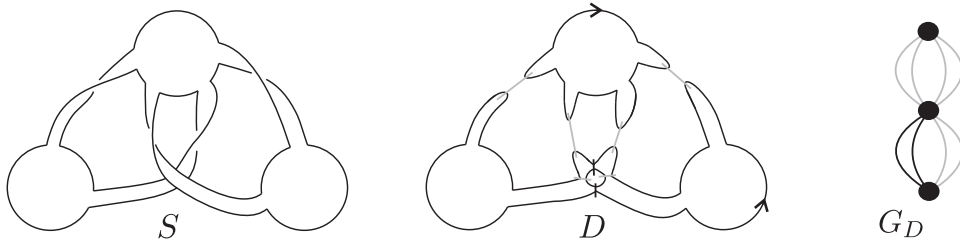
Figure 1.9: The surface S spans the diagram D. Although S is a generalized flat surface, D is not homogeneous, as its associated Seifert graph G_D is not homogeneous.

G_D, D_i the subdiagram of D (together with some arcs) associated to the subgraph $G_i \subset G_D$, and $S_{D_i} \subset S_D$ the projection surface constructed from D_i.

Then, $S_D = S_{D_1} * \ldots * S_{D_n}$. Since blocks in a graph do not contain cut vertices, each S_{D_i} is a primitive flat surface: all bands are twisted in the same way and the discs on the surface are either not nested, or there exists a single disc containing the other ones (and this situation is isotopic to the previous one). Hence, S_D is a generalized flat surface.   □

Although the canonical surface constructed from a homogeneous diagram is a generalized flat surface, the converse is not necessarily true, that is, a diagram spanned by a generalized flat surface is not necessarily homogeneous, as can be seen in Figure 1.9. Remember that something similar happened with alternative and homogeneous diagrams. These two observations together with the next restatement of Conjecture 1.1 are the key points for understanding the difficulties of proving the conjecture, and, at the same time, the "clues" for looking for counterexamples.

Propositions 1.5 and 1.8 imply that Conjecture 1.1 can be restated as follows:

Conjecture 1.9. [27] *The classes of alternative, homogeneous and pseudoalternating links are equal.*

In the following section we present two pseudoalternating links which are not homogeneous, hence non-alternative; these links are counterexamples to Conjecture 1.9, hence to Conjecture 1.1.

1.3   Two counterexamples to the conjecture

The main problem when trying to deal with Kauffman's conjecture is that identifying whether a link is alternative or pseudoalternating is not easy, as these properties are defined in terms of diagrams. Of course, by finding an alternative diagram one shows the alternativity of a link, but this does not help when the link is not alternative. In this sense, working with the intermediate family of homogeneous links will help us.
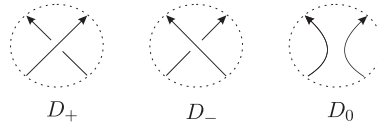
Figure 1.10: Diagrams D_+, D_- and D_0.

Cromwell's paper [19] is devoted to the study of homogeneous links. He provides some sufficient conditions for determining the non-homogeneous character of a link. These properties help to determine the homogeneity within some families of arborescent links, double knots, and all but 5 knots with crossing number at most 10.

In particular, we will use the following result :

Theorem 1.10. [19, Corollary 5.1] *If L is a homogeneous link and the leading coefficient of its Conway polynomial $\nabla(L)$ is ± 1, then the crossing number of L is at most $2 \cdot maxdeg \, \nabla(L)$.*

The Alexander-Conway polynomial (or just Conway polynomial) was re-discovered in 1969 by J. Conway as a normalized version of the Alexander polynomial. Given an oriented link L, its Conway polynomial $\nabla(L)$ is an invariant of L defined as $\nabla(L) = P(1, z)$, where $P(v, z)$ is the HOMFLYPT polynomial ([21] [42]). More precisely, $\nabla(L)$ is the only polynomial in $\mathbb{Z}[z]$ satisfying the skein relation $\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)$, with normalization $\nabla(\text{unknot}) = 1$. Here L_+, L_- and L_0 are links represented by diagrams D_+, D_- and D_0 respectively, which only differ in the neighborhood of a crossing as shown in Figure 1.10.

At this point, we are ready to show the result refuting Conjecture 1.9:

Theorem 1.11. *There exists a pseudoalternating link which is not homogeneous.*

Proof. Consider the oriented link L with two components presented by the oriented diagram D shown in Figure 1.11. This link is $L9n18\{1\}$ in [16], which corresponds to the two-components link $L9n18$ in Thistlethwaite table (or 9^2_{53} in Rolfsen table). Its Conway polynomial is $\nabla(L) = z^3 + 4z$, so by Theorem 1.10, if L were homogeneous its crossing number would be at most $2 \cdot 3 = 6$, yielding a contradiction. Consequently, L is non-homogeneous.

See Figure 1.11. The diagram D can be transformed by a Reidemeister III and a Reidemeister I move into the diagram D'. The new diagram D' allows us to see L as the boundary of a certain surface S. The surface S is clearly isotopic to S', and S' is the result of performing two Stallings plumbings of three surfaces, S_1, S_2, S_3, using discs d_1, d_2 and d_3 as "gluing patches", this is, $S = S_1 * S_2 * S_3$. Each of these surfaces consists on two discs joined by a pair of bands twisted in a positive way, so S_1, S_2 and S_3 are primitive flat surfaces; hence S is a generalized flat surface and L is a pseudoalternating link. □
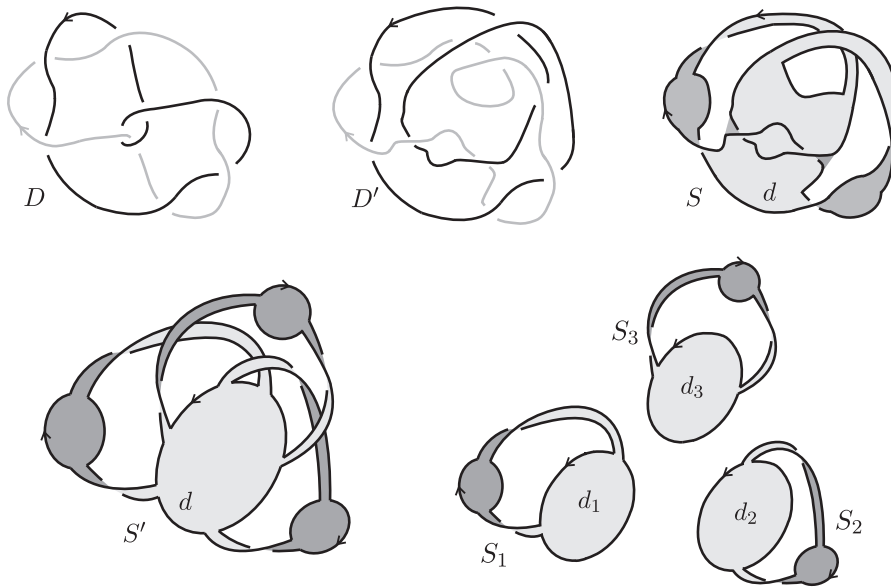
Figure 1.11: D and D' are diagrams of the link $L = L9n18\{1\}$ in [16]; S is a Seifert surface for L, and S' is obtained from S just by overturning the subsurface S_3 over the disc d. Finally $S' = S_1 * S_2 * S_3$.
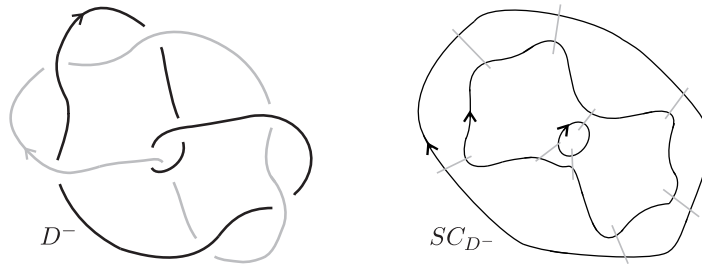
Figure 1.12: Notice that the diagram D in Figure 1.11 becomes negative after reversing orientation of the dark component. Obviously, this diagram is alternative, as its associated Seifert diagram has only edges of negative type.

Since every alternative link is homogeneous, the link $L9n18\{1\}$, with $\beta(L) = 3$, is also a counterexample to Conjecture 1.1.

Corollary 1.12. *There exists a pseudoalternating link which is not alternative.*

Remember that alternative, homogeneous and pseudoalternating characters are orientation dependant. In fact, if one changes the orientation of one component of L (for example, reversing the orientation of the dark component in Figure 1.11) the resulting link is negative, hence alternative, as can be seen in Figure 1.12.

At this point one can wonder if there exist knots or links which are pseudoalternating and non-alternative with any possible orientation of its components. We will show such an example by finding a knot of genus two with these properties, as

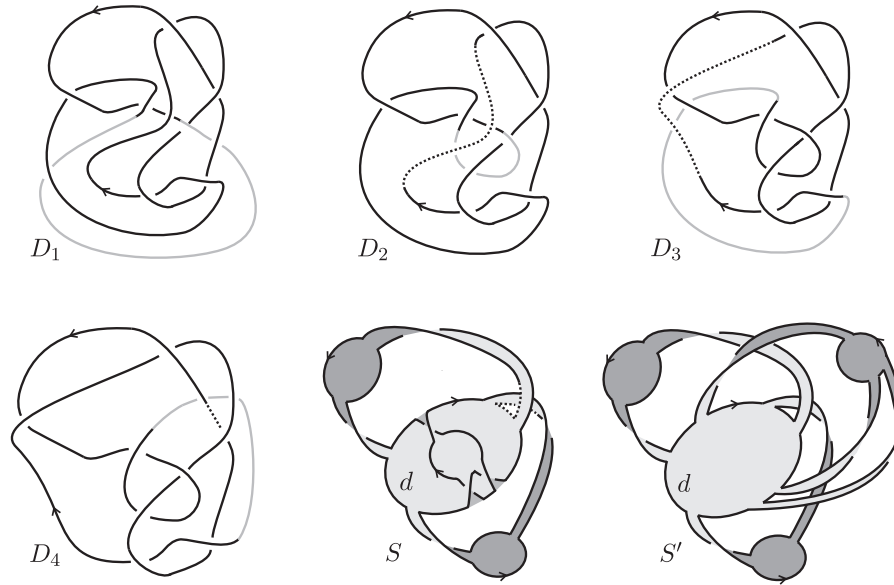Figure 1.13: From D_1 to D_2 and from D_3 to D_4 just move the grey strand, leaving unchanged the rest of the diagram; from D_2 to D_3 perform two Reidemeister III moves on the dotted strand. By performing a Reidemeister I move on the dotted strand in D_4, we see that S is a Seifert surface for K; S' is the result of overturning one of the primitive flat surfaces over the disc d.

pseudoalternating, homogeneous and alternative characters are not orientation-dependant in the case of knots. This knot, whose first Betti number equals 4, is pseudoalternating and non-alternative. It would be interesting to find a link with more than one component being a counterexample to the conjecture with all possible orientations.

Theorem 1.13. *There exists a pseudoalternating knot which is not homogeneous.*

Proof. The proof is analogous to that of Theorem 1.11. Let K be the genus two knot 10_{145} in Rolfsen table, with the orientation given in Figure 1.13. Its Conway polynomial is $\nabla(K) = z^4 + 5z^2 + 1$, so by Theorem 1.10 we deduce that K is not homogeneous, as $10 > 2 \cdot 4$.

See Figure 1.13. By a finite sequence of Reidemeister moves, the classical diagram representing K in [15], D_1, can be transformed into D_4 by performing the following steps: from D_1, we obtain D_2 by leaving unchanged the diagram except for the grey strand; perform two Reidemeister III moves on the dotted strand of D_2 in order to get D_3; finally, transform D_3 into D_4 by moving the grey strand. At this point it is easy to see that the surface S bounds K; S is a generalized flat surface obtained by performing two Stallings plumbings of three primitive flat surfaces (two of them consist on a pair of discs joined by two bands, and the other one consists on two discs together with three bands, as shown in S') using twice the same disc, d, as "gluing patch". As a result, K is a pseudoalternating knot. □
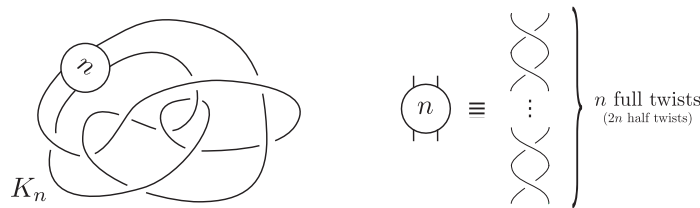
Figure 1.14: Description of knot K_n introduced by Stoimenow in [55, Example 6.1]. The integer value of $n > 0$ represents the number of full twists. As an example, if $n = 2$ the circle in the diagram contains 4 positive crossings.

Corollary 1.14. *There exists a pseudoalternating knot which is not alternative.*

Since 10_{145} is a non-alternating, non-positive knot of genus two, the following result by Stoimenov detects its non-alternative character (note that our result in Theorem 1.13 is stronger, as alternative links are homogeneous). We remark that the definition of homogeneous link given by Stoimenov in [54] is equivalent to our definition of alternative link, taken from Kauffman [27].

Theorem 1.15. [54, Theorem 4.1] *Any alternative genus two knot K is alternating or positive.*

As we said before, alternativity, homogeneity and pseudoalternation depend on the orientation in the case of links but not when working with knots. Moreover, a knot is alternative, homogeneous or pseudoalternating if and only if its mirror image is so (remember that the mirror image of a knot K can be thought as the knot whose diagram is obtained by changing all the crossings in the diagram of K). Since $K = 10_{145}$ oriented as in Figure 1.13 is chiral, its mirror image K^* is another counterexample to Conjectures 1.1 and 1.9.

Theorems 1.11 and 1.13 provided the first counterexamples disproving Conjecture 1.1. After these results were published in our paper [51], Tetsuya Abe and Keiji Tagami found a family of counterexamples for Kauffman's conjecture by using techniques involving Khovanov and Lee homologies and the Rasmussen, Beliakova and Wehrli's s-invariant for links [1].

This family of knots is infinite as it depends on an integer parameter $n > 0$; write K_n for the knot obtained after fixing n. It contains knot 10_{145}, and its definition is given in Figure 1.14.

Their proof lies in the following result:

Proposition 1.16. [1, Corollary 1.7] *Let L be an almost-positive link (that is, it is not positive and it can be represented by a diagram with exactly one negative crossing). Then L is non-homogeneous.*

As every knot K_n in the mentioned family is almost-positive, it is non-homogeneous, hence non-alternative. The procedure given in [1] for getting a
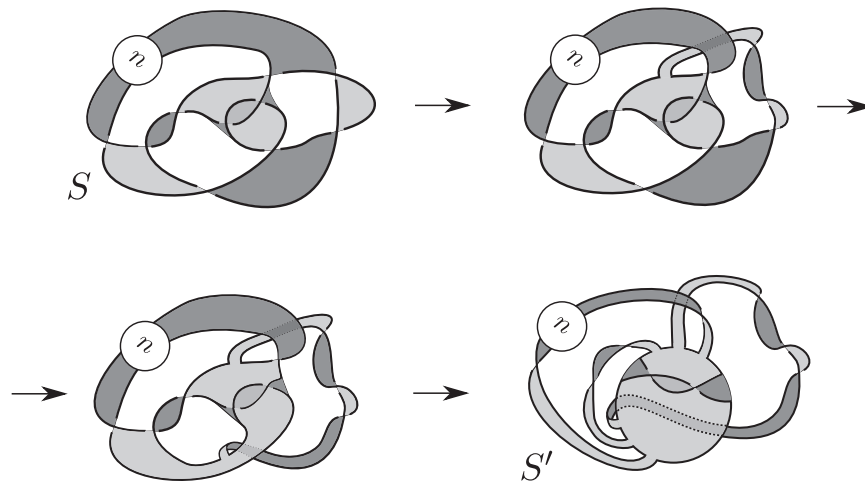
Figure 1.15: Starting from a diagram such those in Figure 1.14, construct its canonical Seifert surface S. After performing some isotopies in the surface one gets the generalized flat surface S'. Figure taken from [1].

generalized flat surface spanning the knot K_n is shown in Figure 1.15. Hence, each K_n is a pseudoalternating non-alternative knot for every $n > 0$.

1.4   The case $\beta(L) \leq 2$

Theorems 1.11 and 1.13 show that Conjecture 1.1 does not hold for links or knots in general. Nevertheless, in this section we prove that all links whose first Betti number is smaller than 3 satisfy Kauffman's conjecture (including the particular case of knots of genus one).

Although in Figures 1.11 and 1.13 we show how to transform the classic diagrams of link $L9n18\{1\}$ and knot 10_{145} in order to get a generalized flat surface bounding each of them, our procedure for finding such counterexamples worked in the opposite way: we started with their generalized flat surfaces, checked that the associated pseudoalternating links were not homogeneous and after determining which links they were using topological properties, we found the sequence of moves transforming the diagrams bounding the surfaces into the ones appearing in the tables.

In general it is not easy to find a generalized flat surface spanning a link given by a diagram, even if one knows that the link is pseudoalternating. Here we give an upper bound for the number of non-trivial (that is, non-isotopic to a disc) primitive flat surfaces which can be plumbed in order to get a generalized flat surface spanned by a pseudoalternating link.

Lemma 1.17. *Let S be a generalized flat surface spanning a pseudoalternating link L with first Betti number $\beta(L)$. If $S = S_1 * S_2 * \ldots * S_n$, with each S_i a*
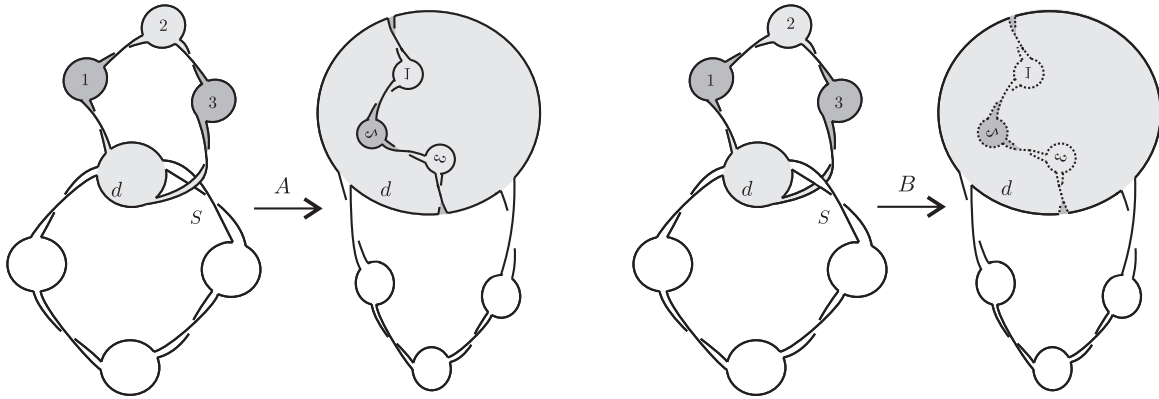
Figure 1.16: $S = S_1 * S_2$, with S_1 and S_2 two primitive flat surfaces plumbed by using d as gluing disc; S_2 has been colored. In A (B), the surface S_2 has been overturned over (under) the disc d.

non-trivial primitive flat surface, then $n \leq \beta(L)$.

Proof. As S is a generalized flat surface, $\beta(S) = \beta(L)$, by Proposition 1.7. As the surface S_i is connected and non-trivial, $\beta(S_i) \geq 1$. Then, after performing n Stallings plumbings, one gets $\beta(S_1 * S_2 * \ldots * S_n) = \beta(S_1) + \beta(S_2) + \ldots + \beta(S_n) \geq n$. $\qquad \square$

Lemma 1.18. *Let S be a generalized flat surface spanning a pseudoalternating link L. If S is either a primitive flat surface or a generalized flat surface constructed as the Stallings plumbing of two primitive flat surfaces, then L is alternative.*

Proof. If S is a primitive flat surface, then L is a positive or negative link, hence it is alternative.

Otherwise $S = S_1 * S_2$, with S_1 and S_2 two primitive flat surfaces. Now just turn S_2 over (or under) the "gluing disc" (see Figure 1.16 for an example of a plumbing of two annuli; notice that any primitive flat surface could be turned over in the same way). Then it is clear that the projection of the boundary of the new surface on the plane that contains the discs of S_1 provides a diagram for L with just two spaces containing edges; this diagram is alternative, as edges related to S_1 and S_2 are in different spaces. $\qquad \square$

Combining Lemmas 1.17 and 1.18 leads to the following result:

Corollary 1.19. *Every pseudoalternating link L with $\beta(L) \leq 2$ is alternative.*

The following corollaries are particular cases of Corollary 1.19:

Corollary 1.20. *Every pseudoalternating genus zero link with three or less components is alternative.*

Proof. Let L be a pseudoalternating genus zero link with μ components, and let S be a generalized flat surface whose boundary is L. As $\beta(L) = 2g(L)+\mu-1 = \mu-1$, by Lemma 1.17 the surface S is plumbing of at most $\mu - 1$ non-trivial primitive flat surfaces. If $\mu = 1$, L is the trivial knot, which bounds a disc, hence it is alternative. If μ is 2 or 3, the result holds by Lemma 1.18. $\square$

We claim that pseudoalternating genus zero links with four or five components are also alternative (notice that their first Betti numbers are 3 and 4, respectively). In order to show this, one must check all different possible ways of gluing primitive flat surfaces to obtain such links, and take into account that, when plumbing a surface S with $\beta(S) = 1$ to a pseudoalternating surface S', the number of boundary components increases (decreases) by one when both bands coming from the gluing disc are attached to the same (different) boundary component of S'. This case by case procedure is straightforward but lengthy, so we include the proof for genus zero links having four components in Section 1.5, and omit the five components case.

Corollary 1.21. *Every pseudoalternating genus one knot is alternative.*

Proof. Let K be a pseudoalternating genus one knot, and let S be a generalized flat surface whose boundary is K. By Lemma 1.17, as $\beta(K) = 2g(K)+\mu-1 = 2$, S is plumbing of at most two non-trivial primitive surfaces. Lemma 1.18 completes the proof. $\square$

Let us digress briefly in order to introduce the well-known family of pretzel links. Start with a set of sequences of half-twists and connect them as in Figure 1.17. The resulting diagram is a pretzel link. One can associate an integer number to each of these half-twists columns representing the number of crossings and the sense of the twist. With this notation, the pretzel link $P(a_1, a_2, \ldots, a_r)$ is the link represented by a diagram with r columns, whose $i-th$ column has $|a_i|$ half-twists (the sense of the twists is given by the sign of a_i).

From Corollary 1.21 it follows that a knot of genus one is pseudoalternating if and only if it is homogeneous. As a consequence, we obtain an alternative proof of the following result:

Theorem 1.22. [33] *A genus one knot is homogeneous if and only if it belongs to one of the two following families of knots:*
1.- Pretzel knots with diagram $P(a, b, c)$, where a, b, c are odd integers with the same sign.
2.- Pretzel knots with diagram $P(m, e, .\overset{k}{.}., e)$, where m and k are non-zero even integers and $e = \pm 1$.
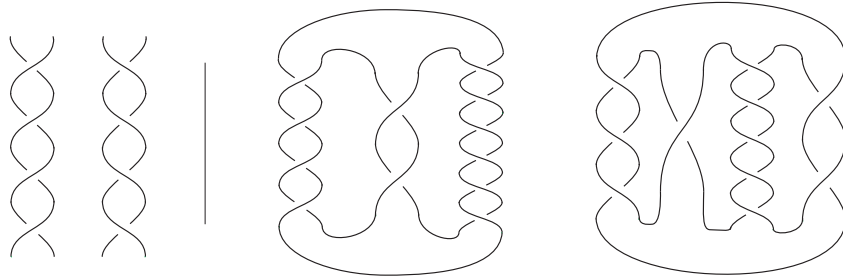
Figure 1.17: The leftmost part shows two sequence of half-twists having positive and negative signs, respectively. In the rightmost the three components pretzel link $P(4, 2, -6)$ and the pretzel knot $P(-3, 1, -5, 2)$ are shown.

Proof. As before, $K = \partial S$, where S is a primitive flat surface with $\beta(S) = 2$ or $S = S_1 * S_2$ is the plumbing of two primitive flat surfaces having $\beta(S_1) = \beta(S_2) = 1$.

In the first case, K is a positive or negative knot. As S is connected and $\beta(S) = 2$, S must be as shown in Figure 1.18 (left). Write A, B, C for each of the three subsurfaces consisting on "a linear path of bands and discs" in S (A, B and C in cyclic order when traveling through the two common discs d_δ and d_γ). As K is a knot, each of these paths must start and end in discs with different orientations (otherwise the link bounding the surface would be a 3-components link). Hence, their respective numbers of bands, a', b', c', are odd. Let ε be their common sign. K is the Pretzel knot $P(\varepsilon \cdot a', \, \varepsilon \cdot b', \, \varepsilon \cdot c')$, as can be seen in Figure 1.18.

Now assume that $S = S_1 * S_2$. Note that after the plumbing, the pair of bands attached to d_2 in S_2 must alternate with the two bands attached to d_1 in S_1 in the gluing disc $d = d_1 = d_2$; otherwise, the resulting surface would span a 3-components link. Let b_i be the number of bands in S_i and ε_i their signs, $i = 1, 2$. $\beta(S_1) = \beta(S_2) = 1$, so both S_1 and S_2 are twisted Hopf-bands, that is, each of them consists on k discs and k bands joined forming a circle. Since they are oriented, b_i is even and it follows (see an example in Figure 1.19) that K is the Pretzel knot $P(\varepsilon_1 \cdot b_1, \varepsilon_2, .^{b_2}., \varepsilon_2)$. Write $m = \varepsilon_1 \cdot b_1, \quad k = b_2$ and $e = \varepsilon_2$. $\square$

1.5   Genus zero four-components links

In this section we prove the following result:

Proposition 1.23. *Every pseudoalternating genus zero link with four components is alternative.*

Proof. Let L be a pseudoalternating genus zero link with four components, and let S_L be a generalized flat surface spanning L. Writing $\mu(L)$ for the number of
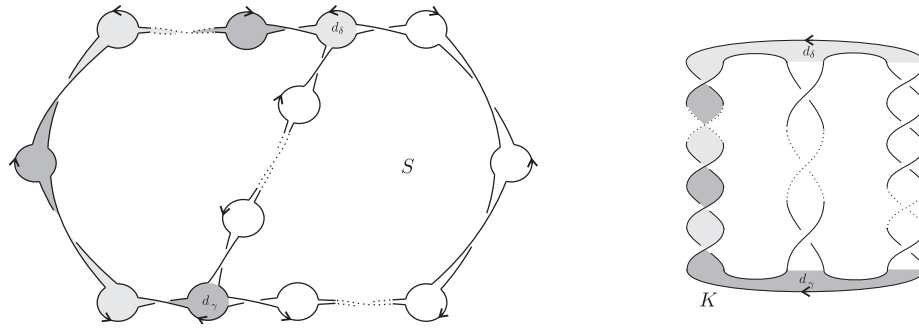
Figure 1.18: S is a primitive flat surface; an even number of discs and bands can be attached in the place of the dotted lines (or removed). One of the three subsurfaces, say A, has been colored. If one thinks on the example in the picture as if all lines were non-dotted, then $a' = 5$, $b' = 3$, $c' = 5$ and $\varepsilon = -$.
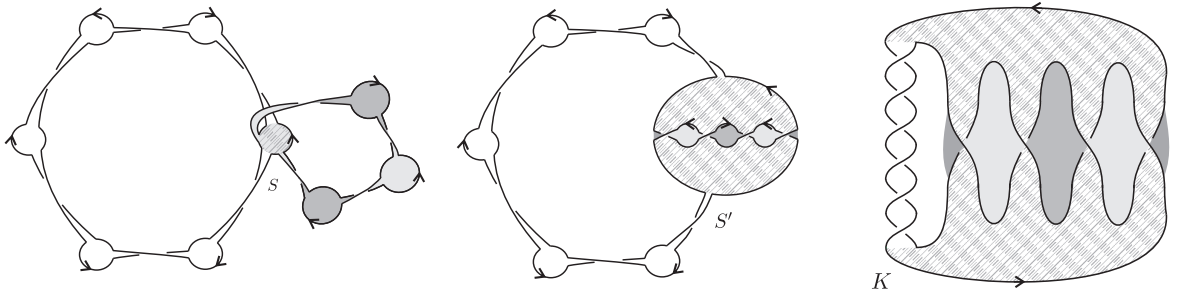

Figure 1.19: $S = S_1 * S_2$ is a generalized flat surface; S_2 has been colored. After over-turning S_2 over the dotted disc, it is clear that K is the Pretzel knot $P = (-6, 1, 1, 1, 1)$

components of L, we have $\beta(S_L) = \beta(L) = 2g(L) + \mu(L) - 1 = 3$, hence Lemma 1.17 implies that S_L is plumbing of at most 3 primitive flat surfaces. We can also say that the number of bands in S_L exceeds in 2 its number of discs, as $\beta(S_L) = b_{S_L} - d_{S_L} + 1$.

If S_L is a primitive flat surface or $S_L = S_1 * S_2$ with S_1 and S_2 being primitive flat surfaces, Lemma 1.18 implies that L is alternative.

We focus now on the case $S_L = S_1 * S_2 * S_3$ with S_1, S_2 and S_3 primitive flat surfaces. Notice that each time two surfaces are plumbed, we keep all the bands but two discs are identified, that is, after the first plumbing $d_{S_1*S_2} = d_{S_1} + d_{S_2} - 1$ and $b_{S_1*S_2} = b_{S_1} + b_{S_2}$. As a consequence, each of S_1, S_2 and S_3 has the same number of discs and bands, so they consist on n discs and n bands joined forming a circle (they are twisted Hopf-bands).

Twisted Hopf bands have two boundary components, and each time one of these surfaces H is plumbed to a generalized flat surface S, the number of boundary components of the resulting surface increases (decreases) by one if the two bands of the gluing disc in H are attached at the same (different) boundary component of S. As S_1, S_2 and S_3 are Hopf bands, the only possible configuration for getting a four components link is increasing by one the number of boundary

components after each plumbing. Combining both facts in this paragraph, after the first plumbing $S_1 * S_2$ the situation must be as shown in Figure 1.20.

At this point, we need to plumb S_3. If the gluing disc in $S_1 * S_2$ is different from the disc used in the first plumbing, then the resulting link is alternative (one just has to overturn the surfaces as in Figure 1.16 if necessary).

Now consider that the gluing disc is used twice. There are three possible options attaching the two bands of S_3 in the same boundary component, as can be seen in Figure 1.20. The key point is noticing that in all of them any surface can be overturned inside the gluing disc, in case its sign is different from the others, and the resulting diagram would be alternative.

<div align="right">□</div>
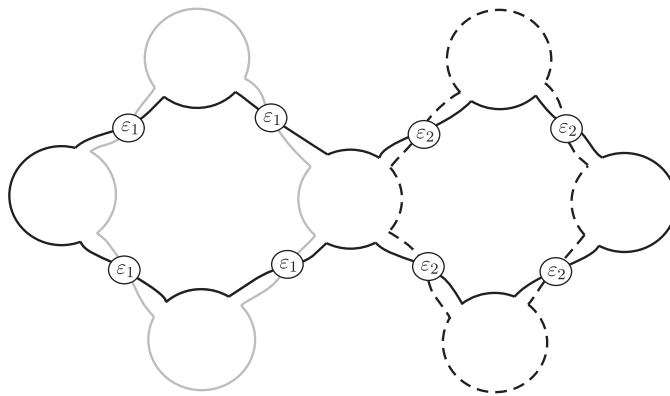

Figure 1.20: The surface $S_1 * S_2$ after the first plumbing in the case of a generalized flat surface constructed as the plumbing of three primitive flat surfaces using the gluing disc twice. The sign of the twists in the bands coming from S_1 (S_2) is ε_1 (ε_2). The number of bands and discs in each of the primitive flat surfaces can be decreased/increased.

Chapter 2

Strongly quasipositive links and their Conway polynomials

2.1   Introduction

The notion of positivity related to a link has been deeply studied from many different points of view. Maybe the simplest class involving this concept is the family of positive links. Recall that an oriented link is said to be positive if it has a positive diagram, that is, a diagram with all crossings being positive (see Figure 2.4).

Many authors have extended this classical notion of positivity by defining new families of links. Frequently, these definitions depend on finding a representative (a diagram, a braid, a Seifert surface...) of the link having some properties. As in general finding such a representative is not easy, giving conditions for the invariants of a link belonging to a specific family becomes useful.

A well-known result by J. Alexander is the popularly known as Alexander's Theorem [2]. It states that every oriented link can be represented as the closure of a braid $\beta \in \mathbb{B}_n$ for some n. Recall that the *braid index* of a link is the smallest n such that the link can be represented as a closed braid on n strands (it can also be defined as the minimum number of Seifert circles in any Seifert diagram of the link). In this chapter we give a necessary condition for a link with braid index 3 to be strongly quasipositive.

Strongly quasipositive links are intrinsically related to positive braids, so let us recall some notions of positivity related to braids. We remark that in the following cases positivity depends on the choice of presentation of the braid group. Roughly speaking, a braid is positive if there exists a positive word representing it, that is a word with all its letters having positive exponents.

Through this chapter we mainly work with two different presentations of the braid group on n strands, $\mathbb{B}_n$. The first one is the standard presentation due to Artin ([4], [5]):
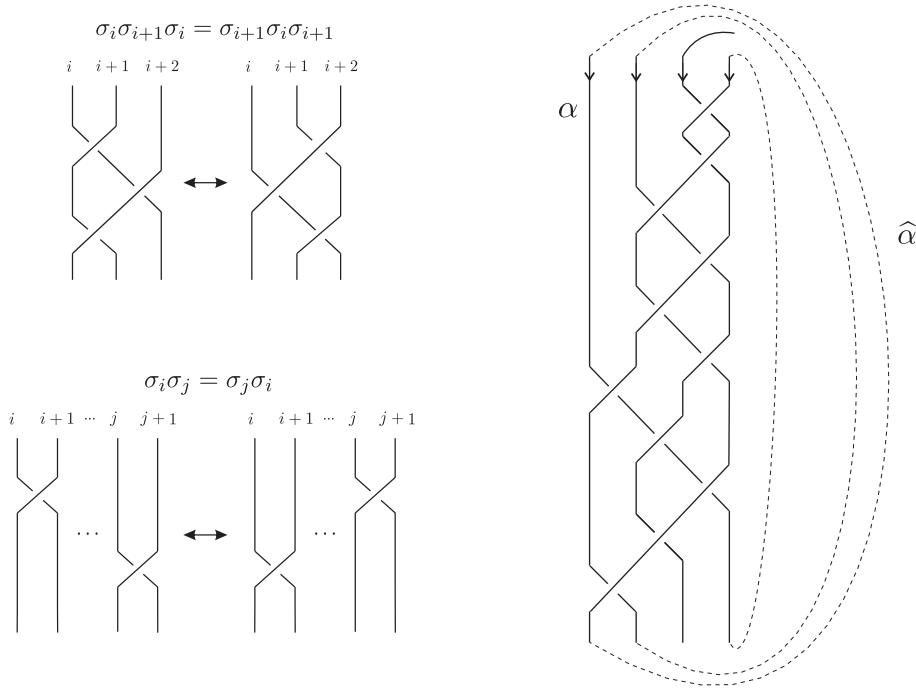
Figure 2.1: Braid relations in the Artin presentation, the Artin-positive braid $\alpha = \sigma_3\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1$ on 4 strands and its closure $\widehat{\alpha}$, an Artin-positive knot.

$$\mathbb{B}_n = \left\langle \sigma_1, \sigma_2, ..., \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & |i-j| = 1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j| > 1 \end{array} \right\rangle$$

Attending to the presentation above, a braid is said to be Artin-positive if it can be represented by a positive standard braid word, that is, a braid word where each Artin generator σ_i appears with positive exponent. The closure of an Artin-positive braid is an Artin-positive link. See Figure 2.1

Artin-positive links are, indeed, positive links. However, the converse is not true. In Proposition 2.2 we show a proof of this well known fact.

The braid group $\mathbb{B}_n$ admits another well known presentation due to Birman, Ko and Lee [12]. The so called BKL generators or *band generators*, σ_{ij}, are related to Artin-generators by the formula $\sigma_{ij} = (\sigma_{j-2}\ldots\sigma_i)^{-1}\sigma_{j-1}(\sigma_{j-2}\ldots\sigma_i)$, with $i < j$. They correspond to a positive crossing of strands in positions i and j passing in front of the other strands, as shown in Figure 2.2.

$$\mathbb{B}_n = \left\langle \sigma_{rs}, 1 \le r < s \le n \;\middle|\; \begin{array}{ll} \sigma_{qr}\sigma_{st} = \sigma_{st}\sigma_{qr} & (t-r)(t-q)(s-r)(s-q) > 0 \\ \sigma_{st}\sigma_{rs} = \sigma_{rt}\sigma_{st} = \sigma_{rs}\sigma_{rt} & 1 \le r < s < t \le n \end{array} \right\rangle$$

Just as before, a BKL-word having only positive exponents is a BKL-positive word.
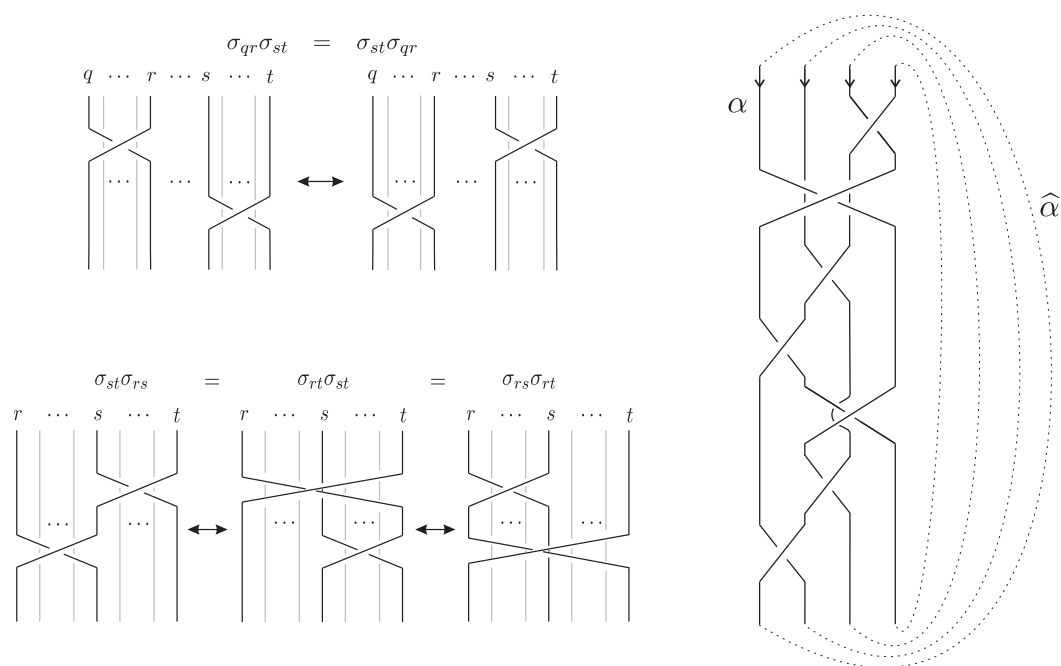
Figure 2.2: Braid relations in the Birman-Ko-Lee presentation, the BKL-positive braid $\alpha = \sigma_{34}\sigma_{14}\sigma_{23}\sigma_{12}\sigma_{24}\sigma_{23}\sigma_{12}$ on 4 strands and its closure $\widehat{\alpha}$, a BKL-positive (or strongly quasipositive) link.

Definition 2.1. *A braid is BKL-positive if it can be expressed by a word written in terms of the generators given by Birman, Ko and Lee, with all letters having positive exponents. The closure of a BKL-positive braid is a BKL-positive link.*

In [46] Rudolph introduced BKL-positive links as the boundaries of certain surfaces, and he called them *strongly quasipositive links*. Rudolph also proved in [48] that positive links are strongly quasipositive, which is not obvious. The converse is not true. In fact, Baader [6] showed that a link is positive if and only if it is strongly quasipositive and homogeneous; in particular, the link $L9n18\{1\}$ appearing in the proof of Theorem 1.11 is strongly quasipositive but not positive [51].

A polynomial with real coefficients is said to be positive if all its coefficients are non-negative. In 1983 Van Buskirk [56] proved that Artin positive links have positive Conway polynomial. Six years later Cromwell [19] extended this result to the class of positive links.

Artin-positive links are indeed positive, and the latter are strongly-quasipositive. It seems natural to wonder if the positivity of Conway polynomial is preserved under this extension. The main result in this chapter is the following:

Theorem 2.8 Strongly quasipositive links with braid index 3 have positive Conway polynomial.

This result cannot be generalized to strongly quasipositive links with arbitrary
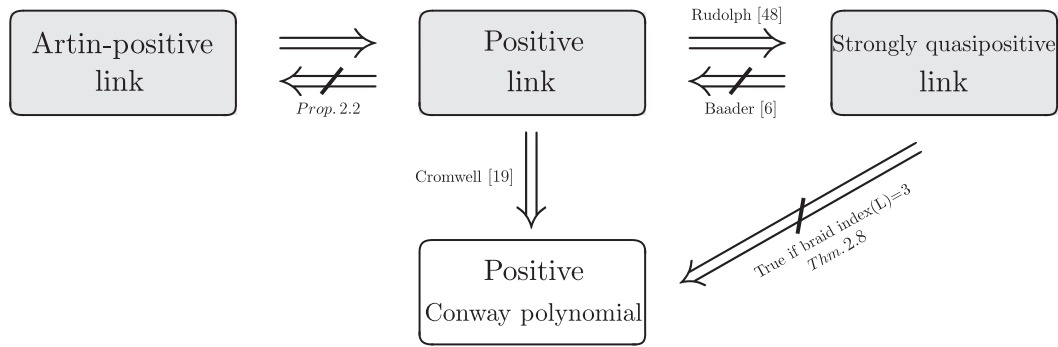
Figure 2.3: Implications between the families of Artin-positive, positive and strongly quasipositive (or BKL-positive) links, and their relation with the positivity of Conway polynomial.

braid index, as we prove in Proposition 2.12. Namely, we give an example showing that for any $n \geq 6$ there exists a BKL-positive braid $\alpha \in \mathbb{B}_n$ whose closure has non-positive Conway polynomial.

2.1.1   Relations and some properties

In this section we review some details about the families of Artin-positive, positive and strongly quasipositive links and their relations.

Proposition 2.2. *There are positive links which are not Artin-positive.*

Proof. In Figure 2.4 a positive diagram of the knot 5_2 is shown; its braid index is 3, as it is the closure of the braid $\alpha = \sigma_2^{-3}\sigma_1^{-1}\sigma_2\sigma_1^{-1}$. Suppose now that 5_2 is the closure of a braid γ represented by a positive Artin-word w. We take w with minimal length. Let D be the associated positive diagram. As projection surfaces constructed from homogeneous diagrams have minimal genus (Propositions 1.7, 1.8) and positive diagrams are homogeneous, $g(S_D) = g(5_2) = 1$ leads to $s+1 = c$, where s and c are the number of Seifert discs and bands in S_D, the projection surface arising from D. Notice that s is the number of strands and c is the number of crossings of γ.

Since $c \geq 5$, γ must have at least 4 strands, and then some generator σ_i must appear at most once. All generators must appear, since 5_2 is a knot (a one-component link), so there exists one generator appearing exactly once, and this is a nugatory crossing. This is a contradiction with the minimality of w since 5_2 is prime. □

The families of strongly quasipositive links and BKL-positive links are equal. The first name reminds that the link is boundary of a nice kind of surface; the second one brings to our mind the algebraic presentation of the braid group.

To be more precise, strongly quasipositive links were introduced as the boundaries of what Rudolph called *quasipositive surfaces* [46]. A quasipositive surface
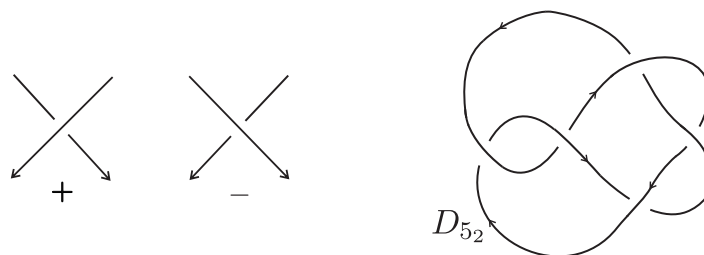
Figure 2.4: We remind the chosen convention of signs and a positive diagram representing the positive knot 5_2.
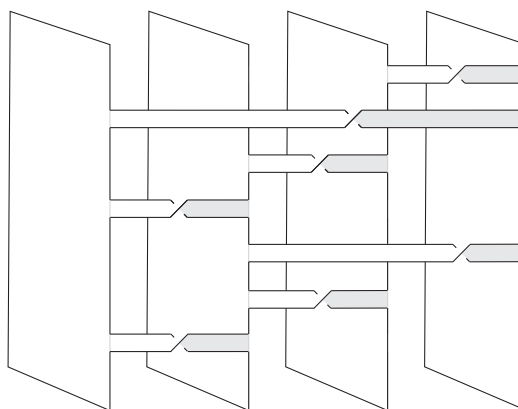
Figure 2.5: The quasipositive surface associated to the closed BKL-positive braid in Figure 2.2. Its boundary is a strongly quasipositive link (or BKL-positive link).

is an orientable surface which is ambient isotopic to a surface consisting of a finite number of parallel discs joined by some bands twisted in a positive way, as in Figure 2.5. Each of these bands corresponds to a BKL-generator.

Strongly quasipositive links should not be confused with quasipositive links, which are closures of products of conjugates of Artin-positive braids. We are not going to deal with quasipositive links through this chapter.

Quasipositive surfaces satisfy many interesting properties, such as being of minimal genus for the strongly quasipositive links they span, or like the following fact:

Theorem 2.3. [47, Theorem 4.3] *The Stallings plumbing (or Murasugi sum) of two surfaces S_1 and S_2 is quasipositive if and only if both S_1 and S_2 are quasipositive.*

Primitive flat surfaces (see Subsection 1.2.3) with their bands twisted in a positive sense are quasipositive, so their boundaries are strongly quasipositive links. Recall that pseudoalternating links are the boundary of generalized flat surfaces, which are constructed as a finite number of plumbings of primitive flat surfaces (see Definition 1.6). Hence Theorem 2.3 implies that a given pseudoalternating link is strongly quasipositive if and only if there exists an associated generalized

flat surface coming from the plumbing of positive primitive flat surfaces. The link $L9n18\{1\}$ is such an example (see Figure 1.11 in Chapter 1).

2.2   Conway polynomial from Burau representation

Burau [14] introduced a linear representation of the braid group on n strands, $\mathbb{B}_n$, by squared matrices of order n over the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$. This representation has been widely studied, being its faithfulness one of the remaining open problems (it is known to be faithful when $n \leq 3$ and unfaithful when $n \geq 5$, but the case $n = 4$ is still unsolved).

We will use the *reduced* Burau representation $\psi : \mathbb{B}_n \to GL(n-1, \mathbb{Z}[s, s^{-1}])$ defined by the formula $\psi(\sigma_i) = A_i$, where

$$A_1 = \begin{pmatrix} -s^2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}, \qquad A_{n-1} = \begin{pmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & s^2 \\ 0 & 0 & -s^2 \end{pmatrix}$$

and for $1 < i < n-1$

$$A_i = \begin{pmatrix} I_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & s^2 & 0 & 0 \\ 0 & 0 & -s^2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-i-2} \end{pmatrix}$$

with I_k being the identity matrix or order k.

In [25] the reduced Burau representation is noted $\psi_n^r : \mathbb{B}_n \to GL(n-1, \mathbb{Z}[t, t^{-1}])$, and it is equal to the above definition after the substitution $t = s^2$. Then Lemma 3.12 and Theorem 3.13 in [25] can be restated to give the following well-known presentation of the Conway polynomial in terms of the reduced Burau representation:

Theorem 2.4. [25] *Let $\alpha \in \mathbb{B}_n$ be a braid and $\widehat{\alpha}$ the link obtained as the closure of α. Then the Conway polynomial of $\widehat{\alpha}$ is given by*

$$\nabla(\widehat{\alpha})(z) = (-1)^{n+1} \frac{s^{-e_\alpha}}{[n]} |\psi(\alpha) - I_{n-1}|$$

after the substitution $s^{-1} - s = z$, where $[n] = \frac{s^{-n} - s^n}{s^{-1} - s}$ and $e_\alpha \in \mathbb{Z}$ denotes the image of α under the homomorphism $\mathbb{B}_n \to \mathbb{Z}$ sending each generator σ_i to 1.

Note that e_α coincides with the exponent sum of any braid word representing the braid α, which is invariant under the (homogeneous) relations of the braid group. Moreover, e_α is invariant under conjugation.

In the proof of Theorem 2.6 we study the case of braids with three strands, so we specify the case $n = 3$. The reduced Burau representation of $\mathbb{B}_3$ is given by the matrices

$$B_1 = \begin{pmatrix} -s^2 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad B_2 = \begin{pmatrix} 1 & s^2 \\ 0 & -s^2 \end{pmatrix}.$$

Since for a square matrix A of order two $|A - xI_2| = x^2 - x\operatorname{Tr}(A) + |A|$, one has $|\psi(\alpha) - I_2| = 1 - \operatorname{Tr}(\psi(\alpha)) + |\psi(\alpha)|$ when working with a braid with 3 strands. Note that, as $|\psi(B_1)| = |\psi(B_2)| = -s^2$, then $|\psi(\alpha)| = (-s)^{2e_\alpha}$, $\forall \alpha \in \mathbb{B}_3$.

At this point, we find useful to show a slight modification of a result by P. V. Koseleff and D. Pecker [30] which simplifies the substitution $s^{-1} - s = z$ in the formula above by using Fibonacci polynomials. Recall that Fibonacci polynomials are defined by the recurrence relation $F_n(z) = z\, F_{n-1}(z) + F_{n-2}(z)$ for $n \geq 2$, starting with $F_0(z) = 0$ and $F_1(z) = 1$.

There exist closed combinatorial formulas to express the Fibonacci polynomials; however, we will not use them through this chapter. For our purposes we need to extend these polynomials to the case when the subindex is negative; we do it in the natural way, by defining $F_{-n}(z) = (-1)^{n+1}F_n(z)$.

Now we are ready to state our adapted version of Lemma 4.1 in [30]:

Lemma 2.5. [30] *Let $F_n(z)$ be the nth Fibonacci polynomial. After the substitution $z = s^{-1} - s$, the identity $(s^{-1})^n + (-s)^n = F_{n+1}(z) + F_{n-1}(z)$ holds for every integer n.*

The following sections in this chapter are fully devoted to present our new results.

2.3   Conway polynomial of 3-braids differing by Δ^2

From now on, we consider the braid group on 3 strands $\mathbb{B}_3$, unless otherwise stated. Although the results contained in this section are interesting on their own, they become really useful in Section 2.4 when proving Corollary 2.10, which completes the proof of the main theorem in this chapter (Therorem 2.8).

Recall that the Garside element $\Delta \in \mathbb{B}_n$ is defined as the half twist of the strands, $\Delta = (\sigma_1\sigma_2 \ldots \sigma_{n-1})(\sigma_1 \ldots \sigma_{n-2}) \ldots (\sigma_1\sigma_2)\sigma_1$. Since $\Delta\sigma_i = \sigma_{n-i}\Delta$ it follows that Δ^2 is in the center of $\mathbb{B}_n$. Moreover, the whole center of $\mathbb{B}_n$ is generated by Δ^2 [18].

The following result provides a relation between the Conway polynomials of two closed braids differing in an even power of the Garside element $\Delta = \sigma_1\sigma_2\sigma_1$ in $\mathbb{B}_3$. See Figure 2.6 for getting an example of two braids α and β as the ones on the statement for the case $k = 1$.
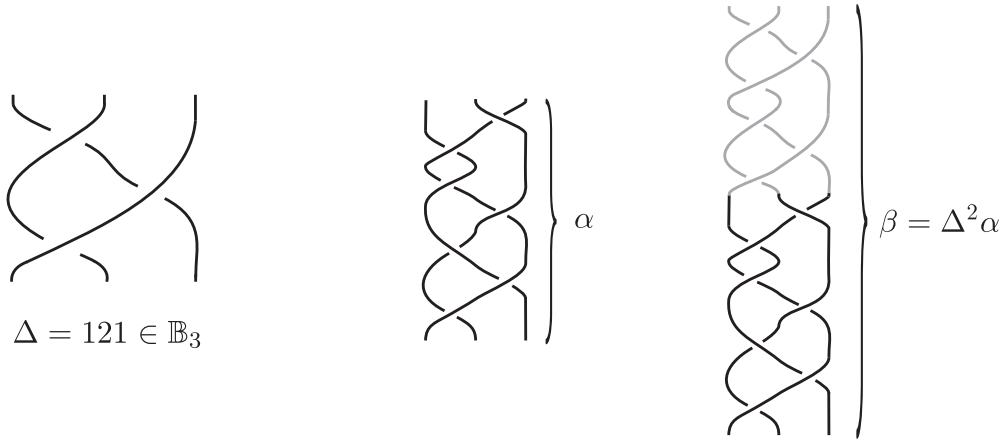
Figure 2.6:   The Garside element in $\mathbb{B}_3$, $\Delta = \sigma_1\sigma_2\sigma_1$, the (dark) braid $\alpha = \sigma_2^{-1}\sigma_1\sigma_1\sigma_2\sigma_1^{-1}\sigma_2\sigma_1 \in \mathbb{B}_3$ and the braid $\beta = \Delta^2\,\alpha$.

Theorem 2.6. *Let $\alpha, \beta \in \mathbb{B}_3$ with $\beta = \Delta^{2k}\,\alpha$ and $k > 0$. Then the difference between the Conway polynomials of their closures is given by*

$$\nabla(\widehat{\beta}) - \nabla(\widehat{\alpha}) = z\sum_{i=0}^{k-1}\left(F_{e_\alpha+6i+4} + F_{e_\alpha+6i+2}\right)$$

with F_n being the nth Fibonacci polynomial for any integer n.

Proof. We proceed by induction on k. We start with the case $k = 1$, that is, the case when the braid β is obtained after multiplying the braid α by a full twist.

As the Garside element in $\mathbb{B}_3$ is $\Delta = \sigma_1\sigma_2\sigma_1$, it follows that $e_\beta = e_\alpha + 6$. Since $\psi(\Delta^2) = s^6 I_2$, we have that $\text{Tr}(\psi(\beta)) = s^6 \cdot \text{Tr}(\psi(\alpha))$.   We now combine these facts with Theorem 2.4 for computing the Conway polynomial of both closed braids.

Taking into account the substitution $s^{-1} - s = z$, we have

$$\nabla(\widehat{\alpha}) = \frac{s^{-e_\alpha}}{[3]}|\psi(\alpha) - I_2| = \frac{s^{-e_\alpha}}{[3]}\left(1 - \text{Tr}(\psi(\alpha)) + |\psi(\alpha)|\right)$$

$$= \frac{1}{[3]}\left(s^{-e_\alpha} - s^{-e_\alpha}\text{Tr}(\psi(\alpha)) + (-1)^{e_\alpha}s^{e_\alpha}\right)$$

and

$$\nabla(\widehat{\beta}) = \frac{s^{-e_\beta}}{[3]}|\psi(\beta) - Id| = \frac{s^{-e_\beta}}{[3]}\left(1 - \text{Tr}(\psi(\beta)) + |\psi(\beta)|\right)$$

$$= \frac{1}{[3]}\left(s^{-e_\alpha-6} - s^{-e_\alpha}\text{Tr}(\psi(\alpha)) + (-1)^{e_\alpha}s^{e_\alpha+6}\right).$$

Now we compute the difference:

$$\nabla(\widehat{\beta}) - \nabla(\widehat{\alpha}) = \frac{1}{[3]}\left(s^{-e_\alpha-6} - s^{-e_\alpha} + (-1)^{e_\alpha}(s^{e_\alpha+6} - s^{e_\alpha})\right)$$

$$= \left((-s)^{e_\alpha+4} + (s^{-1})^{e_\alpha+4}\right) - \left((-s)^{e_\alpha+2} + (s^{-1})^{e_\alpha+2}\right).$$

The second equality comes from the fact that

$$\left(s^{-1} - s\right) \cdot \left(s^{-e_\alpha-6} - s^{-e_\alpha} + (-1)^{e_\alpha}(s^{e_\alpha+6} - s^{e_\alpha})\right)$$

$$= \left(s^{-3} - s^3\right) \cdot \left((-s)^{e_\alpha+4} + s^{-e_\alpha-4} - (-s)^{e_\alpha+2} - s^{-e_\alpha-2}\right).$$

Applying twice Lemma 2.5 we obtain

$$
\begin{aligned}
\nabla(\widehat{\beta}) - \nabla(\widehat{\alpha}) &= (F_{e_\alpha+5} + F_{e_\alpha+3}) - (F_{e_\alpha+3} + F_{e_\alpha+1}) \\
&= (F_{e_\alpha+5} - F_{e_\alpha+3}) + (F_{e_\alpha+3} - F_{e_\alpha+1}) \\
&= z\,(F_{e_\alpha+4} + F_{e_\alpha+2}).
\end{aligned}
$$

Now, suppose the statement true for $1, 2, \ldots, k-1$, and let $\beta = \Delta^{2k}\alpha$. If we write $\gamma = \Delta^{2(k-1)}\alpha$, then by induction

$$
\begin{aligned}
\nabla(\widehat{\beta}) - \nabla(\widehat{\alpha}) &= \nabla(\widehat{\beta}) - \nabla(\widehat{\gamma}) + \nabla(\widehat{\gamma}) - \nabla(\widehat{\alpha}) \\
&= z\left(F_{e_\gamma+4} + F_{e_\gamma+2}\right) + z\sum_{i=0}^{k-2}\left(F_{e_\alpha+6i+4} + F_{e_\alpha+6i+2}\right) \\
&= z\sum_{i=0}^{k-1}\left(F_{e_\alpha+6i+4} + F_{e_\alpha+6i+2}\right)
\end{aligned}
$$

where the third equality holds since $e_\gamma = e_\alpha + 6(k-1)$.

$\square$

With some extra work, the theorem above can be deduced from work by Murasugi in [37, Proposition 4.1]. He compares the normalized Alexander polynomial of two closed braids differing in an even power of Δ and provides an expression for their difference, with an indeterminacy on a power of $-t$. We think that the formula we present in Theorem 2.6 is quite simpler even in the case of the Alexander polynomial (that is, just before the change of variables $s^{-1} - s = z$).

As a consequence of Theorem 2.6 we get an interesting corollary. The result for the even case was proved by Joan Birman in [11]; as far as we know there is no reference for the odd case.

Corollary 2.7. *Let $\alpha \in \mathbb{B}_3$ with $e_\alpha = -3k$ and $k > 0$, and consider $\beta = \Delta^{2k}\alpha$. We have that:*

- *If k is even, then $\nabla(\widehat{\beta}) = \nabla(\widehat{\alpha})$.*

- *If k is odd, then $\nabla(\widehat{\beta}) = \nabla(\widehat{\alpha}) + 2z \sum\limits_{i=0}^{k-1} F_{-3k+6i+4}$.*

Proof. By applying Theorem 2.6 we get

$$
\begin{aligned}
\nabla(\widehat{\beta}) &= \nabla(\widehat{\alpha}) + z \sum_{i=0}^{k-1} \left(F_{-3k+6i+4} + F_{-3k+6i+2} \right) \\
&= \nabla(\widehat{\alpha}) + z \sum_{i=0}^{k-1} \left(F_{-3k+6i+4} + F_{3k-6i-4} \right) \\
&= \nabla(\widehat{\alpha}) + z \sum_{i=0}^{k-1} \left[F_{-3k+6i+4} + (-1)^{k+1} F_{-3k+6i+4} \right] \\
&= \nabla(\widehat{\alpha}) + \left[1 + (-1)^{k+1} \right] z \sum_{i=0}^{k-1} F_{-3k+6i+4}.
\end{aligned}
$$

The second equality holds since $F_{-3k+6i+2} = F_{3k-6j-4}$ when $j = (k-1) - i$. The third one holds since $F_{-n} = (-1)^{n+1} F_n$. This completes the proof.

$\square$

2.4   Strongly quasipositive links with braid index 3

2.4.1   Resolution trees

The skein relation defining the Conway polynomial of a link, $\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)$ (see the first part of Section 1.3), is a recurrence relation for computing the value of this invariant for any link. This process can be codified by using resolution trees.

Given an oriented diagram D representing a link L, we construct a resolution tree rooted at D in the following way. Starting from the root, each node would form a triple (parent, leftchild, rightchild) either of the form (D_+, D_-, D_0) or (D_-, D_+, D_0). See Figure 2.7. In the first case, the edge joining D_+ and D_- is labeled with 1, and the one joining D_+ and D_0 with z; in the second case the edge joining D_- and D_+ is labeled with 1, and the one joining D_- and D_0 with $-z$ (of course this labeling comes from the skein relation). Let $L_1, L_2, \ldots, L_n$ be the leaves in the tree and P_i, $1 \le i \le n$ the product of the labels in the edges on the unique path connecting the leaf L_i and the root of the tree. Then, if we know $\nabla(L_i)$, we can compute the Conway polynomial of the link L:

$$
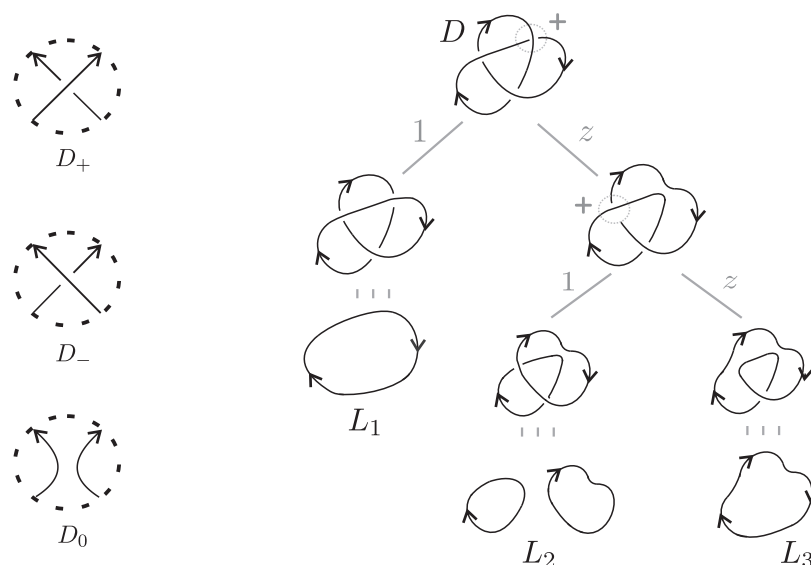\nabla(L) = \sum_{i=1}^{n} P_i \cdot \nabla(L_i).
$$

Figure 2.7: A resolution tree for the trefoil knot 3_1 represented by the diagram D. It has three leaves, L_1, L_2 and L_3, and three paths joining them to the root: $P_1 = 1$, $P_2 = z$ and $P_3 = z^2$. As the value of the Conway polynomial of the unknot and split links is 1 and 0 respectively, one gets $\nabla(3_1) = 1 \cdot 1 + z \cdot 0 + z^2 \cdot 1 = 1 + z^2$.

In [19] P. Cromwell proved that starting from a diagram D, it is possible to construct a resolution tree rooted in D with terminal nodes being trivial links, in such a way that in every path from a terminal node of the tree to the root no crossing is changed more than once. As a consequence, he proved that positive links have positive Conway polynomials [19, Corollary 2.1], that is, a polynomial in the variable z with non-negative coefficients.

In Proposition 2.9 we give an algorithm for constructing a resolution tree whose nodes are D_+ and its terminal nodes are not necessarily trivial links. The reason for choosing D_+ in the nodes position in our construction is marking all branches with positive labels. As a consequence, the fact that all leaves in a resolution tree have positive Conway polynomial ensures the positivity of the Conway polynomial of the link in the root.

2.4.2 Positivity of Conway polynomial

In this section we continue working with braids on 3 strands. First of all, we want to remark that the class of positive links and the class of strongly quasipositive links are not equal even when considering links with braid index 3. We provide such an example in Proposition 2.11, by showing a family of links with braid index 3 which are not positive but strongly quasipositive.

From the point of view of the presentation given by Artin, $\mathbb{B}_3$ has two generators, σ_1 and σ_2. However, if we consider the presentation given by Birman, Ko and Lee, $\mathbb{B}_3$ has three generators: $\sigma_{12} = \sigma_1$, $\sigma_{23} = \sigma_2$ (corresponding to the
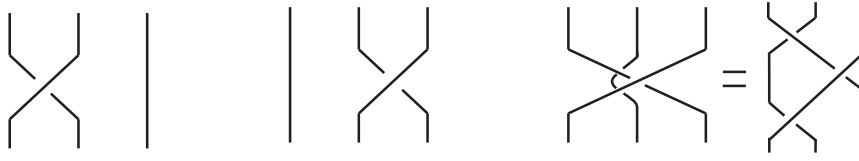
Figure 2.8: BKL-generators $\sigma_{12} = \sigma_1$, $\sigma_{23} = \sigma_2$ and $\sigma_{13} = \sigma_1^{-1}\sigma_2\sigma_1$.

Artin-generators) and $\sigma_{13} = \sigma_1^{-1}\sigma_2\sigma_1$. See Figure 2.8. From now on, we are going to work with BKL-generators, as strongly quasipositive links are closure of positive braids in terms of these generators.

We are now ready to state our main result in this chapter:

Theorem 2.8. *Strongly quasipositive links with braid index 3 have positive Conway polynomial.*

A result by Stoimenov [53] states that any strongly quasipositive link with braid index 3, is the closure of a BKL-positive braid on 3 strands. Hence, we just have to focus on proving the positivity of the Conway polynomial of those closed braids.

The proof of Theorem 2.8 lies on two results. The first one gives a procedure to construct a particular resolution tree starting from a BKL-positive braid word, whose branches have positive labels and each leaf is of one of 4 types. In the second one we show that all possible leaves obtained by following the algorithm above, have positive Conway polynomial. The combination of both results completes the proof of Theorem 2.8.

At this point we find convenient to introduce a special braid on three strands which is going to be used several times in the following results. This braid is $\upsilon = \sigma_1\sigma_2\sigma_{13} \in \mathbb{B}_3$.

Proposition 2.9. *Let $\alpha \in \mathbb{B}_3$ be a BKL-positive braid; then, it is possible to construct a resolution tree for the link $\widehat{\alpha}$, whose branches have positive labels and whose leaves are closed braids belonging to the set $\{\varepsilon, \sigma_1, \sigma_1\sigma_2\} \cup \{\upsilon^k, k > 0\}$, with ε being the trivial braid in $\mathbb{B}_3$.*

Proof. Let w be a BKL-positive word representing the braid α and n its length. If $n = 1$, then w is contained in M_1, the set containing BKL-positive words of length 1. If $n = 2$, then w is either in M_2, the set containing those BKL-positive words of length 2 with two different letters, or it consists on two repeated letters, $w = \sigma_i\sigma_i$, and it can be split into the trivial word, ε, and one word of length one, σ_i, which is in M_1.

Suppose now that $n \geq 3$. If $w = P\sigma_i^2 Q$, with P and Q BKL-positive words, split it by writing w as a node whose left child and right child are $w_1 = PQ$ and $w_2 = P\sigma_i Q$ respectively; the left branch would be labeled with 1, and the right one with z. Note that w_1 and w_2 have length $n-2$ and $n-1$ respectively. Repeat

$w = 31232212$

z

312312
$(123)^2$

3123212
3122112

z

31112

312112

z

312
123

3112

3122

31212
31322

z

32

312
123

z

31

312
123

z

313
331

3132
3322

z

1

31
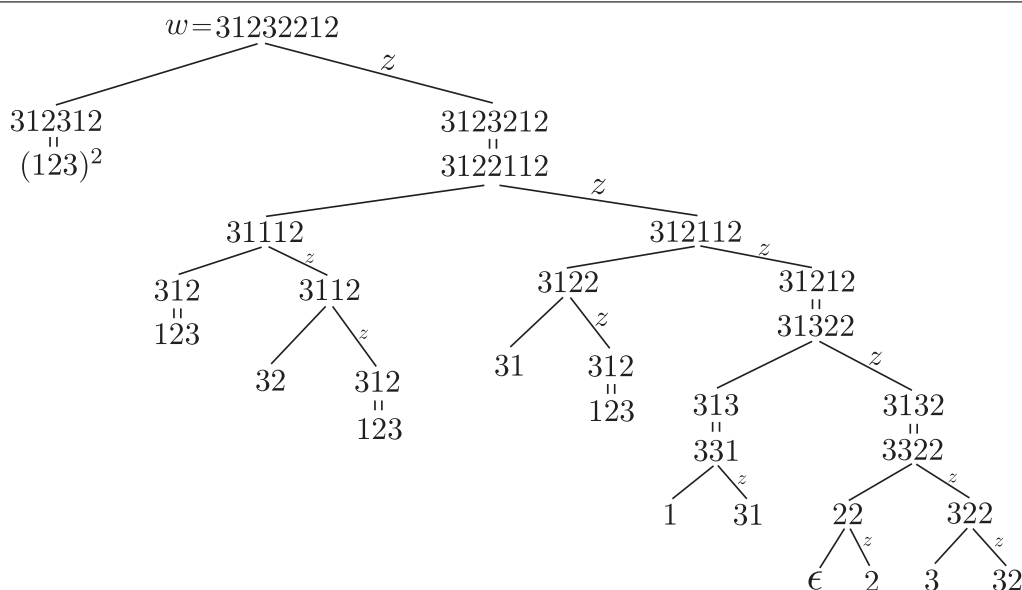
z

22

322

z

ϵ

2

z

3

32

Figure 2.9: This image illustrates the algorithm in the proof of Proposition 2.9. The sign "=" represents a cyclic move in the order of the letters (that is, a conjugation). Letters 1, 2 and 3 represent generators σ_1, σ_2 and σ_{13}, respectively.

this procedure with the BKL-positive words obtained each time. Notice that we are not interested in each particular word, but in the link closure of the braid represented by the word (that is, each node in the tree is a BKL-positive word up to conjugation). Hence, we can replace any positive word by another positive word representing the same braid, or by another positive word corresponding to a cyclic permutation of its letters (which corresponds to a conjugate braid, hence to the same link).

Now let us see that every braid which does not belong to $\{\varepsilon\} \cup M_1 \cup M_2 \cup \{v^k,\ k > 0\}$ can be split by the above procedure.

Consider a BKL-positive braid word of length at least 3, with no equal consecutive letters. Note that the braids $\sigma_2\sigma_1 = \sigma_{13}\sigma_2 = \sigma_1\sigma_{13}$ are equivalent. Now, start reading the braid word from the left; each time you find either a $\sigma_2\sigma_1$, $\sigma_{13}\sigma_2$ or $\sigma_1\sigma_{13}$ occurrence, write this syllable in such a way that its last letter equals the first letter after it (in cyclic order), so you get two repeated generators together. If no occurrence of $\sigma_2\sigma_1, \sigma_{13}\sigma_2, \sigma_1\sigma_{13}, \sigma_1\sigma_1, \sigma_2\sigma_2, \sigma_{13}\sigma_{13}$ appears in any cyclic permutation of the word, then the letter after every σ_1 must be σ_2, the letter after every σ_2 must be σ_{13}, and the letter after every σ_{13} must be σ_1, in every cyclic permutation of the word. Therefore, up to a cyclic permutation, the word equals v^k for some $k > 0$.

As all the BKL-words in M_1 and M_2 are conjugated to σ_1 and $\sigma_1\sigma_2$ respectively, this procedure allows us to construct a resolution tree rooted in w, where all the branches have positive labels (either 1 or z), and all the leaves belong to the set $\{\varepsilon, \sigma_1, \sigma_1\sigma_2\} \cup \{v^k,\ k > 0\}$. $\qquad\square$

At this point, we just need to show that the closure of the braids in the above set have positive Conway polynomials. As closing the trivial braid or σ_1 gives a split link, their Conway polynomials are null. The closure of a braid represented by a word with two different letters, lets say $\sigma_1\sigma_2$, is the trivial knot, so its Conway polynomial is 1.

It remains to prove the case of links which are closure of braids of the form v^k. These are non-split links with 2 or 3 components, depending on the parity of k, and for large enough n computing their Conway polynomial is not trivial. Corollary 2.7 applied to this particular case allows us to give the following result (the case k even was also computed by Stoimenov in [53]):

Corollary 2.10. *The closure of the braid on 3 strands v^k has positive Conway polynomial for any integer $k > 0$. In fact,*

$$\nabla(\widehat{v^k}) = 2z \sum_{i=0}^{k-1} F_{-3k+6i-4}$$

when k is odd, and it is null when k is even.

Proof. It is easy to check that $v = \Delta^2(\sigma_2^{-1})^3$, so $v^k = \Delta^{2k}(\sigma_2^{-1})^{3k}$, since Δ^2 is central. Now apply Corollary 2.7, with $\alpha = (\sigma_2^{-1})^{3k}$ (hence $e_\alpha = -3k$) and $\beta = \Delta^{2k}(\sigma_2^{-1})^{3k} = v^k$.

If k is even, then $\nabla(\widehat{v^k}) = \nabla(\widehat{(\sigma_2^{-1})^{3k}}) = 0$, since the closure of $(\sigma_2^{-1})^{3k}$ in $\mathbb{B}_3$ is a split link.

If k is odd, then $\nabla(\widehat{v^k}) = 2z \sum_{i=0}^{k-1} F_{-3k+6i+4}$. As the Fibonacci polynomials in the summation have odd subindices, all their coefficients are positive, since these polynomials satisfy $F_{-n}(z) = (-1)^{n+1} F_n(z)$. $\square$

This completes the proof of Theorem 2.8.

2.4.3 Non-positive but strongly quasipositive links

We conclude this section by showing a family of links with braid index 3 which is not positive but strongly quasipositive, as we promised in the first lines of this section. This ensures that Theorem 2.8 is a real extension of the well known result about positivity of the Conway polynomial of positive links when working with links having braid index equals 3.

Proposition 2.11. *If $k > 0$ is even then the closure of the braid $v^k \in \mathbb{B}_3$ is a strongly quasipositive but non-positive link.*

Proof. Let k be a fixed even integer number and L the closure of the braid $v^k \in \mathbb{B}_3$, that is, $L = \widehat{v^k}$. From the definition, it is immediate to check that the link L is strongly quasipositive. The proof of its non-positivity lies in a couple of additional results.
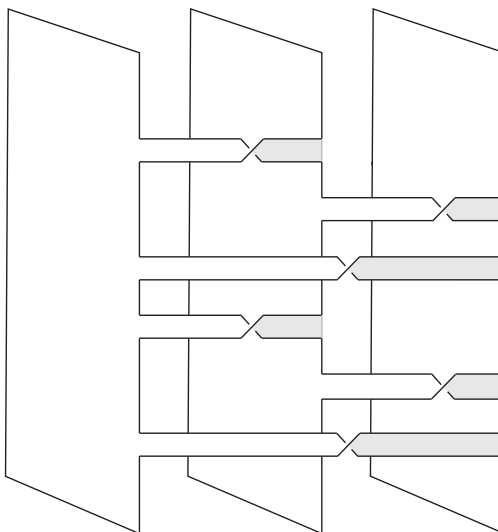
Figure 2.10: The quasipositive surface S spanning the link $(\widehat{\sigma_1\sigma_2\sigma_{13}})^2$.

The first one is a result by Baader [6] which states that a strongly quasipositive link is positive if and only if it is homogeneous. Peter Cromwell proved in [19] that, given an homogeneous link K of $\mu(K)$ components, its Conway polynomial is related to its genus by the formula $2g(K) = \mathrm{maxdeg}\,(\nabla(K)) - \mu(K) + 1$, where $\mathrm{maxdeg}\,(\nabla(K))$ is the maximum degree of the Conway polynomial of K.

We proceed by contradiction. Suppose that L is positive. Since L has 3 components, as a consequence of the above results L should satisfy $2g(L) = \mathrm{maxdeg}\,(\nabla(L)) - 2$.

Let S be the quasipositive surface associated to L (see Figure 2.10). The Euler characteristic of this kind of surfaces can be computed as $\chi(S) = d_S - b_S$, with b_S and d_S being the number of bands and discs in S respectively; as a consequence, $2g(S) = 2 - \mu(L) + b_S - d_S = -1 + b_S - d_S$.

Rudolph proved in [47] that quasipositive surfaces have minimal genus for the link they are spanning. Hence, L should satisfy $\mathrm{maxdeg}\,(\nabla(L)) = 1 + b_S - d_S$. As we are working in $\mathbb{B}_3$, there are 3 discs in S, so $\mathrm{maxdeg}\,(\nabla(L)) + 2 = b_S$.

Hence, if L were positive the number of bands in S and its Conway polynomial should be related in the way above. We proved in Corollary 2.10 that $\nabla(L) = 0$. However, the number of bands in S equals $3k$, yielding a contradiction. $\qquad\square$

2.5 The result cannot be extended

In this section we consider the problem of extending the previous result to a higher number of strands, that is, we study whether every strongly quasipositive link
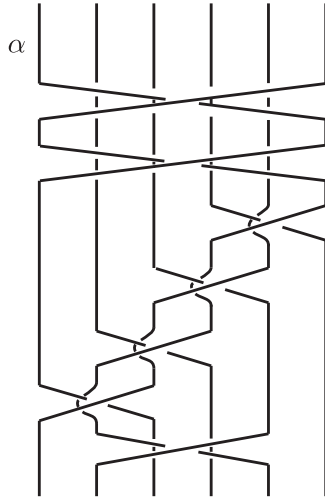
Figure 2.11: The BKL-positive braid $\alpha = \sigma_{16}\sigma_{16}\sigma_{46}\sigma_{35}\sigma_{24}\sigma_{13}\sigma_{25}$. Its closure is a strongly quasipositive link having non-positive Conway polynomial.

has positive Conway polynomial. The following result gives a negative answer to this question by showing a counterexample:

Proposition 2.12. *There are strongly quasipositive links having non-positive Conway polynomial.*

Proof. Consider the BKL-positive braid on 6 strands $\alpha = \sigma_{16}\sigma_{16}\sigma_{46}\sigma_{35}\sigma_{24}\sigma_{13}\sigma_{25}$. Its closure (Figure 2.11) is a strongly quasipositive link, whose Conway polynomial is $\nabla(\widehat{\alpha}) = -z^2 + 1$, which is non-positive. (We computed $\nabla(\widehat{\alpha})$ by using a C++ version of the program br9z.p, developed by Short and Morton in 1985: `http://www.liv.ac.uk/` $\sim$ `su14/knotprogs.html`). $\qquad\square$

In [49], in the proof of Corollary 88, Rudolph stated that every Seifert matrix of a given link can be obtained as the Seifert matrix of a strongly quasipositive link. As a consequence, given a link L with Conway polynomial $\nabla(L)$, there would exist a strongly quasipositive link L' having the same Conway polynomial, that is, satisfying $\nabla(L) = \nabla(L')$. He also gives a procedure for constructing L' as a closed BKL-positive braid, starting from a braid diagram of L. This result would provide an infinite family of examples of strongly quasipositive links having non-positive Conway polynomial. However, we think that there is a problem with the proof of this result: it is claimed that the procedure for obtaining L' starting from L (a sequence of *doubled-delta moves*, also called *trefoil insertion*) preserves the Seifert matrix. After applying this move to the braid $\beta = \sigma_1\sigma_1\sigma_1^{-1} \in \mathbb{B}_2$, one obtains $\beta' = \sigma_{16}\sigma_{16}\sigma_{25}\sigma_{13}\sigma_{24}\sigma_{35}\sigma_{46}$. The closure of β is the trivial knot, hence $\nabla(\widehat{\beta}) = 1$; however $\nabla(\widehat{\beta'}) = 7z^2 + 1$. This contradicts the fact that doubled-delta moves preserve the Seifert matrix.

It follows from Proposition 2.12 that for any $n \geq 6$ there exist BKL-positive braids $\beta \in \mathbb{B}_n$ whose closures have non-positive Conway polynomial. To get such

an example, it suffices to take the braid in the proof of Proposition 2.12 and perform iterated positive stabilizations (that is, replace $\alpha \in \mathbb{B}_6$ with $\beta\sigma_6\sigma_7\ldots\sigma_{n-1} \in \mathbb{B}_n$).

Despite applying some well-known results for giving a boundary on the braid index of a link, we do not know whether the braid index of the closure of the braid α in the proof of Proposition 2.12 is 4, 5 or 6. At Knotinfo [15] one can find 103 and 90 strongly quasipositive knots having braid index equal to 4 and 5 respectively; all of these knots have positive Conway polynomial. Using the computer program above, we have also checked many examples of links being the closure of BKL-positive braids on 4 strands and all of them have positive Conway polynomial. Hence, determining if the result in Theorem 2.8 can be extended to the case of 4 and 5 strands is an open question.

Chapter 3

Knot Floer Homology and FKT states

3.1 Introduction

In this chapter we present our contribution to a project by some authors, aiming to find a combinatorial definition of Knot Floer homology in terms of the states defined in *Formal Knot Theory* [27] (from now on FKT states), in a similar way as that given by Viro [57] for the Khovanov homology in terms of Jones states (see Section 4.2). More specifically, taking FKT states as the generators of the chain complex the goal is to try to find an *appropriate* differential in such a way that the resulting homology matches with Knot Floer homology.

The aim of this chapter is not to give an extensive formal definition of Knot Floer homology, but to explain its relations with the classical concepts introduced in FKT.

Knot Floer homology is a relatively new link invariant developed independently by Ozsváth and Szabó [40] and Rasmussen [43]. There has been an intensive study of this theory, as it provides important geometric information about a link, such as its genus or its fibredness, and even characterizes the trivial knot [31].

One of the main properties about Knot Floer Homology is that it categorifies the Alexander polynomial. To be more specific, Knot Floer homology is a bigraded homology whose Euler characteristic is the Alexander polynomial, that is:

$$\sum_{i,j} (-1)^j \cdot \mathrm{rank}(H_{i,j}(K)) \cdot t^i = \Delta_K(t),$$

where $H_{i,j}(K)$ is the Knot Floer homology module of the knot K and $\Delta_K(t)$ is the normalized version of the Alexander-Conway polynomial of K as defined in Sections 1.3 and 3.2.

There are several approaches for proving this fact. For example, in [10]

M. Benheddi and D. Cimasoni used the purely combinatorial version of Knot Floer Homology (the one based on grid diagrams) for verifying it. But this version is not directly related to the FKT model.

For our purposes, the most interesting proof of this categorification is the one given by Ozsváth and Szabó in [39, 41], as they identify the graded Euler characteristic of Knot Floer homology theory as a state summation related to the Formal Knot Theory model. There is a technical difference in that they sum over different local weights and use a different sign convention. They claim that the resulting state sum is the Alexander-Conway polynomial by reference to the FKT model (Theorem 11.3 in [41]). In section 3.2 we include both definitions of the polynomial and prove their equality by providing a proof using classical concepts. The key point in this proof is Proposition 3.8, where we relate the parity of the *white and black holes* for the states in the FKT model. These results are contained in a joint paper with Louis Kauffman [26].

Going back to the problem of finding a definition of Knot Floer Homology in terms of FKT states, there have been many attempts to find such a differential. In particular, we want to present a differential proposed by Y. Rong [45]. As far as we know, they could prove neither its validity nor find an example where its invariance fails; we found such an example, revealing that this differential is not correct. In Section 3.3 we present this counterexample and give an extended explanation of the ideas behind Rong's differential, as his approach could be the starting point for finding the "good differential".

3.2   Parities of black and white holes

At the level of combinatorial knot theory, this section shows that a reformulation of the Alexander-Conway polynomial following the FKT model using white hole rather than black hole counts gives the same topological result. In proving this we obtain an interesting result (Proposition 3.8) relating the parities of the white and black holes for the states in the FKT model. We include in this section some of the results contained in a joint paper with Louis Kauffman [26].

We start by explaining classical notions such as marker, region or state.

Given an oriented non-split link L in S^3, fix an associated diagram D, and write $v_1, \ldots, v_n$ for its crossings. If one forgets the under-over information, one obtains a planar diagram P_D dividing the plane into $n + 2$ regions, one of them unbounded. Choose two adjacent regions and mark each of them with a star.

Definition 3.1. *In the previous conditions, a (FKT) state S associated to D is an assignment of markers to each v_i (that is, a choice of an adjacent unstarred region for each crossing) so that no region of P_D contains more than one marker. Let $\mathcal{S}$ denote the set of all possible states in D.*

In Figure 3.1 we present two types of labels for the crossings of an oriented link
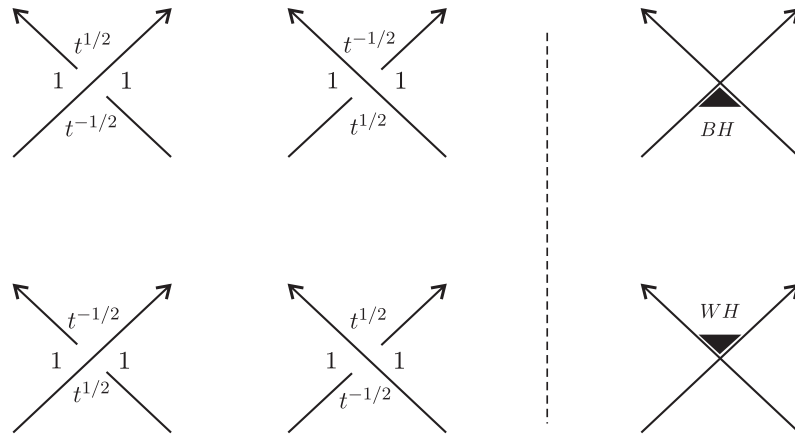
Figure 3.1: The left part of the first (second) row shows the Alexander-Conway labeling defining $\langle \cdot \,|\, \cdot \rangle$ ($[\cdot \,|\, \cdot]$); the rightmost part shows the state marker producing a black (white) hole.

diagram. The labeling shown in the first row will be called the *Alexander-Conway labeling*. It gives rise, as we will see in Theorem 3.2, to a normalized version of the Alexander polynomial and gives a state summation for the Alexander-Conway polynomial in $z = t^{1/2} - t^{-1/2}$. We call these Alexander-Conway labels because they are different from labels that derive directly from Alexander's original paper on his polynomial [3]. In [27] Kauffman reformulates the Alexander polynomial as a state summation by using the Alexander-Conway labels.

More precisely, given a state S, let $\langle D|S \rangle$ denote the product of the Alexander-Conway labels associated to each marker in S (see Figure 3.1). A marker is called a black hole if it occurs between two ingoing lines to the corresponding crossing. A marker is called a white hole if it occurs between outgoing lines at a crossing. Write B_S (W_S) for the set of double points whose markers are black (white) holes.

Theorem 3.2. [27, Theorem 4.3] *Let D be a diagram representing an oriented link L. Then the polynomial in $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$*

$$\Delta_L(t) = \sum_{S \in \mathcal{S}} (-1)^{\sharp B_S} \langle D|S \rangle.$$

is independent of the choice of the diagram D, and it is in fact, up to unit $\pm t^n$, the original Alexander polynomial of L as defined in [3]. In particular $\Delta_{\bigcirc}(t) = 1$ and if L_+, L_- and L_0 are three links with identical diagrams D_+, D_- and D_0 except in a small neighborhood where they differ as shown in Figure 1.10, then they satisfy the skein relation

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{1/2} - t^{-1/2})\Delta_{L_0}(t).$$

In [41] an alternative definition of the Alexander polynomial is given in the following terms:

$$\Theta_L(t) = \sum_{S \in \mathcal{S}} (-1)^{\sharp W_S}[D|S],$$

where D is a diagram representing the oriented link L and the square bracket is defined in terms of the labels in the second row of Figure 3.1. This definition is a state sum using white holes and labels that are essentially the inverses of the ones in Theorem 3.2.

We will show that this alternative definition of the Alexander polynomial is, in fact, equal to the Alexander-Conway polynomial as expressed in Theorem 3.2. Before proving that fact, we will show an interesting result relating the parities of white and black holes in a given state in Proposition 3.8.

Definition 3.3. *A flat Reidemeister move is a move performed in a planar diagram representing a link (a link diagram with double points, that is with no specification of over or under crossings). Such diagrams are called universes in [27]. Flat Reidemeister moves are diagrammatically the same as the usual Reidemeister moves, but have no constraint about patterns of over and under crossings, as shown in Figure 3.2.*

Lemma 3.4. *Let D be a planar diagram representing an oriented link L; write $s(D)$ and $c(D)$ for the number of (Seifert) circles and edges in the Seifert diagram associated to D (defined in Subsection 1.2.1). The parity of $e(D) = c(D) - s(D) + 1$ is invariant under flat Reidemeister moves. Hence, the parity is a link invariant which will be called $E(L)$.*

Proof. The proof is given in Figure 3.2. □

Corollary 3.5. *Let L be an oriented link with μ components. Then the parity of μ is opposite to $E(L)$.*

Proof. By a finite sequence of flat Reidemeister moves, any planar diagram representing L can be transformed into the trivial link with μ components, which satisfies $e(U_\mu) = 1 - \mu$. □

Lemma 3.6. *Let D be a planar diagram of an oriented link L. There always exists an FKT state S for D satisfying $\sharp B_S + \sharp W_S = c(D) - s(D) + 1$, where B_S and W_S are the sets of black and white holes in S respectively.*

Proof. We give an algorithm for constructing such an state, following some ideas in [27]. Starting from D, construct its associated Seifert diagram. Choose an innermost Seifert circle s_i and choose one of the edges joining s_i to another circle s_j. Joint the two circles in the place of the edge by considering the opposite smoothing as the one given by Seifert's algorithm. The resulting configuration has one fewer circle, and after repeating this procedure $s(D) - 1$ times using each

$$s(D) = s(D_0) + 1$$
$$c(D) = c(D_0) + 1$$

$$s(D) = s(D_0)$$
$$c(D) = c(D_0) + 2$$

$$s(D) = s(D_0) + 2$$
$$c(D) = c(D_0) + 2$$

$$s(D) = s(D_0)$$
$$c(D) = c(D_0) + 2$$
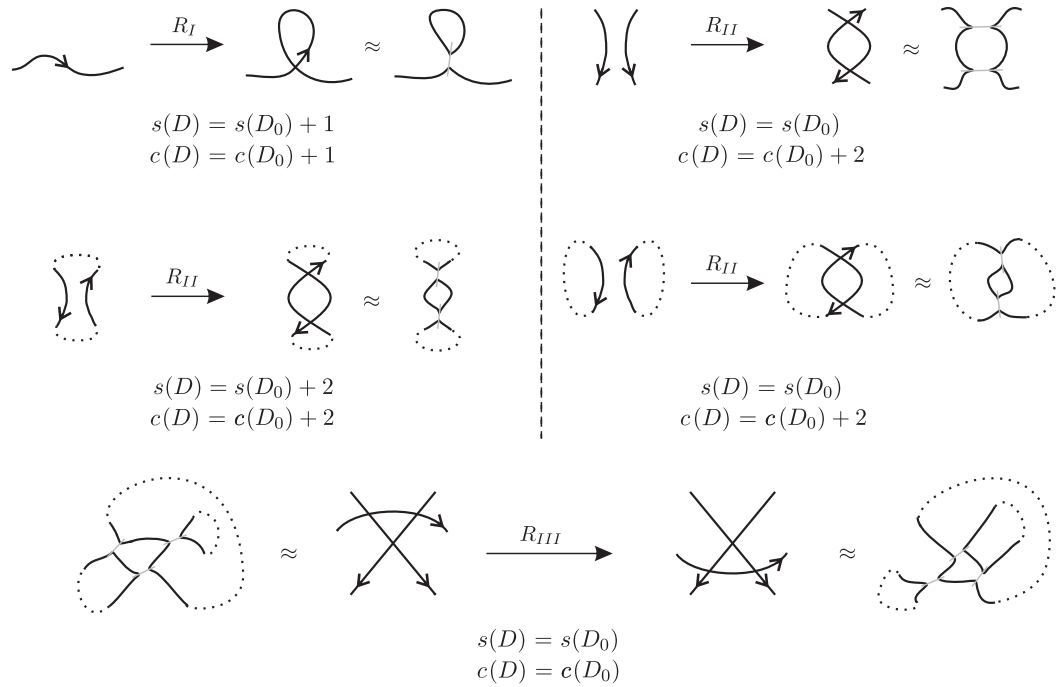
$$s(D) = s(D_0)$$
$$c(D) = c(D_0)$$

Figure 3.2: The proof of Lemma 3.4 for first and second Reidemeister moves with all possible orientations and Seifert circles configurations are shown in this picture (D_0 and D are the diagrams before and after the corresponding Reidemeister move, respectively). For the third Reidemeister move the many different possibilities can be checked as in the example.
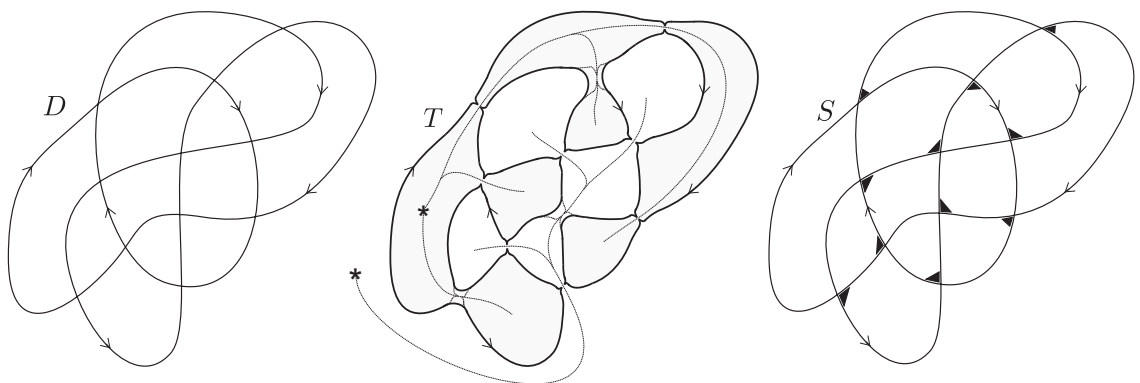

Figure 3.3: The figure shows the trail T associated to the planar diagram D together with a choice of the starred regions and the two rooted trees. It also shows the associated Kauffman state S.
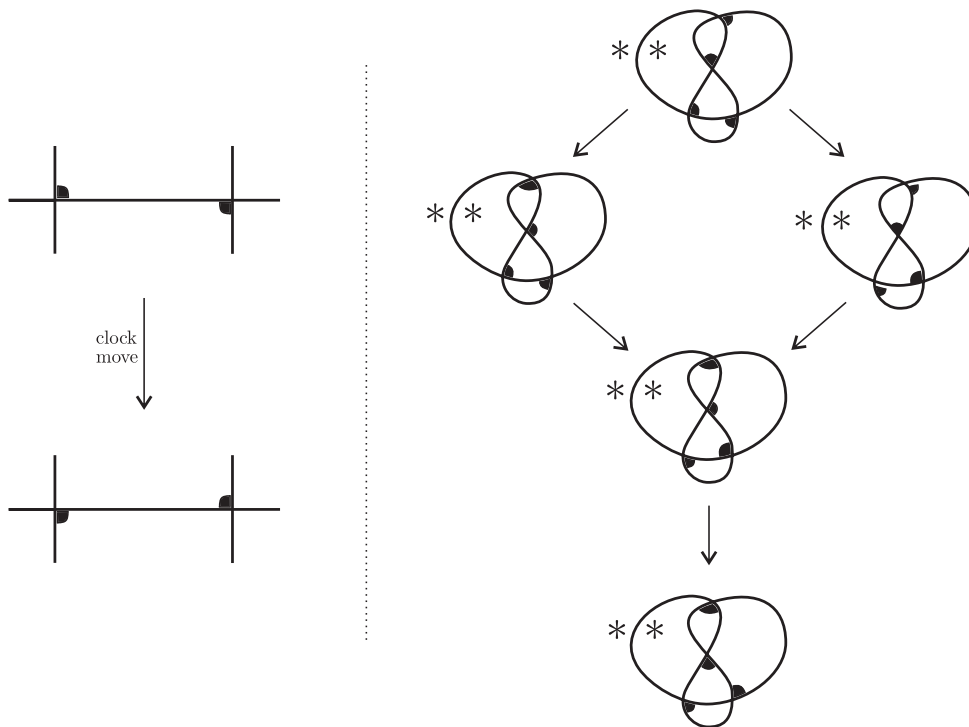
Figure 3.4: A clock move transposition and the lattice obtained from the 4_1 classic diagram with a fixed choice of unmarked regions.

edge at most once, we obtain a simple closed curve or Jordan trail T, as shown in Figure 3.3.

Kauffman proved that there exists a bijection between the collection of states of D sharing a fixed choice of adjacent stars and the collection on all Jordan trails arising from D. Choose two adjacent regions in D and draw an star in each of them; let S be the Kauffman state associated to T constructed in the following way: construct two trees, each one rooted at one of the stars, in such a way that their branches visit each region once (see Figure 3.3). If we consider the tree starting at the root, draw in each vertex a marker in the place where the branch enters a new region.

Let us see that S satisfies the condition in the statement. The markers coming from a re-smoothing are neither black nor white holes. Each marker coming from a smoothing preserving orientation is either a black or a white hole. As we did $s(D) - 1$ re-smoothings, it follows that $\sharp(B_S) + \sharp(W_S) = c(D) - s(D) + 1$.   □

A *clock move* is a modification of two adjacent markers as shown in the left-most part of Figure 3.4. In [27] Kauffman proved that the collection of possible states arising from a given planar diagram with a fixed choice of adjacent un-marked regions is a lattice, where all the states are related by a finite sequence of clock moves. In Figure 3.4 we show the example of knot 4_1.

Corollary 3.7. *Let D be a planar diagram of an oriented link L. Every FKT state S for D satisfies $\sharp B_S + \sharp W_S = c(D) - s(D) + 1$, where B_S and W_S are the sets of black and white holes in S respectively.*

Proof. Notice that clock moves preserve the number of black plus white holes. This remark together with Lemma 3.6 leads to the fact that every state S arising from a diagram D satisfies

$$\sharp B_S + \sharp W_S = c(D) - s(D) + 1.$$

$\square$

Corollary 3.5 together with Corollary 3.7 leads to the following result:

Proposition 3.8. *Let L be an oriented link with μ components and S any FKT state of a diagram D representing L. Then, the parities of the number of black and white holes in S are equal when μ is odd, and opposite when μ is even. That is, $(-1)^{\sharp W_S} = (-1)^{\mu+1}(-1)^{\sharp B_S}$.*

The following result shows that the definition of Knot Floer Homology given by Ozsváth and Szabó in [41] categorifies the Alexander-Conway polynomial via the Formal Knot Theory state sum shown in [27].

Theorem 3.9. *Both versions of the Alexander polynomial, that given by Ozsváth and Szabó [41] and that by Kauffman [27] coincide. Precisely,*

$$\Theta_L(t) = \Delta_L(t).$$

Proof. Let $\overline{L}$ denote the mirror image of L. Since $\Delta_L(t)$ satisfies the skein relation in Theorem 3.2, after substituting $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ and using an inductive argument, it follows

$$\Delta_L(z) = \Delta_{\overline{L}}(-z) = (-1)^{\mu+1}\Delta_{\overline{L}}(z).$$

Hence $\Delta_L(z) = (-1)^{\mu+1}\Delta_{\overline{L}}(z)$.

Note that the labels used in $[\cdot|\cdot]$ for positive (negative) crossings are equal to those used in $\langle\cdot|\cdot\rangle$ for negative (positive) crossings. As a consequence

$$\Delta_{\overline{L}}(z) = \sum_{S\in\mathcal{S}}(-1)^{\sharp B_S}\langle\overline{D}|S\rangle = \sum_{S\in\mathcal{S}}(-1)^{\sharp B_S}[D|S].$$

Combining these facts with Proposition 3.8 one gets:

$$\Delta_L(t) = (-1)^{\mu+1}\Delta_{\overline{L}}(t) =$$
$$= (-1)^{\mu+1}\sum_{S\in\mathcal{S}}(-1)^{\sharp B_S}[D|S] = \sum_{S\in\mathcal{S}}(-1)^{\mu+1}(-1)^{\sharp B_S}[D|S] =$$
$$= \sum_{S\in\mathcal{S}}(-1)^{\sharp W_S}[L|S] = \Theta_L(t).$$

$\square$

3.3   Disproving a proposed differential

In this section we focus on the attempts of giving a combinatorial definition of Knot Floer homology in terms of FKT states. Namely, we focus on a model proposed by Y. Rong, presented in an AMS meeting at Ohio in 2007. As far as we know, he could not prove or disprove his model. In this subsection we disprove his model by finding an example where the proposed differentials do not work, namely, we prove that the model is not invariant under Reidemeister moves.

However, we think that the ideas in the mentioned model are interesting, and they can be thought as a starting point for finding appropriate gradings in order to define the groups in the chain complex, and specially for finding an accurate differential in terms of FKT states in such a way that Knot Floer homology coincides with the homology of the complex.

The original definition of Knot Floer homology was given by Ozsváth and Szabó in 2002. We will not review this definition, as its understanding requires some familiarity with symplectic geometry. However, since Proposition 3.11 can be thought as an evidence of the relation between Knot Floer homology and FKT states, we find interesting to introduce the generators of Knot Floer homology as they were introduced in its first definition, that is, in terms of Heegaard diagrams.

Definition 3.10. *Given an oriented knot* $K \in S^3$, *a doubly-pointed Heegaard diagram* $(\Sigma, \alpha, \beta, w, z)$ *for* K *consists of*
· *A surface* $\Sigma \subset S^3$ *of genus* $g \geq 0$, *splitting* S^3 *into two handlebodies* U_0 *and* U_1, *with* Σ *oriented as the boundary of* U_0.
· *A collection* $\alpha = \{\alpha_1, \ldots, \alpha_g\}$ *(*$\beta = \{\beta_1, \ldots, \beta_g\}$ *consisting of* g *pairwise disjoint, simple closed curves on* Σ, *such that each* α_i *(*β_i*) bounds a properly embedded disk in* U_0 *(*U_1*).*
· *Two distinct points* w *and* z *on* $\Sigma \backslash (\alpha_1 \cup \ldots \cup \alpha_g \cup \beta_1 \cup \ldots \cup \beta_g)$.
We require that K *intersects* Σ *exactly at* w *and* z *and that the intersections of* K *with* U_0 *and* U_1 *consist of two properly embedded intervals.*

Every knot admits a doubly-pointed Heegaard diagram. Although there are several ways of finding such a Heegaard diagram, we are interested in the following one:

See Figure 3.5. Let D be a knot diagram and consider its planar projection P_D, which splits the plane into g regions $R_0, \ldots R_g$, with R_0 being unbounded. Take Σ as a boundary of a tubular neighborhood of P_D in $\mathbb{R}^3$. Let α_i be a curve parallel to the boundary of the region R_i, for $i = 1, \ldots, g$, and let β_i be a curve in a neighborhood of the i-th crossing of D, as shown in Figure 3.5, for $i = 1, \ldots, g-1$ (notice that D has $g-1$ crossings). Draw an additional curve β_g as a meridian of Σ next to an edge on the boundary of R_0. Finally, place the basepoints w and z on each side of β_g.

Following Manolescu [35], we review the formal definition of the generators in the original version of Knot Floer Homology. Let $\Sigma^{\times g}$ be the Cartesian product
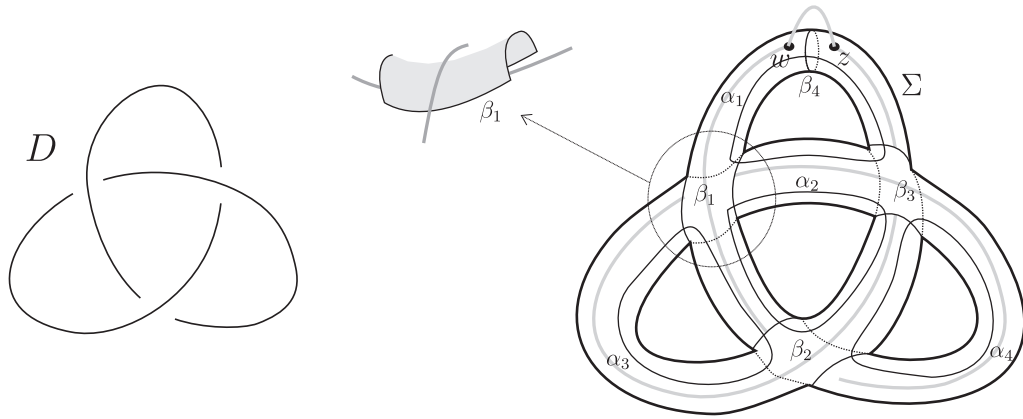
Figure 3.5: A doubly-pointed Heegaard diagram for the trefoil knot. Diagram D is represented in the Heegaard diagram in light color. The curve β_1 is detailed (β_2 and β_3 are similar).

of g copies of Σ. The symmetric group S_g acts on $\Sigma^{\times g}$ by permuting the factors. The quotient is the symmetric product $\mathrm{Sym}^g(\Sigma) := \Sigma^{\times g}/S_g$. Consider the submanifolds $T_\alpha = \alpha_1 \times \ldots \times \alpha_g$ and $T_\beta = \beta_1 \times \ldots \times \beta_g$ obtained by projection from $\Sigma^{\times g}$.

Every intersection point $x \in \mathbb{T}_\alpha \bigcap \mathbb{T}_\beta$ consists of an unordered g-tuple of points of Σ, one on each α curve and one on each β curve. These are the generators of Knot Floer complex. Notice that each intersection point x corresponds to a collection of markers as in Definition 3.1, hence to a FKT state. Hence, the above construction of a doubly-pointed Heegaard diagram for any given knot diagram leads to the following result:

Proposition 3.11. [41, Proposition 12.1] *Let K be a knot in S^3. Fix a diagram D representing K together with a choice of starred regions in the planar diagram P_D. Then there is a Heegaard diagram for K, whose generators are in one-to-one correspondence with the FKT states of D.*

The previous theorem reveals that Knot Floer homology could be defined in terms of FKT states in a similar way to that introduced by Viro for defining Khovanov homology (we explain his approach in Section 4.2). However, as the differentials in the original definitions are defined in terms of pseudo-holomorphic disks counting, translating them to the language of FKT states is not easy.

One could think that it may be easier to translate the differentials of Knot Floer homology defined in terms of grid diagrams, as this combinatorial version can be easily understood. However, in this approach the relation between generators in the grid and FKT states is not clear.

In fact, finding an appropriate differential in terms of KFT states is a hard problem. There have been some attempts to give such a definition, but they failed. We focus now on an attempt by Y. Rong.

Figure 3.6: The labelings associated to Alexander and Markov gradings.

Taking as starting point the fact that Knot Heegaard Floer homology categorifies Alexander polynomial and the definition of this polynomial in terms of FKT states (Theorem 3.2), Rong's model was introduced in a talk entitled *"A homology theory via clock moves"* presented at the AMS Spring Central Sectional Meeting in Ohio (March 2007) [45].

Let D be an oriented diagram representing the knot K. Any state S of D has two associated gradings. Write $c(D)$ for the number of crossings of D, $w(D)$ for its writhe and $rot(D)$ for its rotation number (also known as winding number), and define the Alexander and Maslov gradings of S as

$$\mathbb{A}(S) = \sum_{i=1}^{c(D)} a_i(S) - \frac{1}{2}(c(D) + rot(D))$$

$$\mathbb{M}(S) = \sum_{i=1}^{c(D)} m_i(S) - \frac{1}{4}(3c(D) + 2rot(D) - w(D)),$$

where a_i and m_i are given by the labels in Figure 3.6.

Note that in both Alexander and Maslov gradings, the terms $c(D), w(D)$ and $rot(D)$ only depend on the chosen diagram D representing K. For this reason, we will sometimes forget these terms and write

$$A(S) = \sum_{i=1}^{c(D)} a_i(S) \qquad \text{and} \qquad M(S) = \sum_{i=1}^{c(D)} m_i(S)$$

for the reduced versions of Alexander and Maslov gradings.

The state T is adjacent to S if both states assign the same labels except in two (changing) crossings, where they are related as shown in Figure 3.7. Notice that if T is adjacent to S, then the relations between their gradings do not depend on the orientation of the strands involved in the changing crossings, namely $A(T) = A(S)$ and $M(T) = M(S) - 1$.

We define now the chain complex in Rong's model. Let $C_{i,j}(D)$ be the free abelian group generated by the set of states S of D with $A(s) = j$ and $M(s) = i$. Now fix an integer j and consider the descendant complex

$$\ldots \quad \rightarrow \quad C_{i,j}(D) \quad \xrightarrow{d_i} \quad C_{i-1,j}(D) \quad \longrightarrow \quad \ldots$$
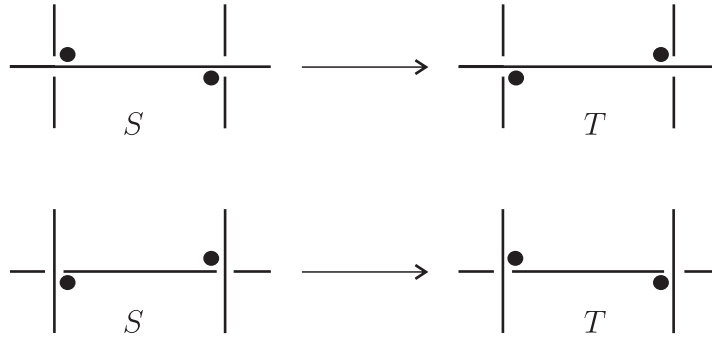
Figure 3.7: T is adjacent to S if they differ in one of these two situations.

with differential

$$d_i(S) = \sum_{T \text{ adjacent to } S} T.$$

Taking coefficients in $\mathbb{Z}_2$ it is not hard to check that $d_{i-1} \circ d_i = 0$, so one can consider the corresponding homology modules over $\mathbb{Z}_2$

$$H^{i,j}(D) = \frac{\ker(d_{i-1})}{\operatorname{im}(d_i)}.$$

Note that the only effect of considering the reduced gradings $A(S)$ and $M(S)$ instead of $\mathbb{A}(S)$ and $\mathbb{M}(S)$ is a shifting in the gradings of the homology modules.

In was claimed in [45] that, up to some shiftings in the gradings, these homology groups were independent on the diagram representing the knot, that is, they were link invariants, and they categorify Alexander polynomial in the same way as Knot Floer homology:

$$\sum_{i,j}(-1)^i \cdot \operatorname{rank}(H_{i,j}(D)) \cdot t^j = \Delta_K(t).$$

However, we show this is not true by finding the following counterexample which is not invariant under Reidemeister moves.

Example 3.12. *Let D and D' be the diagrams shown in Figure 3.8 representing the knot $K = 5_1$. They are related by a Reidemeister II move, and they have 7 and 9 crossings respectively.*

Let us start by studying Rong's model when considering diagram D. Fix two starred regions in its planar projection P_D: the unbounded and the shaded regions in Figure 3.8.

As there are 21 possible states associated to D, its chain complex has 21 generators. If one considers the reduced gradings $A(S)$ and $M(S)$ the chain complex is the following (we draw the generators of each module, together with arrows indicating adjacent states):
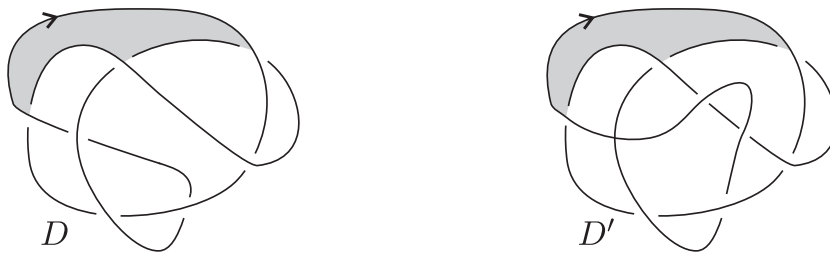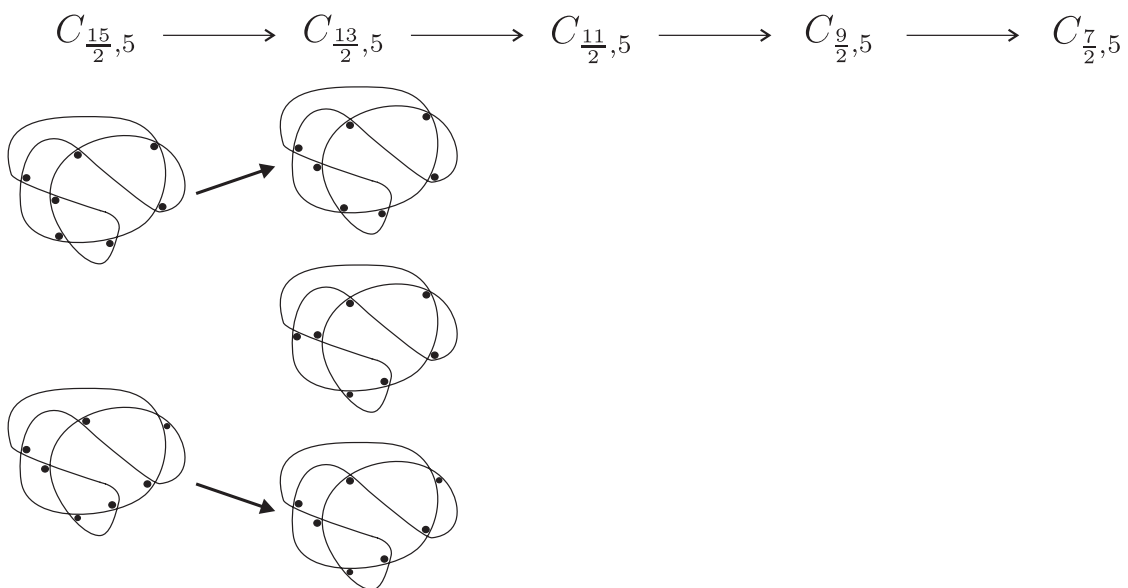
Figure 3.8: Diagrams D and D' representing knot 5_1. The starred regions in the associated planar diagrams P_D and $P_{D'}$ are the unbounded and the shaded ones.

$$C_{\frac{15}{2},6} \longrightarrow C_{\frac{13}{2},6} \longrightarrow C_{\frac{11}{2},6} \longrightarrow C_{\frac{9}{2},6} \longrightarrow C_{\frac{7}{2},6}$$

$$C_{\frac{15}{2},5} \longrightarrow C_{\frac{13}{2},5} \longrightarrow C_{\frac{11}{2},5} \longrightarrow C_{\frac{9}{2},5} \longrightarrow C_{\frac{7}{2},5}$$
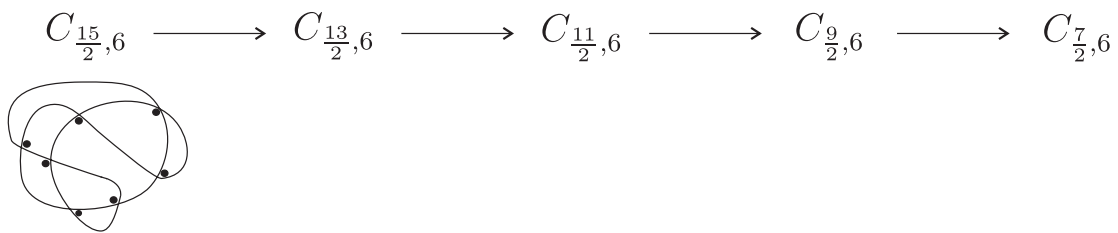
$$C_{\frac{15}{2},4} \longrightarrow C_{\frac{13}{2},4} \longrightarrow C_{\frac{11}{2},4} \longrightarrow C_{\frac{9}{2},4} \longrightarrow C_{\frac{7}{2},4}$$
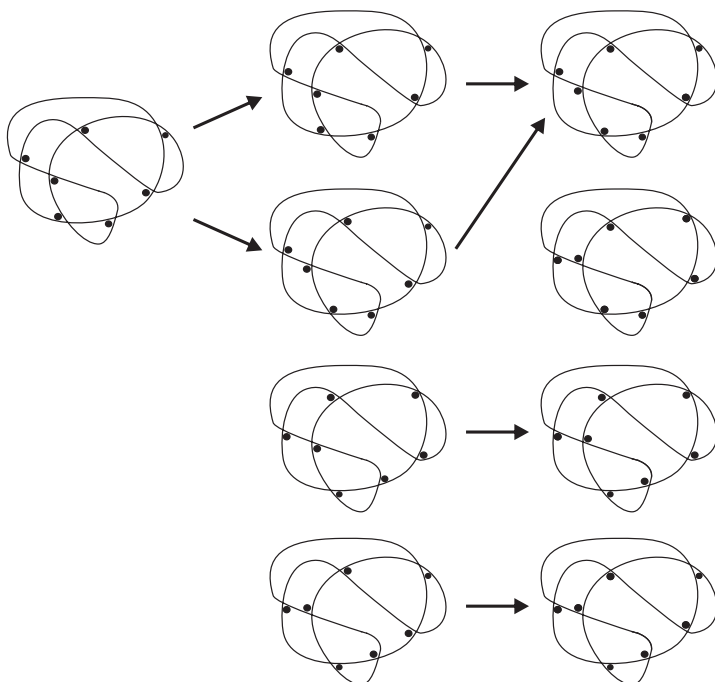
$$C_{\frac{15}{2},3} \longrightarrow C_{\frac{13}{2},3} \longrightarrow C_{\frac{11}{2},3} \longrightarrow C_{\frac{9}{2},3} \longrightarrow C_{\frac{7}{2},3}$$
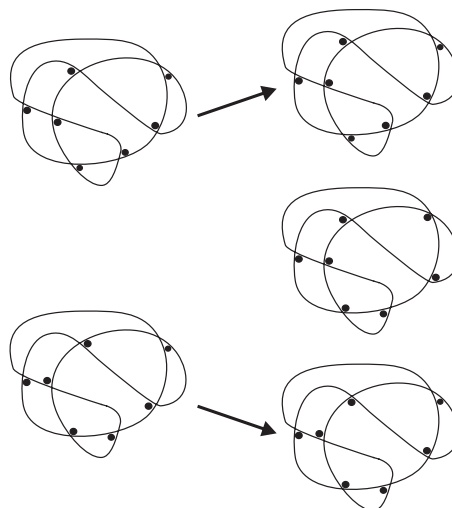
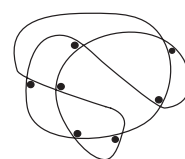$$C_{\frac{15}{2},2} \longrightarrow C_{\frac{13}{2},2} \longrightarrow C_{\frac{11}{2},2} \longrightarrow C_{\frac{9}{2},2} \longrightarrow C_{\frac{7}{2},2}$$
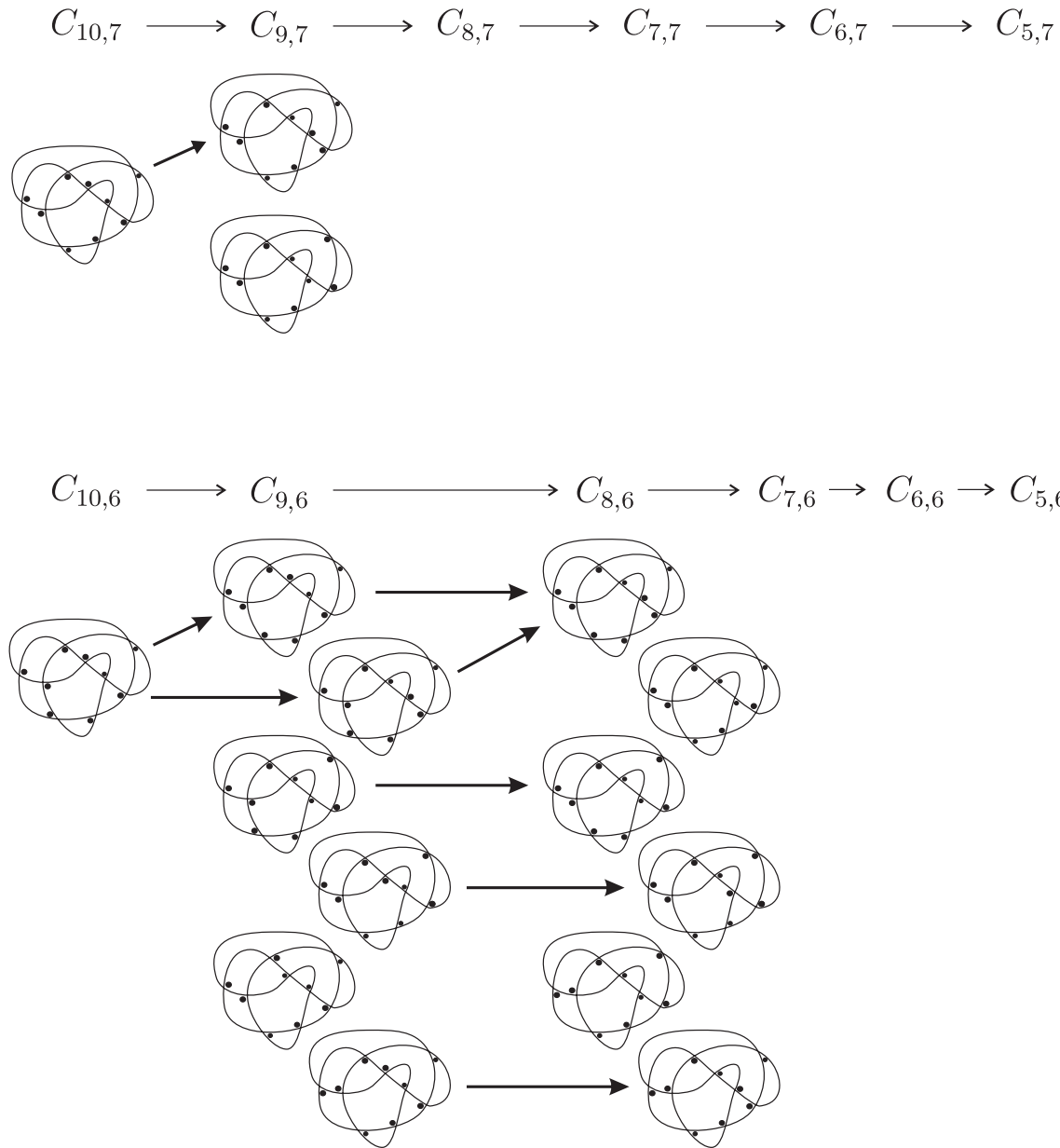
Hence the non-trivial homology groups associated to D are

$$H_{\frac{15}{2},6}(D) \simeq H_{\frac{13}{2},5}(D) \simeq H_{\frac{11}{2},4}(D) \simeq H_{\frac{9}{2},3}(D) \simeq H_{\frac{7}{2},2}(D) \simeq \mathbb{Z}_2.$$

In fact, this computation matches with the one expected for knot 5_1: since this knot is alternating, its Knot Floer homology is known to have five non-trivial groups disposed in a diagonal (that is, appearing in $H_{i,i+k}$ with k being a constant).

Now consider the diagram D'. According to Rong's model its chain complex, which has 49 generators, is the following:

$$C_{10,7} \longrightarrow C_{9,7} \longrightarrow C_{8,7} \longrightarrow C_{7,7} \longrightarrow C_{6,7} \longrightarrow C_{5,7}$$

$$C_{10,6} \longrightarrow C_{9,6} \longrightarrow C_{8,6} \longrightarrow C_{7,6} \rightarrow C_{6,6} \rightarrow C_{5,6}$$

$C_{10,5} \longrightarrow C_{9,5} \xrightarrow{\hspace{3cm}} C_{8,5} \xrightarrow{\hspace{4cm}} C_{7,5} \longrightarrow C_{6,5} \rightarrow C_{5,5}$
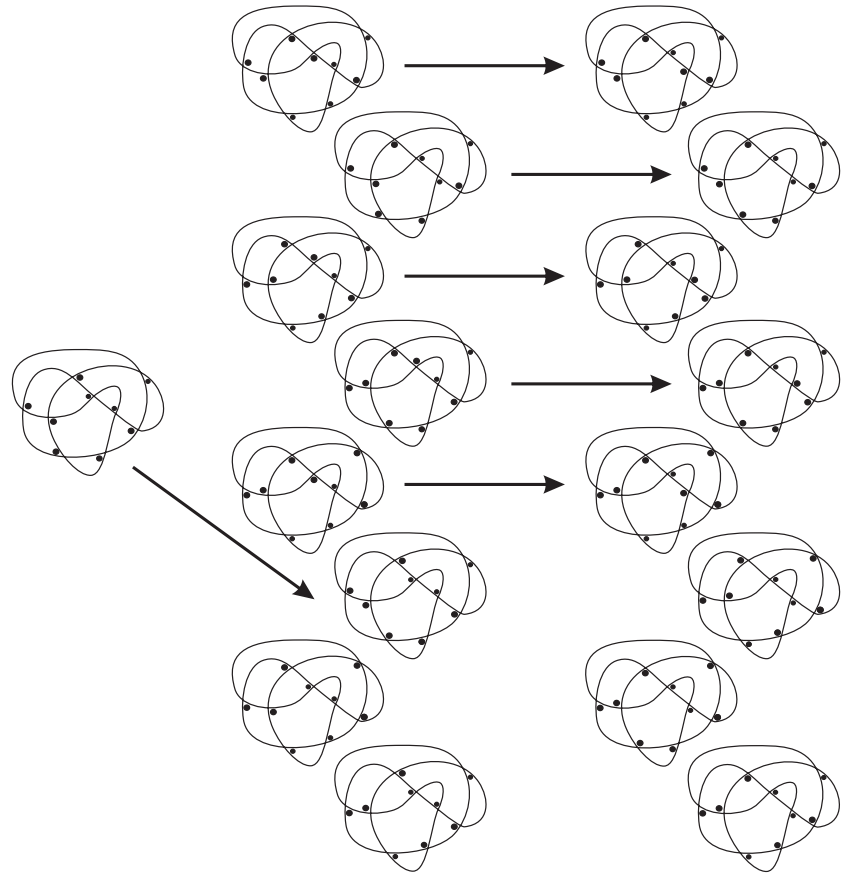
$$C_{10,4} \to C_{9,4} \to C_{8,4} \longrightarrow C_{7,4} \longrightarrow C_{6,4} \longrightarrow C_{5,4}$$
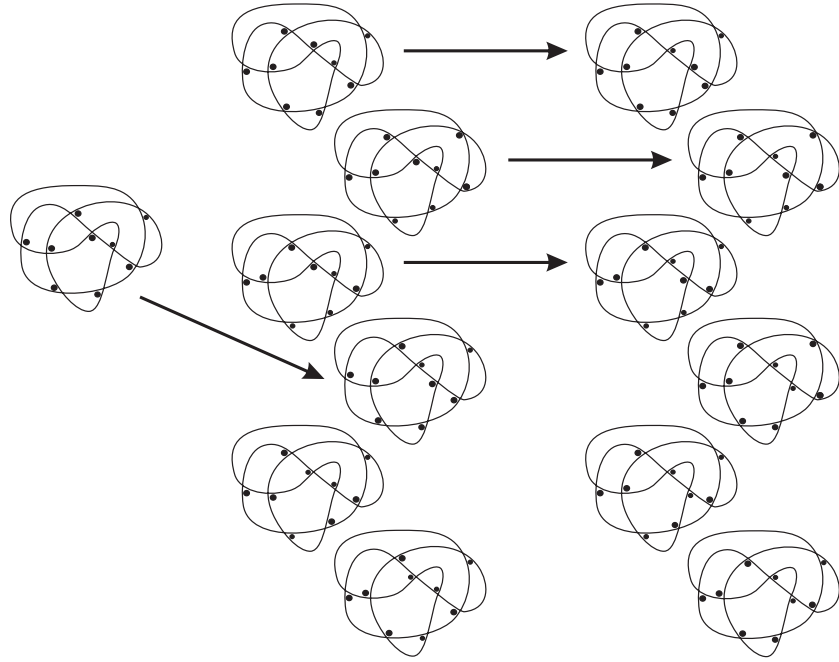
$$C_{10,3} \longrightarrow C_{9,3} \longrightarrow C_{8,3} \longrightarrow C_{7,3} \longrightarrow C_{6,3} \longrightarrow C_{5,3}$$
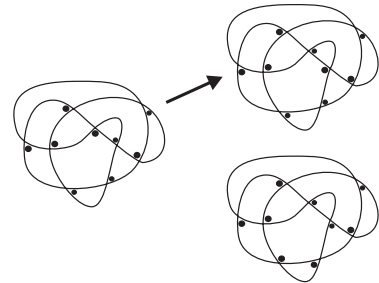
If one computes its homology the non-trivial groups are

$$H_{9,7}(D') \simeq H_{9,6}(D') \simeq H_{5,3}(D') \simeq \mathbb{Z}_2, \qquad H_{8,6}(D') \simeq H_{8,5}(D') \simeq H_{7,4}(D') \simeq \mathbb{Z}_2^2,$$

$$H_{7,5}(D') \simeq H_{6,4}(D') \simeq \mathbb{Z}_2^3.$$

The computations of the homology groups in these examples are summarized in Figure 3.9. One can see that both homologies are not equivalent, even when allowing some shifting on the gradings.

First board

$(\frac{15}{2},6)\,\mathbb{Z}_2$	$(\frac{12}{2},6)$	$(\frac{11}{2},6)$	$(\frac{9}{2},6)$	$(\frac{7}{2},6)$
$(\frac{15}{2},5)$	$(\frac{12}{2},5)\,\mathbb{Z}_2$	$(\frac{11}{2},5)$	$(\frac{9}{2},5)$	$(\frac{7}{2},5)$
$(\frac{15}{2},4)$	$(\frac{12}{2},4)$	$(\frac{11}{2},4)\,\mathbb{Z}_2$	$(\frac{9}{2},4)$	$(\frac{7}{2},4)$
$(\frac{15}{2},3)$	$(\frac{12}{2},3)$	$(\frac{11}{2},3)$	$(\frac{9}{2},3)\,\mathbb{Z}_2$	$(\frac{7}{2},3)$
$(\frac{15}{2},2)$	$(\frac{12}{2},2)$	$(\frac{11}{2},2)$	$(\frac{9}{2},2)$	$(\frac{7}{2},2)\,\mathbb{Z}_2$

Second board

$(10,7)$	$(9,7)\,\mathbb{Z}_2$	$(8,7)$	$(7,7)$	$(6,7)$	$(5,7)$
$(10,6)$	$(9,6)\,\mathbb{Z}_2$	$(8,6)\,\mathbb{Z}_2^2$	$(7,6)$	$(6,6)$	$(5,6)$
$(10,5)$	$(9,5)$	$(8,5)\,\mathbb{Z}_2^2$	$(7,5)\,\mathbb{Z}_2^3$	$(6,5)$	$(5,5)$
$(10,4)$	$(9,4)$	$(8,4)$	$(7,4)\,\mathbb{Z}_2^2$	$(6,4)\,\mathbb{Z}_2^3$	$(5,4)$
$(10,3)$	$(9,3)$	$(8,3)$	$(7,3)$	$(6,3)$	$(5,3)\,\mathbb{Z}_2$

Figure 3.9: The first (second) board summarizes $H_{i,j}(D)$ ($H_{i,j}(D')$). A cell indexed by (i,j) represents $H_{i,j}$. Blank cells correspond to trivial groups.

Although we have shown that this model is not accurate enough, we think that there are some interesting ideas in Rong's approach which could be used as a start point for finding an appropriate differential. Focusing on the obstructions found in Example 3.12, our idea is to extend the definition of adjacency between two different states, that is, allowing other moves in addition to those shown in Figure 3.7. I have been discussing these ideas with Louis Kauffman and this is the topic of an ongoing joint project.

Chapter 4

A geometric realization of the extreme Khovanov cohomology

4.1 Introduction

The Khovanov cohomology of knots and links was introduced by Mikhail Khovanov at the end of last century (see [28]) and nicely explained by Bar-Natan in [8]. In [57] Viro interpreted it in terms of enhanced states of diagrams. This chapter is devoted to explain a new approach to the extreme Khovanov cohomology of a link in terms of the independence complex of a graph. More precisely, using Viro's point of view, in Theorem 4.4 we prove that the hypothetical extreme Khovanov cohomology of a link is equal to the cohomology of the independence simplicial complex of its Lando graph.

The Lando graph [17] of a link diagram was studied by Morton and Bae in [7], where they proved that the hypothetical extreme coefficient of the Jones polynomial is equal to certain numerical invariant of the graph, named in Section 4.3 independence number. Hence, on one hand the Jones polynomial can be seen as the bigraded Euler characteristic of the Khovanov cohomology, and on the other hand the formula for the independence number certainly suggests the formula of an Euler characteristic. Both ideas together gave us the key for understanding the extreme Khovanov cohomology in terms of Lando graph. This new approach to Khovanov cohomology is detailed in Section 4.4.

There are many other interesting constructions of graphs starting with a link diagram (see for example [20]), not to be confused with the Lando graph. In addition, there are other very interesting ways of trying to understand the Khovanov cohomology as the cohomology of *something*. For example in [32] Lipshitz and Sarkar construct, in an explicit and combinatorial way, a chain bicomplex that produces the Khovanov cohomology.

In [34] it was shown that the independence number can take any value, hence there are links (in fact knots) with arbitrary extreme coefficients. We also extended this idea to Khovanov cohomology in Theorem 4.16, by proving that there
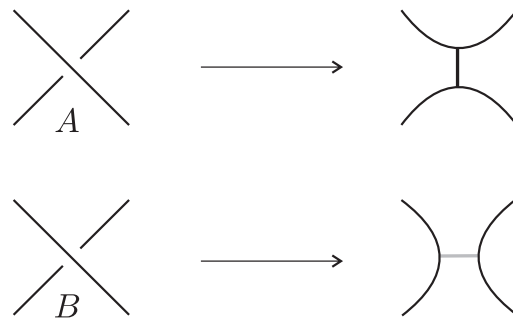
Figure 4.1: The smoothing of a crossing according to its A or B-label. A-chords (B-chords) are represented by dark (light) segments.

are links (in fact knots) with an arbitrary number of non-trivial extreme Khovanov cohomology modules. These knots are examples of H-thick knots (see [29]). The basic piece of these examples is a link with exactly two non-trivial extreme Khovanov cohomology modules, constructed in Theorem 4.13. Not only this result but the construction of the example is interesting in itself, as it shows an interesting relation between the homology of a simplicial complex and the extreme Khovanov cohomology of a link. We develop these results in Sections 4.5 and 4.6.

The next two sections are mainly devoted to review concepts as Khovanov cohomology, independence complex or Lando graph, in order to fix the conventions about signs, gradings and notation that we are going to use through the chapter.

4.2   Khovanov cohomology

In this section we review Khovanov cohomology in a concise way by following Viro's approach in terms of enhanced states [57] and precise some concepts like extreme complexes and extreme cohomology modules.

Let D be an oriented diagram of a link L with c crossings and writhe $w = p - n$, with p and n being the number of positive and negative crossings in D, according to the sign convention shown in Figure 1.2. A *state* s assigns a label, A or B, to each crossing of D. Let $\mathcal{S}$ be the collection of 2^c possible states of D. For $s \in \mathcal{S}$ assigning $a(s)$ A-labels and $b(s)$ B-labels, write $\sigma = \sigma(s) = a(s) - b(s)$. The result of smoothing each crossing of D according to its label following Figure 4.1 is a collection sD of disjoint circles embedded in the plane together with some A and B-chords (segments joining the circles in the site where there was a crossing). We represent A-chords as dark segments, and B-chords as light ones. See Figure 4.2.

Enhance the state s with a map e which associates a sign $\epsilon_i = \pm 1$ to each of the $|s|$ circles in sD. Unless otherwise stated, we will keep the letter s for the enhanced state (s, e) to avoid cumbersome notation. Write $\tau = \tau(s) = \sum_{i=1}^{|s|} \epsilon_i$,
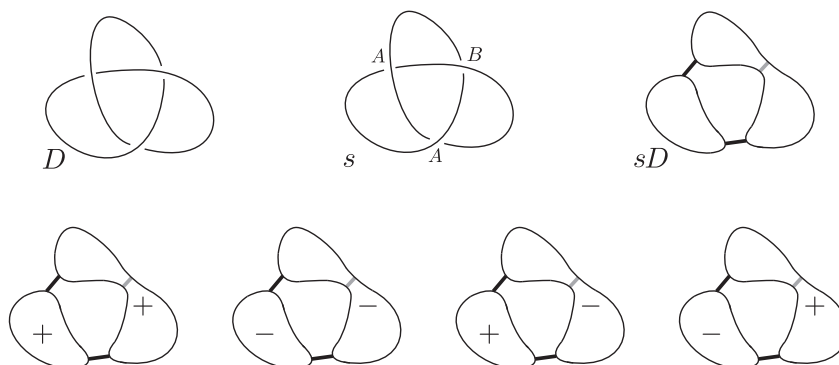
Figure 4.2: The first row shows a diagram D representing 3_1, a state s and sD. Here $|s| = 2$. The four related enhanced states are shown in the second row. From left to right the values of τ are $2, -2, 0, 0$.

and define, for the enhanced state s, the integers

$$i = i(s) = \frac{w - \sigma}{2}, \qquad j = j(s) = w + i + \tau.$$

The enhanced state t is adjacent to s if the following conditions are satisfied:

1. $i(t) = i(s) + 1$ and $j(t) = j(s)$.

2. The labels assigned by t are identical to those assigned by s except in one (change) crossing $x = x(s, t)$, where s assigns an A-label and t a B-label.

3. The signs assigned to the common circles in sD and tD are equal.

Note that the circles which are not common to sD and tD are those touching the crossing x. In fact, passing from sD to tD can be realized by melting two circles into one, or splitting one circle into two. The different possibilities according to the previous points are shown in Figure 4.3.
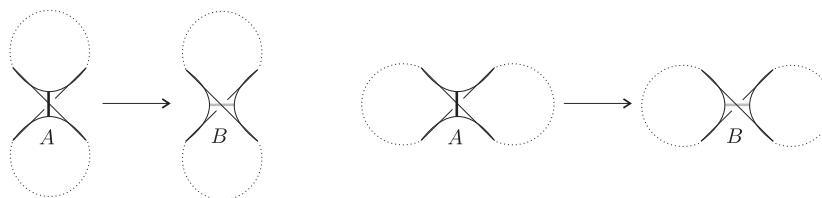

Figure 4.3: All possible enhancements when melting two circles are: $(++ \rightarrow +)$, $(+- \rightarrow -)$, $(-+ \rightarrow -)$. The possibilities for the splitting are: $(- \rightarrow --)$, $(+ \rightarrow +-)$ or $(+ \rightarrow -+)$.

We review now the chain complex and cohomology modules associated to a diagram D representing a link as presented by Viro in [57].

Let R be a commutative ring with unit. Let $C^{i,j}(D)$ be the free module over R generated by the set of enhanced states s of D with $i(s) = i$ and $j(s) = j$. Order the crossings in D. Now fix an integer j and consider the ascendant complex

$$\ldots \quad \to \quad C^{i,j}(D) \quad \xrightarrow{d_i} \quad C^{i+1,j}(D) \quad \longrightarrow \quad \ldots$$

with differential $d_i(s) = \sum(s:t)t$, where $(s:t) = 0$ if t is not adjacent to s and $(s:t) = (-1)^k$ otherwise, with k being the number of B-labeled crossings coming after the change crossing x. It turns out that $d_{i+1} \circ d_i = 0$ and the corresponding cohomology modules over R

$$H^{i,j}(D) = \frac{\ker(d_i)}{\text{im}(d_{i-1})}$$

are independent of the diagram D representing the link L and the ordering of its crossings, that is, these modules are link invariants. They are the *Khovanov cohomology modules* $H^{i,j}(L)$ of L ([28], [8]), as presented by Viro in [57] in terms of enhanced states.

Let $j_{\min} = j_{\min}(D) = \min\{j(s) \ / \ s \text{ is a enhanced state of } D\}$. We will refer to the complex $\{C^{i,j_{\min}}(D), d_i\}$ as the *extreme Khovanov complex*, and to the corresponding modules $H^{i,j_{\min}}(D)$ as the (hypothetical) *extreme Khovanov modules*. Indeed, there are analogous definitions for a certain $j_{\max}$.

Note that the integers $j_{\min}$ and $j_{\max}$ depend on the diagram D, and may differ for two different diagrams representing the same link, as can be seen in Figure 4.4.

Figure 4.4: Two diagrams D and D' representing knot 4_1. $j_{\min}(D) = -5$ and $j_{\min}(D') = -7$. (Proposition 4.1 makes these computations easier.)

4.3 Lando graph and its cohomology

Let G be a graph. A set σ of vertices of G is said to be independent if no vertices in σ are adjacent. The independence number of G is defined to be

$$I(G) = \sum_{\sigma}(-1)^{|\sigma|}$$

where the sum is taken all over the independent sets of vertices of G. The empty set is considered as an independent set of vertices with $|\emptyset| = 0$. A point has independence number 0, an hexagon 2.
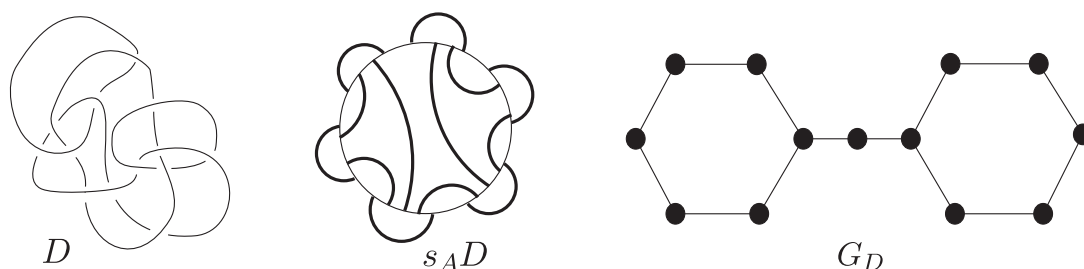
Figure 4.5: A diagram D representing a link with 4 trivial components, $s_A D$ and the corresponding Lando graph G_D.

Starting from a link diagram D we review now the construction of its Lando graph ([7], [34]). An A or B-chord of sD is *admissible* if its ends are both in the same circle of sD. Let s_A be the state assigning A-labels to all the crossings of D. The Lando graph associated to D is constructed from $s_A D$ by considering a vertex for every admissible A-chord, and an edge joining two vertices if the ends of the corresponding A-chords alternate in the same circle. Figure 4.5 exhibits a diagram D, the corresponding $s_A D$ and its Lando graph G_D.

In [7] it was proved that the coefficient of the hypothetical lowest degree monomial of the Jones polynomial of a link L is, up to sign, the independence number $I(G_D)$ of the Lando graph G_D of D, where D is a diagram of L. In [34] it was shown that $I(G_D)$ can take arbitrary values.

The Jones polynomial can be seen as the bigraded Euler characteristic of the Khovanov cohomology [28]. The formula for the independence number suggests that each extreme coefficient of the Jones polynomial is the Euler characteristic of a certain cohomology given in terms of independent sets of vertices of the Lando graph. Combining both facts was the key for thinking in the approach of the extreme Khovanov cohomology in terms of Lando graph that we present in Section 4.4.

Let X_D be the independence simplicial complex of the graph G_D. By definition, the simplices σ of X_D are the independent subsets of vertices of G_D. Let $C^i(X_D)$ be the free module over R generated by the simplices of dimension i, where the dimension of a simplex σ is the number of its vertices minus one. Associated to this simplicial complex we have the (standard) cochain complex

$$\ldots \to C^i(X_D) \xrightarrow{\delta_i} C^{i+1}(X_D) \to \ldots$$

with differential $\delta_i(\sigma) = \sum_v (-1)^k \sigma \cup \{v\}$ where v runs over the set of vertices of G_D which are not adjacent to any vertex of σ, and $k = k(\sigma, v)$ is the number of vertices of σ coming after v in the predetermined order of the vertices of G_D. It turns out that $\delta_{i+1} \circ \delta_i = 0$ and the corresponding reduced cohomology modules of X_D are

$$H^i(X_D) = \frac{\ker(\delta_i)}{\text{im}(\delta_{i-1})}.$$

We will refer to $\{C^i(X_D), \delta_i\}$ as the *Lando ascendant complex*, and to $H^i(X_D)$ as the *Lando cohomology modules*.

4.4   Lando and extreme Khovanov cohomologies

In this section we show our new approach to the extreme Khovanov cohomology of a link diagram in terms of the independence complex of its Lando graph. This idea is synthesized in Theorem 4.4, which is the key point for developing Sections 4.5 and 4.6 in this chapter.

Let D be an oriented link diagram and s_A^- the enhanced state assigning an A-label to each crossing and the sign -1 to each circle of $s_A D$. Let

$$S_{\min} = \{\text{enhanced states } s \ / \ |s| = |s_A| + b(s), \ \tau(s) = -|s|\}.$$

Proposition 4.1. *With the previous notation $j_{\min} = j(s_A^-)$, and $j(s) = j_{\min}$ if and only if $s \in S_{\min}$.*

Proof. Recall that $j(s) = \frac{3w - \sigma}{2} + \tau$ with $\sigma = a(s) - b(s)$ and $\tau(s) = \sum_{i=1}^{|s|} \epsilon_i$, where ϵ_i is the sign (+1 or -1) associated to the circle c_i in sD.

Given a diagram D let s be an enhanced state associating a positive sign to at least one of the circles in sD. Then $j(s) \neq j_{\min}$, as the state given by associating negative signs to every circle in sD has a smaller value of j. Hence, all states s realizing $j_{\min}$ assign -1 to their circles, or equivalently $\tau(s) = -|s|$ (the second condition in the definition of $S_{\min}$).

Now identify a state with the set of crossings of D where the state assigns a B-label. Let $s = \{y_1, \ldots, y_b\}$ be an enhanced state assigning $b = b(s)$ B-labels (at the crossings $y_1, \ldots, y_b$) and negative signs to all its circles. Consider the sequence of enhanced states

$$s_0 = s_A^-, s_1, \ldots, s_b = s$$

where $s_k = \{y_1, \ldots, y_k\}$ for $k = 1, \ldots, b$, and all circles in $s_k D$ have sign -1.

Since w is invariant and $\sigma(s_k) = \sigma(s_{k-1}) - 2$, there are two possibilities: if $|s_k| = |s_{k-1}| + 1$, then $\tau(s_k) = \tau(s_{k-1}) - 1$ and $j(s_k) = j(s_{k-1})$; if $|s_k| = |s_{k-1}| - 1$, then $\tau(s_k) = \tau(s_{k-1}) + 1$ and $j(s_k) = j(s_{k-1}) + 2$. Note that this is independent of the ordering of the crossings.

A first consequence is that $j(s_A^-) \leq j(s)$, where s was taken to be any state assigning -1 to all circles in sD, so $j(s_A^-) = j_{\min}$. A second consequence is that $j(s) = j_{\min}$ if and only if $|s_k| = |s_{k-1}| + 1$ for each $k \in \{1, \ldots, b\}$, that is, if and only if $s \in S_{\min}$. $\square$

There are analogous s_B^+, $j_{\max}$ and S_{max}, with $j(s_B^+) = j_{\max}$ and $s \in S_{max}$ if and only if $j(s) = j_{\max}$.
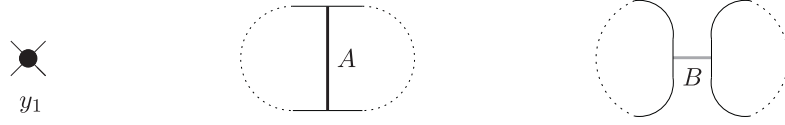
Figure 4.6: The vertex y_1 corresponds to a splitting from $s_A D = s_0 D$ to $s_1 D$.

Corollary 4.2. *Fix an oriented link diagram D with c crossings, n negative and p positive. Then $j_{\min} = c - 3n - |s_A|$ and $j_{\max} = -c + 3p + |s_B|$.*

Proof. Since $w = p - n = c - 2n$ and $\sigma(s) = c - 2b(s)$ we deduce that $i(s) = b(s) - n$. In particular $i(s_A) = -n$. It follows that

$$
\begin{aligned}
j_{\min} &= j(s_A^-) \\
&= w + i(s_A^-) + \tau(s_A^-) \\
&= (c - 2n) - n - |s_A| \\
&= c - 3n - |s_A|.
\end{aligned}
$$

A similar argument works for $j_{\max}$ using s_B^+ instead of s_A^-. $\qquad\square$

Recall that the vertices in the Lando graph of D, G_D, are associated to the admissible A-chords in $s_A D$ (the ones having both ends in the same circle of $s_A D$). Given an enhanced state s, let V_s be the set of vertices of G_D corresponding to the crossings of D to which s associates a B-label. Note that V_s can have less than $b(s)$ vertices, or even be empty.

Proposition 4.3. *The map that assigns V_s to each enhanced state s defines a bijection between S_{min} and the set of independent sets of vertices of G_D. Moreover, if $s \in S_{min}$ then the cardinal of V_s is exactly $b(s)$.*

Proof. Let $s = \{y_1, \ldots, y_b\}$ be an enhanced state in $S_{\min}$ with $b = b(s)$ B-labels (at the crossings $y_1, \ldots, y_b$). Consider the sequence of enhanced states

$$
s_0 = s_A^-, s_1, \ldots, s_b = s
$$

where $s_k = \{y_1, \ldots, y_k\}$ for $k = 1, \ldots, b$, and all circles in $s_k D$ have sign -1. As $s \in S_{min}$, according to the proof of Proposition 4.1 $|s_k| = |s_{k-1}| + 1$ for each $k \in \{1, \ldots, b\}$, or equivalently, one passes from $s_{k-1} D$ to $s_k D$ by splitting one circle into two circles.

Note that the A-chord of $s_A D$ corresponding to the crossing y_1 of D is admissible, since otherwise $|s_1| = |s_0| - 1$ (see Figure 4.6). As the construction in the previous sequence does not depend on the order of the crossings, it follows that any A-chord of $s_A D$ corresponding to a crossing y_i is admissible in $s_A D$, so G_D contains its associated vertex.

Moreover there is no pair of A-chords in $s_A D$ corresponding to B-labels of s with their ends alternating in the same circle, since otherwise two B-smoothings
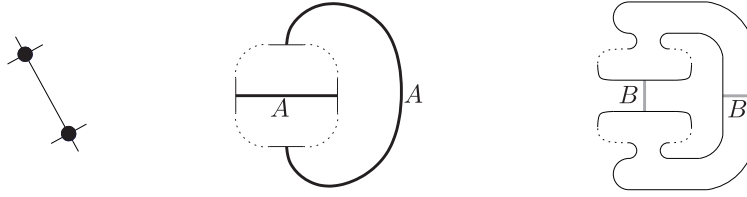
Figure 4.7: Adjacent vertices in G_D correspond to $|s_A| = |s|$ when smoothing.

in these two crossings would not increase the number of circles by two, as Figure 4.7 shows schematically. This implies that the corresponding vertices in V_s are independent.

Conversely, if C is an independent set of vertices of G_D, consider the state s that assigns B-labels exactly in the corresponding crossings. In particular $b(s) = |C|$. Enhance this state assigning -1 to each circle of sD. Since C is independent, $|s| = |s_A| + b(s)$, hence $s \in S_{min}$ as we wanted to show.   □

The extreme Khovanov cohomology is constructed, according to Proposition 4.1, in terms of the states in S_{min}. For these states the definition of adjacency given in Section 4.2 is reduced to the second condition given there. Namely, if $s, t \in S_{min}$ then t is adjacent to s if and only if t assigns the same labels as s except in one crossing x, where $s(x)$ and $t(x)$ are an A-label and a B-label, respectively. We introduce now the main result in this chapter:

Theorem 4.4. *Let L be an oriented link represented by a diagram D having n negative crossings. Let G_D be the Lando graph of D and let $j = j_{\min}(D)$. Then the Lando ascendant complex $\{C^i(X_D), \delta_i\}$ is a copy of the extreme Khovanov complex $\{C^{i,j}(D), d_i\}$, shifted by $n - 1$. Hence*

$$H^{i,j}(D) \approx H^{i-1+n}(X_D).$$

Proof. According to Proposition 4.1, the extreme Khovanov cohomology is constructed with the states in S_{min}. Suppose that $s \in S_{min}$ and let V_s be the corresponding independent set of vertices of G_D. Since $i(s) = b(s) - n$ and $\dim(V_s) = |V_s| - 1 = b(s) - 1$, the bijection between S_{min} and the set of independent sets of vertices of G_D established in Proposition 4.3 provides an isomorphism

$$C^{i,j}(D) \approx C^{i-1+n}(X_D).$$

One just needs to show that this isomorphism respects the differential of both complexes (in other words, that the assignment s to V_s defines a chain isomorphism). Recall that for two enhanced states $s, t \in S_{min}$, t is adjacent to s if and only if t assigns the same labels as s except in one crossing x, where $s(x)$ and $t(x)$ are an A-label and a B-label, respectively. Moreover, in this case only a splitting is possible at the change crossing x when passing from sD to tD, since the degree $j_{\min}$ must be preserved and the degree i must be increased by
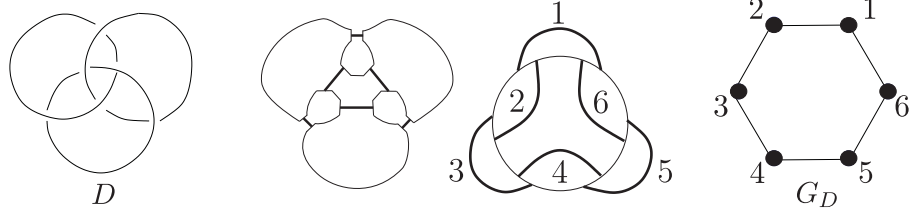
Figure 4.8: A link diagram D, two versions of $s_A D$ and its associated Lando graph G_D.

one (note that $\tau(s) = -|s|$ and $\tau(t) = -|t|$). It follows that $V_t = V_s \cup \{v_x\}$ where v_x is the vertex in G_D corresponding to x.

In addition, if we order the vertices of G_D according to the order of the crossings in D we get that the number of B-labeled crossings of D coming after the crossing x, is exactly the number of vertices of V_s coming after the vertex v_x. $\square$

Example 4.5. *Consider the link L represented by the diagram D shown in the leftmost part of Figure 4.8. The Lando graph G_D is the hexagon shown at the right hand side of Figure 4.8. Number its vertices consecutively, from 1 to 6. Then $C^{-1}(X_D)$, $C^0(X_D)$, $C^1(X_D)$ and $C^2(X_D)$ have ranks 1, 6, 9 and 2 respectively, with respective basis*

$$\{\emptyset\}, \{1,2,3,4,5,6\}, \{13,14,15,24,25,26,35,36,46\}, \{135,246\},$$

the other modules being trivial (note that we write, for example, 135 instead of $\{1,3,5\}$). The Lando ascendant complex is

$$0 \longrightarrow C^{-1}(X_D) \xrightarrow{\delta_{-1}} C^0(X_D) \xrightarrow{\delta_0} C^1(X_D) \xrightarrow{\delta_1} C^2(X_D) \longrightarrow 0,$$

with differentials δ_{-1}, δ_0 and δ_1 given respectively by the matrices

$$
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},
\begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1
\end{pmatrix}.
$$

Let R be the field of rational numbers. The ranks of these matrices are 1, 5 and 2 respectively. In particular $H^1(X_D)$ has dimension two as a rational vector space, the other Lando cohomology vector spaces being trivial.

If we orient now the three components of D in a counterclockwise sense, by Corollary 4.2 $j_{\min} = c - 3n - |s_A| = 6 - 3 \cdot 6 - 1 = -13$, and by Theorem 4.4 the complexes are shifted by $n - 1 = 5$, hence $H^{-4,-13}(L) \approx H^1(X_D)$ is two-dimensional, being trivial the rest of extreme Khovanov cohomology vector spaces.

This example shows that, in general, for different orientations of the components of a link, we obtain the same extreme Khovanov cohomology modules ($j_{\min}$ may change), with some shifting in the index i.

Definition 4.6. *A graph G is bipartite if its vertices can be divided into two disjoint sets V and W in such a way that every edge of G connects a vertex in V to one in W. A bipartite graph is complete if every pair of vertices $v \in V$ and $w \in W$ are connected by an edge.*

Note that if one colors the regions of $s_A D$ in a chess fashion, the vertices corresponding to admissible A-chords in white regions are not adjacent to those associated to admissible A-chords in black regions. Hence, the following Proposition holds:

Proposition 4.7. *For any link diagram D its associated Lando graph G_D is bipartite.*

Corollary 4.8. *Let L be an oriented link represented by a diagram D having n negative crossings. Let G_D be the Lando graph of D and let $j = j_{\min}(D)$. Then $H^{1-n,j}(L) \approx R$ if G_D is the complete bipartite graph $K_{r,s}$ and it is trivial otherwise.*

Proof. According to Theorem 4.4 we just have to prove that $H^0(X_D) \approx R$ if G_D is $K_{r,s}$, and it is trivial otherwise. Let G_D^c be the complementary graph of G_D. Any Lando graph is a bipartite graph, and it is complete if and only if G_D^c has exactly two connected components; otherwise G_D^c is connected. The key observation is now that the connected components of G_D^c coincide exactly with the elements of a basis of $\ker(\delta_0)$. The fact that $\delta_{-1}(\emptyset) = 1 + 2 + \cdots + c \in C^0(X_D)$ completes the proof. $\square$

4.5   Lando cohomology as homology of a simplicial complex

In this section we show how to construct a simplicial complex whose homology is equal to the cohomology of the independence complex of a Lando graph G_D up to some shifting. This fact together with Theorem 4.4 implies that the homology of the simplicial complex determines the extreme Khovanov cohomology of the link represented by D. A key point is the following result by Jonsson [24]:

Theorem 4.9. [24, Theorem 3.1] *Let G be a bipartite graph with nonempty parts V and W. Then there exists a simplicial complex $X_{G,V}$ with ground set V, whose suspension is homotopy equivalent to the independence complex of G.*

In [24] Jonsson also gave the procedure for constructing the complex $X_{G,V}$. Starting with the bipartite graph G, a set $\sigma \subseteq V$ belongs to $X_{G,V}$ if and only if there is a vertex $w \in W$ such that $\sigma \bigcup \{w\}$ is an independent set in G. In other words, $\sigma \subseteq V$ is a face of $X_{G,V}$ if and only if σ is not adjacent to every $w \in W$.

Recall that the Alexander dual of a simplicial complex X with ground set V is a simplicial complex X^* whose faces are the complements of the nonfaces of X. The combinatorial Alexander duality (see for example [13]) relates the homology and cohomology of a given simplicial complex and its Alexander dual:

Theorem 4.10. *Let X be a simplicial complex with a ground set of size n. Then the reduced homology of X in degree i is equal to the reduced cohomology of the dual complex X^* in degree $n - i - 3$.*

As Lando graphs are bipartite, these two results together with the fact that a simplicial complex X and its suspension $\mathrm{S}(X)$ have the same reduced homology and cohomology with the indices shifted by one, provide an algorithm for computing the cohomology of the independence complex associated to a Lando graph G_D (or equivalently, the extreme Khovanov cohomology of the link represented by D) from the homology of a specific simplicial complex.

Theorem 4.11. *Let D be a diagram of an oriented link L with n negative crossings. Let $j = j_{\min}(D)$. Let $Y_D = (X_{G,V})^*$, where $G = G_D$ is the Lando graph of D, with parts V and W. Then*

$$H^{i,j}(L) \approx \widetilde{H}_{|V|-i-1-n}(Y_D).$$

Proof. Let $Z = X_{G,V}$ hence $Y_D = Z^*$. Then

$$H^{i+1-n,j}(L) \approx \widetilde{H}^i(X_D) \approx \widetilde{H}^i(\mathrm{S}(Z)) \approx \widetilde{H}^{i-1}(Z) \approx \widetilde{H}_{|V|-i-2}(Y_D),$$

where we have applied Theorem 4.4 (recall that $X_D = X_G$ is the independence complex of the Lando graph $G = G_D$), the homotopy equivalence $X_G \approx S(X_{G,V})$ given by Theorem 4.9, the relation between the reduced cohomology of a simplicial complex and its suspension, and finally the combinatorial Alexander duality theorem. □

The complex Y_D can also be described in terms of $s_A D$, avoiding any reference to the Lando graph G_D. This description will be useful in Section 4.6, more precisely in Theorem 4.16. Start by coloring the regions of $s_A D$ in a chess fashion. Call an A-chord white (black) if it is in a white (black) region. The ground set of Y_D is the set of admissible white arcs of $s_A D$, and a set of admissible white arcs

σ is a simplex of Y_D if and only if for any admissible black arc there is at least an admissible white arc which is not in σ whose ends alternate with the ends of the black arc in the same circle of $s_A D$. Note that there are two different choices when coloring the regions; in order to get the simplest ground set of Y_D, choose colors in such a way that white regions contain a lower number of admissible A-chords than black regions.

We are now interested in reversing the process above, namely, starting with any simplicial complex, we will construct a bipartite graph with an associated independence simplicial complex whose cohomology is equal to the homology of the original simplicial complex shifted by some degree. Again, the key point is the Alexander duality theorem together with the following result by Jonsson:

Theorem 4.12. [24, Theorem 3.2] *Let X be a simplicial complex. Then there is a bipartite graph G whose independence complex is homotopy equivalent to the suspension of X.*

The bipartite graph G can be constructed taking as set of vertices the disjoint union of the ground set V of the complex X, and the set M of maximal faces of X. The edges of G are all pairs $\{v, \mu\}$ such that $v \in V$, $\mu \in M$ and $v \notin \mu$.

We want to remark that, although Theorem 4.12 holds for any simplicial complex, the graph obtained by the above procedure is not necessarily the Lando graph associated to a link diagram. A graph G is said to be realizable if there is a link diagram D such that $G = G_D$ (in [34] these graphs were originally called convertible).

In the next theorem we show a realizable graph whose associated independence complex has two non-trivial cohomology groups, by starting with a simplicial complex whose homology has the same property. As the graph is realizable, Theorem 4.4 implies that there exists a link L whose extreme Khovanov cohomology $H^{i,j\min}$ is non-trivial for two different values of i. In the following section this example will be a basic piece to obtain interesting families of H-thick knots.

Theorem 4.13. *There exist oriented link diagrams whose extreme Khovanov cohomology modules are non-trivial for two different values of i, that is, $H^{i,j\min}(D)$ is non-trivial for two different values of i.*

Proof. Although our argument is equally valid for any commutative ring R with unit, just for convenience set $R = \mathbb{Z}$, the ring of integers.

Let $X = \{\emptyset, 1, 2, 3, 4, 5, 12, 23, 34, 41\}$ be a simplicial complex with ground set $V = \{1, 2, 3, 4, 5\}$. Its topological realization is the disjoint union of a point and a square, as shown in Figure 4.9, hence its reduced homology is $\widetilde{H}_0(X) \approx \widetilde{H}_1(X) \approx \mathbb{Z}$ (the other homology groups being trivial).
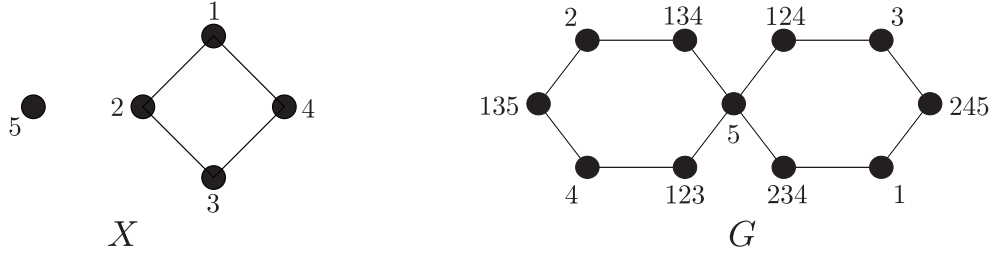
Figure 4.9: The topological realization of the simplicial complex X and the graph G. The independence complex of G is homotopy equivalent to the suspension of X^*.
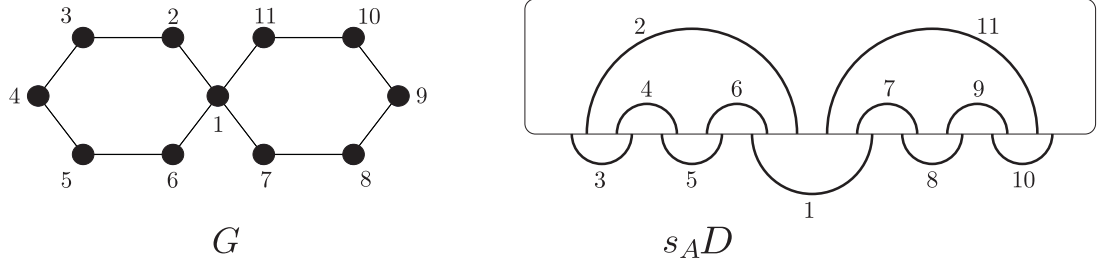

Figure 4.10: The graph G is realizable, the correspondence between its vertices and the A-chords in $s_A D$ is shown.

Consider now the Alexander dual of X, $X^* = \{\emptyset, 1, 2, 3, 4, 5, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45, 123, 124, 134, 135, 234, 245\}$. Note that $|X| + |X^*| = 32 = 2^{|V|}$. Applying the combinatorial Alexander duality leads to

$$\widetilde{H}_i(X) \approx \widetilde{H}^{|V|-i-3}(X^*) = \widetilde{H}^{2-i}(X^*),$$

which implies that $\widetilde{H}^2(X^*) \approx \widetilde{H}^1(X^*) \approx \mathbb{Z}$, the other groups being trivial.

Applying Theorem 4.12 (and the construction described right after) to the simplicial complex X^* leads to a graph G consisting in two hexagons sharing a common vertex as shown in Figure 4.9, whose independence complex X_G is homotopy equivalent to the suspension of X^*. In particular,

$$\widetilde{H}^{i-1}(X^*) \approx \widetilde{H}^i(S(X^*)) \approx \widetilde{H}^i(X_G).$$

Hence, $\widetilde{H}^3(X_G) \approx \widetilde{H}^2(X_G) \approx \mathbb{Z}$ are the only non-trivial groups in the reduced cohomology of X_G. In fact, as the indices are different from zero, this is still true for the (non-reduced) cohomology, so $H^2(X_G) \approx H^3(X_G) \approx \mathbb{Z}$.

An important point now is the fact that the graph G is realizable. In fact, $G = G_D$ with D being the link diagram in Figure 4.11. Indeed, Figure 4.10 shows the correspondence between the vertices of G and the A-chords in $s_A D$.

Consider now the link L represented by the diagram D oriented as shown in Figure 4.11. Then applying Corollary 4.2 we get that $j_{\min} = c - 3n - |s_A| = 11 - 3 \cdot 3 - 1 = 1$, and by Theorem 4.4 one gets $H^{i,1}(L) \approx \mathbb{Z}$ for $i = 0, 1$, the other cohomology groups being trivial. This concludes the proof. $\qquad\square$
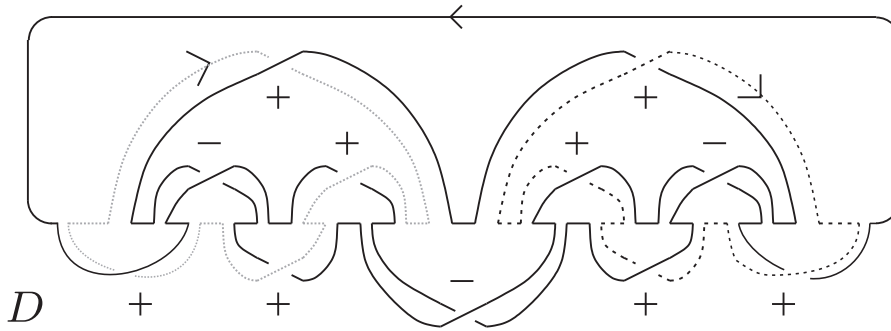
Figure 4.11: The oriented diagram D representing the link L. Each of its three components has been drawn in a different way.

For the example in the previous proof we have checked with computer assistance that the ranks of the chain groups $C^i = C^i(X_D)$ and differentials δ_i are

$$1(1)11(10)43(33)73(39)52(12)13(1)1,$$

where the rank of the differentials are parenthesized. This means that the Lando ascendant complex is

$$0 \longrightarrow C^{-1} \xrightarrow{\delta_{-1}} C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} C^2 \xrightarrow{\delta_2} C^3 \xrightarrow{\delta_3} C^4 \xrightarrow{\delta_4} C^5 \xrightarrow{\delta_5} 0,$$

with $\mathrm{rk}(C^{-1}) = 1$, $\mathrm{rk}(\delta_{-1}) = 1$, $\mathrm{rk}(C^0) = 11$, $\mathrm{rk}(\delta_0) = 10$ and so on. Hence

$$\mathrm{rk}(H^2(X_D)) = 73 - 39 - 33 = 1 \quad \text{and} \quad \mathrm{rk}(H^3(X_D)) = 52 - 12 - 39 = 1.$$

Remark 4.14. *The proof of Theorem 4.13 does not work if, for example, we start with the simplicial complex whose topological realization is a point plus a triangle. Although one gets again a graph G such that X_G has two non-trivial cohomology groups, G consists of two hexagons with four common consecutive edges (a total of eight vertices), which is no longer a realizable graph.*

4.6   Families of H-thick knots

Citing Khovanov [29], there are 249 prime unoriented knots with at most 10 crossings (not counting mirror images). It is known that for all but 12 of these knots the non-trivial Khovanov cohomology groups lie on two adjacent diagonals, in a matrix where rows are indexed by j and columns by i. Such knots are called H-thin. An H-thick knot is a knot which is not H-thin. For example, any non-split alternating link is H-thin, and in the opposite direction, any adequate non-alternating knot is H-thick (see [29], Theorem 2.1 and Proposition 5.1).

Up to eleven crossings, there are no knots with more that one non-trivial cohomology group in the rows corresponding to the hypothetical extreme $j_{\max}$ or $j_{\min}$ obtained from the associated diagrams in [9]. There are examples which

seem to contradict this statement. For example knot 10_{132}, whose Khovanov cohomology groups are trivial for $j > -1$ and has two non-trivial groups for $j = -1$, but for the diagram of 10_{132} taken from [9] $j_{\max}(D) = -c + 3p + |s_B| = -10 + 3 \cdot 3 + 2 = 1$. We do not know if there exists a diagram D representing the knot 10_{132} with $j_{\max}(D) = -1$.

In this section we show examples of H-thick knots having any arbitrary number of non-trivial cohomology groups in the non-trivial row of smallest possible index. More precisely, we will provide a diagram D whose row indexed by $j_{\min}(D)$ is non-trivial, and hence corresponds to the non-trivial row of smallest possible index. Moreover, this row has as many non-trivial cohomology groups as desired. The basic piece in our construction is the link given in the proof of Theorem 4.13.

We want to remark that Theorem 4.4 allows us to compute the extreme Khovanov cohomology of any link diagram D by considering independently each of the circles appearing in $s_A D$, as the non-admissible $A-$chords do not take part in the construction of the simplicial complex Y_D described in Section 4.5. More precisely, let D be a link diagram and $c_1, \ldots c_n$ the circles of $s_A D$. Write C_i for the circle c_i together with the admissible $A-$chords having both ends in the circle c_i, and let D_i be the diagram reconstructed from C_i by reversing the corresponding smoothings. Then, from the construction right after Theorem 4.11 it follows that $Y_D = Y_{D_1} * \ldots * Y_{D_n}$, with $*$ being the join of simplicial complexes. Recall that the join $X * Y$ of two simplicial complexes X and Y is defined as the simplicial complex whose simplices are the disjoint unions of simplices of X and Y.

The reduced homology of the join of two simplicial complexes can be computed directly from the reduced homology of each of the complexes, namely

$$\widetilde{H}_i(X * Y) = \sum_{r+s=i-1} \widetilde{H}_r(X) \otimes \widetilde{H}_s(Y) \oplus \sum_{r+s=i-2} \mathrm{Tor}(\widetilde{H}_r(X), \widetilde{H}_s(Y)).$$

Taking copies of the example in the proof of Theorem 4.13 one obtains a link which, by Theorem 4.11 and the above formula, has any number of non-trivial extreme Khovanov cohomology groups. Although one could also use the general formula for the Khovanov cohomology of a split link ([28, Corollary 12]), we think that our techniques are more useful in order to make computations. Even more, our understanding of extreme Khovanov cohomology in terms of Lando cohomology allows us to slightly modify a link in such a way that one obtains a knot with the same extreme Khovanov cohomology. We explain this construction in detail in Theorem 4.16 and Remark 4.17. We need first the following result:

Proposition 4.15. *Let $*_n X$ be the join of n copies of the simplicial complex $X = \{\emptyset, 1, 2, 3, 4, 5, 12, 23, 34, 41\}$. Then $\widetilde{H}_i(*_n X) \approx \mathbb{Z}^{\binom{n}{i-n+1}}$ if $n-1 \le i \le 2n-1$, and it is trivial otherwise.*

Proof. By induction on n. In the proof of Theorem 4.13 we saw that $\widetilde{H}_0(X) \approx \widetilde{H}_1(X) \approx \mathbb{Z}$, which is the case $n = 1$. For $n > 1$ we apply the formula for the

homology of a join (torsion terms do not appear in any case):

$$
\begin{aligned}
\widetilde{H}_i(*_n X) \;&\approx\; \bigoplus_{r+s=i-1}\left[\widetilde{H}_r(*_{n-1}X)\otimes\widetilde{H}_s(X)\right]\\[2mm]
&\approx\; \widetilde{H}_{i-1}(*_{n-1}X)\bigoplus\widetilde{H}_{i-2}(*_{n-1}X)\\[2mm]
&\approx\; \mathbb{Z}^{\left(\binom{n-1}{(i-1)-(n-1)+1}\right)}\bigoplus\mathbb{Z}^{\left(\binom{n-1}{(i-2)-(n-1)+1}\right)}\\[2mm]
&\approx\; \mathbb{Z}^{\left(\binom{n-1}{i-n+1}\right)}\bigoplus\mathbb{Z}^{\left(\binom{n-1}{i-n}\right)}\\[2mm]
&\approx\; \mathbb{Z}^{\left(\binom{n}{i-n+1}\right)}.
\end{aligned}
$$

$\square$

Theorem 4.16. *For every $n > 0$ there exists an oriented knot diagram D with exactly $n+1$ non-trivial extreme Khovanov integer cohomology groups $H^{i,j_{\min}}(D)$.*

Proof. Let L be the oriented link represented by the diagram D in Figure 4.11. Considering as ground set the chords in the unbounded region of $s_A D$, the associated simplicial complex Y_D is the simplicial complex X appearing in the proof of Theorem 4.13, whose topological realization is the disjoint union of a point and a square (Figure 4.9). Hence it has two non-trivial reduced homology groups, $\widetilde{H}_0(Y_D)\approx\widetilde{H}_1(Y_D)\approx\mathbb{Z}$.

Now consider the link L_n consisting of the split union of n copies of L. It can be represented by D_n, the disjoint union of n copies of D, so $s_A D_n$ is the disjoint union of n copies of $s_A D$, shown in Figure 4.10. Hence its associated simplicial complex is $Y_{D_n} = *_n Y_D = *_n X$, that is, the join of n copies of X. Applying Proposition 4.15 to $*_n X$ one gets

$$
\widetilde{H}_i(Y_{D_n})\approx\mathbb{Z}^{\left(\binom{n}{i-n+1}\right)}
$$

for $n-1\le i\le 2n-1$.

This fact together with Theorem 4.11 shows that the extreme Khovanov cohomology of L_n has $n+1$ non-trivial groups.

Now we will construct a knot having the same extreme Khovanov cohomology groups as L_n (the value of $j_{\min}$ changes in general). Starting from the diagram D in Figure 4.11, add four crossings, as shown in Figure 4.12, in such a way that the resulting diagram D' has one component. Note that $s_A D'$ is obtained from $s_A D$ by adding two circles with four A-chords. Consider now n copies of D' and join them as shown in Figure 4.13. The resulting diagram D'_n is a knot diagram. Since $s_A D'_n$ just adds $5n-1$ non-admissible A-chords to $s_A D_n$, both diagrams D_n and D'_n share the same Lando graph. Hence D'_n represents a knot having $n+1$ non-trivial groups in its extreme Khovanov cohomology. $\square$
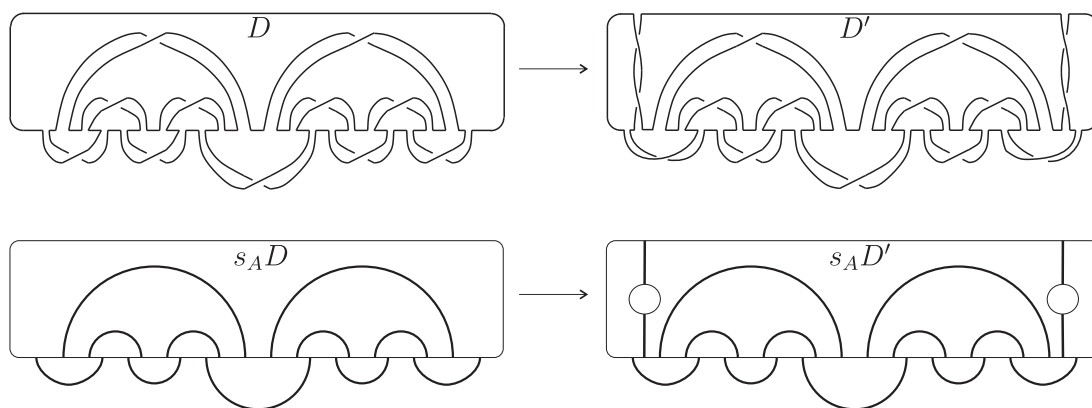
Figure 4.12: The first row shows D and D'. The corresponding $s_A D$ and $s_A D'$ are shown in the second row. Note that D' has one component.
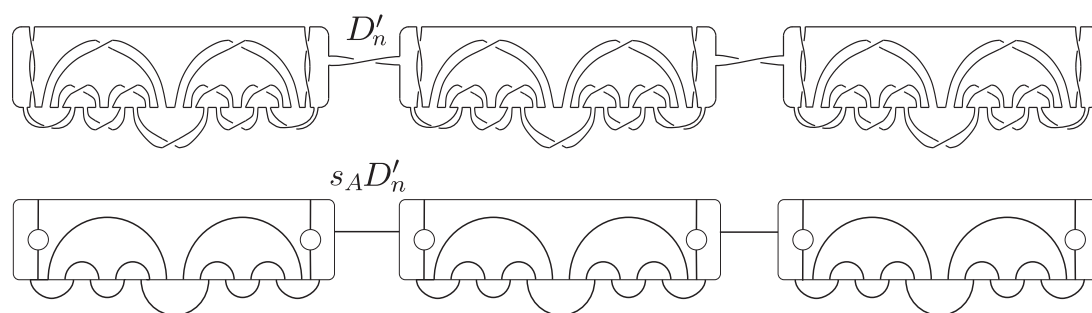
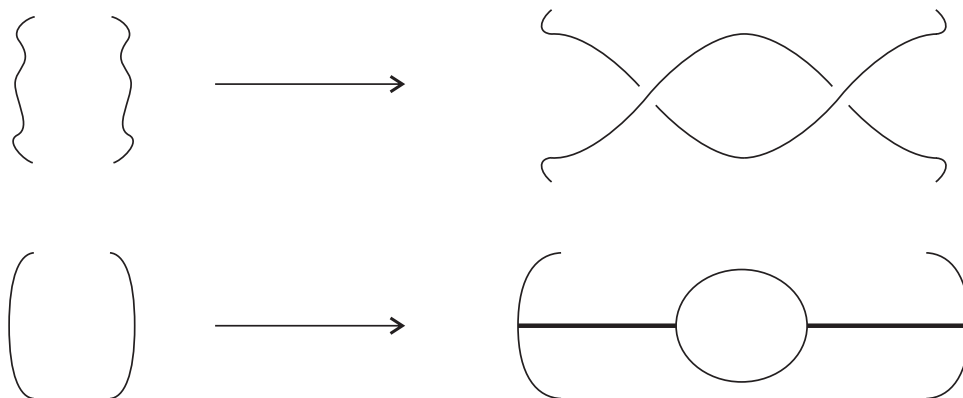Figure 4.13: D'_n and $s_A D'_n$ are shown for the case $n = 3$.

Figure 4.14: This transformation reduces the number of components of the link by one, and the extreme Khovanov cohomology is preserved.

The proof of Theorem 4.16 provides a family of knots having as many non-trivial extreme Khovanov cohomology groups as desired. These are examples of H-thick knots as far of being H-thin as desired.

Remark 4.17. *Every link L with μ components can be turned into a knot preserving its extreme Khovanov cohomology (j_{min} can change). One just needs to consider a diagram D of L and add two extra crossings melting two different components into one, as shown in Figure 4.14. Since $s_A D'$ just adds two non-admissible A-chords to $s_A D$, both diagrams share the same Lando graph. After repeating this procedure $\mu - 1$ times, the link L is transformed into a knot.*

The approach to extreme Khovanov cohomology given in Theorem 4.4 leads some open questions. A first problem is studying the realization of other degrees (not just the extreme ones) of Khovanov cohomology by using similar techniques, replacing the Lando graph, which is related to the state where all the crossings are labeled with an A-label, with a graph related to states having all but one crossings marked with an A-label.

An open conjecture states that the extreme Khovanov cohomology of a link diagram is homotopy equivalent to a wedge of spheres. Theorem 4.4 leads to a new approach to study this conjecture. More precisely, this conjecture can be restated in the following way

Conjecture 4.18. *The independence simplicial complex of the Lando graph of any link diagram is homotopy equivalent to a wedge of spheres.*

In this sense, giving some properties of even characterizing the possible Lando graphs which can be obtained from any link diagram seems to be an interesting problem.

A more specific question would be whether torsion can appear in the extreme Khovanov cohomology of oriented links. The techniques used in the proof of Theorem 4.13 could be useful when considering this problem. A negative answer to this question would be a supporting argument for the above conjecture.

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