Global in time solution and time-periodicity for a smectic-A liquid crystal model

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Abstract

In this paper some results are obtained for a smectic-A liquid crystal model with time-dependent boundary Dirichlet data for the so-called layer variable φ (the level sets of φ describe the layer structure of the smectic-A liquid crystal). First, the initial-boundary problem for arbitrary initial data is considered, obtaining the existence of weak solutions which are bounded up to infinity time. Second, the existence of time-periodic weak solutions is proved. Afterwards, the problem of the global in time regularity is attacked, obtaining the existence and uniqueness of regular solutions (up to infinity time) for both problems, i.e. the initial-valued problem and the time-periodic one, but assuming a dominant viscosity coefficient in the linear part of the diffusion tensor.

Keywords: solution bounded up to infinity time, time-periodic solutions, global in time regular solutions, Navier-Stokes equations, Smectic-A liquid crystal, coupled non-linear parabolic system.

1 Introduction

In this work, we study the time evolution of a smectic-A liquid crystal model proposed in [E'97]. Smectic crystals are in a liquid-crystalline phase, where the molecules of the liquid crystal not only have a certain orientational order (as in the nematic case) but also have a

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certain positional order (layer structure); the molecules are arranged in almost incompressible layers of almost constant width. Within each layer, the smectic-A system consists of a single optical axis $\boldsymbol{n}$ perpendicular to the layer such that $\nabla \times \boldsymbol{n} = 0$. In this case, $\boldsymbol{n} = \nabla\varphi$ for a potential function φ, and the level sets of φ will represent the layer structure in the sample.

This study is motivated by the following problem in liquid crystals. The usual nematic molecule configuration is determined by minimizing the *Oseen Frank energy*, which in the more simple case of equal constants derives to *Dirichlet energy* $\displaystyle\int_\Omega |\nabla \boldsymbol{d}|^2$. Here, the unit vector $\boldsymbol{d}$ stands for the orientation of liquid crystal molecules.

Now, in the smectic-A case, this orientation $\boldsymbol{d}$ coincides with the normal vector $\boldsymbol{n}$ of each layer. Then, in order to study the energy $\displaystyle\int_\Omega |\nabla(\nabla\varphi)|^2$ under the constraint $|\nabla\varphi| = 1$, it is natural to introduce the penalized energy

$$\int_\Omega \frac{1}{2}|\Delta\varphi|^2 + \boldsymbol{f}(\nabla\varphi)$$

where $\boldsymbol{f}$ is the Ginzburg-Landau penalization function

$$\boldsymbol{f}(\boldsymbol{n}) = \frac{1}{\varepsilon^2}(|\boldsymbol{n}|^2 - 1)\boldsymbol{n},$$

which has the potential function

$$F(\boldsymbol{n}) = \frac{1}{4\varepsilon^2}(|\boldsymbol{n}|^2 - 1)^2$$

verifying $\boldsymbol{f}(\boldsymbol{n}) = \nabla_{\boldsymbol{n}} F(\boldsymbol{n})$ for each $\boldsymbol{n} \in \mathbb{R}^N$. As $\varepsilon \to 0$, one can hope that the minimizer of the penalized energy, or the solution of the corresponding Euler-Lagrange equation

$$\Delta^2\varphi - \nabla\cdot\boldsymbol{f}(\nabla\varphi) = 0,$$

will be convergent to the minimizer of the energy $\int_\Omega \frac{1}{2}|\Delta\varphi|^2$ with the non-convex constraint $|\nabla\varphi| = 1$ (cf. [Kinderlehrer,Liu'96] and [E'97]). Thus, it is important to study the asymptotic behavior as $\varepsilon \to 0$ (cf. [Guillén,Rojas'02] for nematic crystal models). However, very little is known about this.

We assume the smectic-A liquid crystal confined in an open bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3) with regular boundary $\partial\Omega$. We consider the following PDE system in $\Omega \times (0, +\infty)$:

$$\begin{cases} \rho(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) - \nabla\cdot(\sigma^d + \lambda\,\sigma^e) + \nabla p = 0, \quad \nabla\cdot\boldsymbol{u} = 0, \\ \partial_t\varphi + \boldsymbol{u}\cdot\nabla\varphi + \gamma(\Delta^2\varphi - \nabla\cdot\boldsymbol{f}(\nabla\varphi)) = 0, \end{cases} \tag{1}$$

where $\boldsymbol{u} : \Omega \times [0, +\infty) \mapsto \mathbb{R}^N$ is the flow velocity, $p : \Omega \times [0, +\infty) \mapsto \mathbb{R}$ describes the fluid pressure and $\varphi : \Omega \times [0, +\infty) \mapsto \mathbb{R}$ is the layer variable. The constants ρ, λ, and γ are positive, representing respectively, the density of the fluid, the ratio between the kinetic energy and the elastic one, and the elastic relaxation time. Moreover, we consider the same constitutive laws for the dissipative stress tensor σ^d and the elastic stress tensor σ^e as in [Liu'00]:

$$\sigma^d = \mu_1(\boldsymbol{n}^t D(\boldsymbol{u})\boldsymbol{n})\boldsymbol{n}\otimes\boldsymbol{n} + \mu_4 D(\boldsymbol{u}) + \mu_5(D(\boldsymbol{u})\boldsymbol{n}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes D(\boldsymbol{u})\boldsymbol{n}),$$

$$\sigma^e = -\boldsymbol{f}(\boldsymbol{n})\otimes\boldsymbol{n} + \nabla(\nabla\cdot\boldsymbol{n})\otimes\boldsymbol{n} - (\nabla\cdot\boldsymbol{n})\nabla\boldsymbol{n}$$

where $\mu_1 \geq 0$, $\mu_4 > 0, \mu_5 \geq 0$ are dissipative constant coefficients, $\boldsymbol{n} = \nabla\varphi$ and $D(\boldsymbol{u})$ denotes the symmetric tensor of the velocity gradient: $D(\boldsymbol{u}) = \dfrac{1}{2}(\nabla\boldsymbol{u} + \nabla^t\boldsymbol{u})$.

The problem (1) is completed with the (Dirichlet) boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = 0, \quad \varphi|_{\partial\Omega} = \varphi_1, \quad \partial_{\mathrm{n}}\varphi|_{\partial\Omega} = \varphi_2 \tag{2}$$

(assuming time-depending boundary data $\varphi_1, \varphi_2 : \partial\Omega \times (0, +\infty) \mapsto \mathbb{R}^N$) and one of the following conditions:

- either the initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \varphi(0) = \varphi_0 \quad \text{in } \Omega \tag{3}$$

- or the time-periodic conditions:

$$\boldsymbol{u}(0) = \boldsymbol{u}(T), \qquad \varphi(0) = \varphi(T) \quad \text{in } \Omega, \tag{4}$$

  where $T > 0$ is a given final time.

In the first case, the compatibility condition $\varphi_0|_{\partial\Omega} = \varphi_1(0)$ must be assumed. In this last case, one assumes $\varphi_1(0) = \varphi_1(T)$ and $\varphi_2(0) = \varphi_2(T)$.

By splitting the symmetric dissipative tensor into the linear and nonlinear part

$$\sigma^d = \mu_4 D(\boldsymbol{u}) + \sigma^d_{nl}(D(\boldsymbol{u}), \nabla\varphi),$$

where $\sigma^d_{nl} := \mu_1(\boldsymbol{n}^t D(\boldsymbol{u})\boldsymbol{n})\boldsymbol{n}\otimes\boldsymbol{n} + \mu_5(D(\boldsymbol{u})\boldsymbol{n}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes D(\boldsymbol{u})\boldsymbol{n})$, notice that

$$-\nabla\cdot\sigma^d = -\mu_4\nabla\cdot D(\boldsymbol{u}) - \nabla\cdot\sigma^d_{nl} = -\frac{\mu_4}{2}\Delta\boldsymbol{u} - \nabla\cdot\sigma^d_{nl}$$

since $\nabla\cdot\boldsymbol{u} = 0$. By decomposing the term due to the penalization to the other terms in the elastic tensor as follows

$$\sigma^e = -\boldsymbol{f}(\boldsymbol{n})\otimes\boldsymbol{n} + \sigma^e_{np}(\boldsymbol{n}),$$

where $\sigma^e_{np} := \nabla(\nabla\cdot\boldsymbol{n})\otimes\boldsymbol{n} - (\nabla\cdot\boldsymbol{n})\nabla\boldsymbol{n}$ is the non-penalized tensor, and taking into account that

$$\nabla\cdot(\boldsymbol{f}(\boldsymbol{n})\otimes\boldsymbol{n}) = (\nabla\cdot\boldsymbol{f}(\nabla\varphi))\nabla\varphi + \boldsymbol{f}_i(\nabla\varphi)\partial_i\nabla\varphi = (\nabla\cdot\boldsymbol{f}(\nabla\varphi))\nabla\varphi + \nabla F(\nabla\varphi)$$

and

$$
\begin{aligned}
(\nabla\cdot\sigma^e_{np})_j &= (\nabla\cdot(\nabla(\nabla\cdot\boldsymbol{n})\otimes\boldsymbol{n}-(\nabla\cdot\boldsymbol{n})\nabla\boldsymbol{n}))_j = (\nabla\cdot(\nabla(\Delta\varphi)\otimes\nabla\varphi-\Delta\varphi\nabla^2\varphi))_j\\
&= \partial_i(\partial_i(\Delta\varphi)\partial_j\varphi-\Delta\varphi\partial_{ij}\varphi) = \Delta^2\varphi\partial_j\varphi+\partial_i(\Delta\varphi)\partial_i\partial_j\varphi-\partial_i(\Delta\varphi)\partial_{ij}\varphi-\Delta\varphi\partial_i\partial_{ij}\varphi\\
&= \Delta^2\varphi\partial_j\varphi-\Delta\varphi\partial_j\Delta\varphi = \Delta^2\varphi\partial_j\varphi-\frac{1}{2}\partial_j(|\Delta\varphi|^2),
\end{aligned}
$$

we have

$$
-\nabla\cdot\sigma^e = (\nabla\cdot\boldsymbol{f}(\nabla\varphi))\nabla\varphi+\nabla F(\nabla\varphi)-\Delta^2\varphi\nabla\varphi+\nabla\left(\frac{|\Delta\varphi|^2}{2}\right).
$$

Then, joining together all the gradient terms, the momentum system of (1) can be written as:

$$
\rho(\partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u})-\frac{\mu_4}{2}\Delta\boldsymbol{u}-\nabla\cdot\sigma^d_{nl}-\lambda(\Delta^2\varphi-\nabla\cdot\boldsymbol{f}(\nabla\varphi))\nabla\varphi+\nabla q=0 \tag{5}
$$

where q is the potential function $q=p+\lambda F(\nabla\varphi)+\lambda\frac{|\Delta\varphi|^2}{2}$.

One observes that the liquid crystal model (1)-(2) lacks of maximum or comparison principles (for $\nabla\varphi$) so one looses one of the strongest tools in analyzing nonlinear pdes.

Assuming time-independent boundary data φ_1,φ_2, an important fact of the model (1)-(2) is its dissipative character, because this system admits (at least formally) the following energy equality:

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(\frac{1}{2}|\mathbf{u}|^2+\lambda\left(\frac{1}{2}|\Delta\varphi|^2+F(\nabla\varphi)\right)\right)\\
&+\int_\Omega\left(\frac{\mu_4}{2}|\nabla\mathbf{u}|^2+\sigma^d_{nl}:D(\boldsymbol{u})+\lambda\gamma|\Delta^2\varphi-\nabla\cdot\boldsymbol{f}(\nabla\varphi)|^2\right)=0.
\end{aligned} \tag{6}
$$

This equality is obtained multiplying the φ-equation by $-\lambda(\Delta^2\varphi-\nabla\cdot\boldsymbol{f}(\nabla\varphi))$, the $\mathbf{u}$-system (5) by $\mathbf{u}$ and integrating by parts, because all the nonlinear convective and elastic terms cancel and the boundary terms vanish by using that $\boldsymbol{u}|_{\partial\Omega}=0$, $\partial_t\varphi|_{\partial\Omega}=0$ and $\partial_t\partial_n\varphi|_{\partial\Omega}=0$ (see (20) below for an energy equality related to a system with time-dependent boundary data). In particular, since $\int_\Omega\sigma^d_{nl}:D(\boldsymbol{u})\geq 0$ (see (22) below), this equality implies that the total energy (that is, the kinetic energy $\frac{1}{2}\int_\Omega|\mathbf{u}|^2$ plus the elastic energy $\lambda\int_\Omega\frac{1}{2}|\Delta\varphi|^2+F(\nabla\varphi)$) decreases respect to the time. Now, since time-dependent boundary data φ_1,φ_2 will be considered, (6) must be modified with a right hand side depending on time derivatives of φ_1,φ_2 which act as force terms, see (20).

If we consider again time-independent boundary data, it is important to remark that the following (static) critical points are particular solutions of the time-periodic problem:

$$
\begin{aligned}
&\boldsymbol{u}=0,\\
&\varphi:\ \text{any solution of the problem: } \Delta^2\varphi-\nabla\cdot\boldsymbol{f}(\nabla\varphi)=0 \text{ in } \Omega,\ \varphi=\varphi_1,\ \partial_{\rm n}\varphi=\varphi_2 \text{ on } \partial\Omega,\\
&p=-\lambda F(\nabla\varphi)-\lambda\frac{|\Delta\varphi|^2}{2}.
\end{aligned}
$$

Therefore, in order to consider a nontrivial time-periodic problem, it will be essential to assume time-dependant boundary data for φ.

Definition 1 *We say that* $(\boldsymbol{u},\varphi)$ *is a weak solution of* (1)-(3) *in* $[0,T)$, $0<T<+\infty$ *if*

$$\nabla\cdot\boldsymbol{u}=0 \ \textit{in}\ Q,\quad \boldsymbol{u}|_\Sigma=0,\quad \varphi|_\Sigma=\varphi_1,\quad \partial_{\rm n}\varphi|_\Sigma=\varphi_2$$

$$\boldsymbol{u}\in L^\infty(0,T;\boldsymbol{L}^2(\Omega))\cap L^2(0,T;\boldsymbol{H}^1(\Omega)),\qquad \varphi\in L^\infty(0,T;\boldsymbol{H}^2(\Omega))\cap L^2(0,T;\boldsymbol{H}^4(\Omega)),\tag{7}$$

verifying

$$\langle\partial_t\boldsymbol{u},\boldsymbol{v}\rangle+((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{v})+((\mu_4/2)\nabla\boldsymbol{u}+\sigma^d_{nl},\nabla\boldsymbol{v})-\lambda((\Delta^2\varphi-\nabla\cdot\boldsymbol{f}(\nabla\varphi))\nabla\varphi,\boldsymbol{v})=0\quad\forall\,\boldsymbol{v}\in\boldsymbol{V},$$
$$\partial_t\varphi+(\boldsymbol{u}\cdot\nabla)\varphi+\gamma(\Delta^2\varphi-\nabla\cdot\boldsymbol{f}(\nabla\varphi))=0,\quad \textit{a.e. in}\ Q$$
$$\boldsymbol{u}(0)=\boldsymbol{u}_0,\quad \varphi(0)=\varphi_0\quad \textit{in}\ \Omega.$$

Definition 2 *We say that a weak solution* $(\boldsymbol{u},\varphi)$ *is a strong solution of* (1)-(3) *in* $[0,T)$ *if*

$$\boldsymbol{u}\in L^\infty(0,T;\boldsymbol{H}^1(\Omega))\cap L^2(0,T;\boldsymbol{H}^2(\Omega)),\qquad \varphi\in L^\infty(0,T;\boldsymbol{H}^4(\Omega))\cap L^2(0,T;\boldsymbol{H}^6(\Omega)),\tag{8}$$

verifying point-wise the fully differential system (1)*.*

In [Liu'00], considering a problem like (1)-(3), with variable density and time-independent boundary conditions for φ, author proves the existence of weak solutions using a semi-Galerkin procedure keeping the transport equation for density and the φ-equation at infinity dimension. Moreover, the global regularity of weak solutions (for big enough μ_4 if $N=3$) is deduced in [Liu'00], and a preliminary analysis about the asymptotic behavior in time is made (see also [Lai,Liu'06] for other asymptotic behavior study for a related model).

The main results of present paper are the following, always for boundary data φ_1 and φ_2 depending on the time:

1. the uniqueness of weak/strong solutions of the initial-value problem (1)-(3),
2. the existence of global weak solutions of problem (1)-(3), which is bounded up to infinity time (with an exponential weighted norm for the $L^2(0,+\infty)$-norm, see (31)),
3. the existence of weak time-periodic solutions,
4. the existence of regular solutions for both previous cases, the initial-valued problem and the time-periodic one, but assuming a dominant viscosity coefficient μ_4 in the linear part of the diffusion tensor.

The results obtained in this paper are in a certain sense similar to the results presented in [Climent et al.'06] and [Climent et al.] for the weak solutions and the regular solutions of a nematic liquid crystal model, respectively.

The paper is organized as follows. In Section 2, some differential inequalities are deduced, which will be used in the rest of the paper. In Section 3, the uniqueness of weak/strong solutions of the initial-value problem (1)-(3) is analyzed. In Section 4, the global in time solution of the initial valued problem is studied at infinity time and the existence of weak time-periodic solutions is obtained in Section 5. Finally, under the constraints of viscosity coefficient μ_4 big enough, the existence and uniqueness of global regular solutions of the initial valued problem is proved in Section 6 and the existence of regular time-periodic solutions is deduced in Section 7.

For simplicity we fix the constants excepting the viscosity μ_4, taking

$$\rho = \lambda = \gamma = \mu_1 = \mu_5 = 1, \qquad \nu = \mu_4/2.$$

Notation

- We denote $Q = (0, +\infty) \times \Omega$, $Q_T = (0, T) \times \Omega$, $\Sigma = (0, +\infty) \times \partial\Omega$ and $\Sigma_T = (0, T) \times \partial\Omega$.

- In general, the notation will be abridged. We set $L^p = L^p(\Omega)$, $p \geq 1$, $H_0^1 = H_0^1(\Omega)$, etc. If $X = X(\Omega)$ is a space of functions defined in the open set Ω, we denote by $L^p(X)$ the Banach space $L^p(0, T; X)$. Also, boldface letters will be used for vectorial spaces, for instance $\mathbf{L}^2 = L^2(\Omega)^N$.

- The L^p norm is denoted by $|\cdot|_p$, $1 \leq p \leq \infty$, the H^m norm by $\|\cdot\|_m$ (in particular $|\cdot|_2 = \|\cdot\|_0$) and the product norm in $H^n \times H^m$ by $\|\cdot\|_{n\times m}$. The inner product of $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$.

- We set $\mathcal{V}$ the space formed by all fields $\boldsymbol{u} \in C_0^\infty(\Omega)^N$ satisfying $\nabla \cdot \boldsymbol{u} = 0$. We denote $\boldsymbol{H}$ (respectively $\boldsymbol{V}$) the closure of $\mathcal{V}$ in $\boldsymbol{L}^2$ (respectively $\boldsymbol{H}^1$). $\boldsymbol{H}$ and $\boldsymbol{V}$ are Hilbert spaces for the norms $|\cdot|_2$ and $\|\cdot\|_1$, respectively. Furthermore,

$$\boldsymbol{H} = \{\boldsymbol{u} \in \boldsymbol{L}^2;\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \qquad \boldsymbol{V} = \{\boldsymbol{u} \in \boldsymbol{H}^1;\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} = 0 \text{ on } \partial\Omega\}$$

- In the sequel, $C, C_1, C_2 > 0$ will denote different constants, depending only on the fixed data of the problem, as Ω, φ_1, φ_2, ε (and $\boldsymbol{u}_0, \varphi_0$ for the initial-value problem).

2 Preliminaries

2.1 A lifting function

We define $\widetilde{\varphi} = \widetilde{\varphi}(t)$ as the weak solution of the problem

$$\begin{cases} -\Delta^2 \widetilde{\varphi} = 0 & \text{in } \Omega, \\ \widetilde{\varphi} = \varphi_1(t) \quad \partial_{\rm n} \widetilde{\varphi} = \varphi_2(t) & \text{on } \partial\Omega. \end{cases} \tag{9}$$

In the time-periodic case, since by hypothesis $\varphi_1(0) = \varphi_1(T)$ and $\varphi_2(0) = \varphi_2(T)$ on $\partial\Omega$, then $\widetilde{\varphi}(0) = \widetilde{\varphi}(T)$ in Ω.

Therefore, if we define $\widehat{\varphi}(t) = \varphi(t) - \widetilde{\varphi}(t)$, then $\Delta^2\widehat{\varphi} = \Delta^2\varphi$ in Q and $\widehat{\varphi} = \nabla\widehat{\varphi} = 0$ on Σ. In the time-periodic case, one has $\varphi(0) = \varphi(T)$ if and only if $\widehat{\varphi}(0) = \widehat{\varphi}(T)$. Then, we can rewrite the problem (1)-(2) respect to the variables $(\boldsymbol{u}, \widehat{\varphi})$ (with $\widehat{\varphi}(t) = \varphi(t) - \widetilde{\varphi}(t)$) as follows (recall that all coefficients have been taken equal to one, excepting viscosity $\nu = \mu_4/2$):

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} - \nabla\cdot\sigma^d_{nl} - (\Delta^2\widehat{\varphi} - \nabla\cdot\boldsymbol{f}(\nabla\varphi))\nabla\varphi + \nabla q = 0 \quad \text{in } Q_T, \\ \nabla\cdot\boldsymbol{u} = 0 \quad \text{in } Q_T, \\ \partial_t\widehat{\varphi} + \boldsymbol{u}\cdot\nabla\varphi + \Delta^2\widehat{\varphi} - \nabla\cdot\boldsymbol{f}(\nabla\varphi) = \partial_t\widetilde{\varphi} \quad \text{in } Q_T, \\ \boldsymbol{u} = 0, \quad \widehat{\varphi} = 0, \quad \partial_n\widehat{\varphi} = 0 \quad \text{on } \Sigma_T \end{cases} \tag{10}$$

jointly with either the initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\widehat{\varphi}(0) = \varphi_0 - \widetilde{\varphi}(0)$ or the time-periodic conditions $\boldsymbol{u}(0) = \boldsymbol{u}(T)$, $\widehat{\varphi}(0) = \widehat{\varphi}(T)$.

Since $\boldsymbol{u} \in \boldsymbol{H}^1_0$ and $\widehat{\varphi} \in \boldsymbol{H}^2_0$, the following norms are equivalents:

$$\|\boldsymbol{u}\|_1 \approx |\nabla\boldsymbol{u}|_2, \qquad \|\widehat{\varphi}\|_2 \approx |\Delta\widehat{\varphi}|_2 \qquad \|\widehat{\varphi}\|_4 \approx |\Delta^2\widehat{\varphi}|_2.$$

2.2 Some inequalities

We will give two inequalities in the next two lemmas, relating the elliptic operator $\Delta^2\widehat{\varphi} - \nabla\cdot\boldsymbol{f}(\nabla\varphi)$ and the penalized energy $\displaystyle\int_\Omega F(\nabla\varphi)$ with some norms.

Lemma 3 *The following inequality holds:*

$$|\Delta\widehat{\varphi}|_2^2 + \frac{1}{2\varepsilon^2}|\nabla\widehat{\varphi}|_4^4 \leq \frac{1}{2}|\Delta^2\widehat{\varphi} - \nabla\cdot\boldsymbol{f}(\nabla\varphi)|_2^2 + C_1, \tag{11}$$

where $C_1 > 0$ is a constant depending on ε, $|\Omega|$, and $\|\nabla\widetilde{\varphi}\|_{L^\infty(L^4)}$.

Proof. We denote $\omega = \Delta^2\widehat{\varphi} - \nabla\cdot\boldsymbol{f}(\nabla\varphi)$. Testing this equality by $\widehat{\varphi}$ one has:

$$(\omega, \widehat{\varphi}) = (\Delta^2\widehat{\varphi}, \widehat{\varphi}) - (\nabla\cdot\boldsymbol{f}(\nabla\varphi), \widehat{\varphi}) = |\Delta\widehat{\varphi}|_2^2 + (\boldsymbol{f}(\nabla\varphi), \nabla\widehat{\varphi}). \tag{12}$$

The last term on the right hand side of (12) can be written as

$$\begin{aligned}(\boldsymbol{f}(\nabla\varphi),\nabla\widehat{\varphi}) &= (\boldsymbol{f}(\nabla\varphi)-\boldsymbol{f}(\nabla\widehat{\varphi}),\nabla\widehat{\varphi})+(\boldsymbol{f}(\nabla\widehat{\varphi}),\nabla\widehat{\varphi})\\ &= (\boldsymbol{f}(\nabla\varphi)-\boldsymbol{f}(\nabla\widehat{\varphi}),\nabla\widehat{\varphi})+\frac{1}{\varepsilon^2}|\nabla\widehat{\varphi}|_4^4-\frac{1}{\varepsilon^2}|\nabla\widehat{\varphi}|_2^2\end{aligned} \tag{13}$$

From (12) and (13), one has

$$|\Delta\widehat{\varphi}|_2^2+\frac{1}{\varepsilon^2}|\nabla\widehat{\varphi}|_4^4=(\omega,\widehat{\varphi})+\frac{1}{\varepsilon^2}|\nabla\widehat{\varphi}|_2^2-(\boldsymbol{f}(\nabla\varphi)-\boldsymbol{f}(\nabla\widehat{\varphi}),\nabla\widehat{\varphi}). \tag{14}$$

The first term on the right hand side of (14) can be bounded as

$$|(\omega,\widehat{\varphi})|\leq|\omega|_2|\widehat{\varphi}|_2\leq\frac{1}{2}|\omega|_2^2+\frac{C}{2}|\nabla\widehat{\varphi}|_4^2\leq\frac{1}{2}|\omega|_2^2+\frac{1}{6\varepsilon^2}|\nabla\widehat{\varphi}|_4^4+C\,\varepsilon^2 \tag{15}$$

where the constant C depends on ε and $|\Omega|$.

On the other hand, the second term on the right hand side of (14) will be bounded as:

$$\frac{1}{\varepsilon^2}|\nabla\widehat{\varphi}|_2^2\leq\frac{1}{6\varepsilon^2}|\nabla\widehat{\varphi}|_4^4+\frac{C}{\varepsilon^2}. \tag{16}$$

Now, we are going to bound the third term of (14). Taking into account that

$$\boldsymbol{f}(\boldsymbol{a})-\boldsymbol{f}(\boldsymbol{b})=\frac{1}{\varepsilon^2}\left(|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+\boldsymbol{a}\cdot\boldsymbol{b}-1\right)(\boldsymbol{a}-\boldsymbol{b})\quad\forall\,\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^N,$$

in particular

$$\boldsymbol{f}(\nabla\varphi)-\boldsymbol{f}(\nabla\widehat{\varphi})=\frac{1}{\varepsilon^2}\left(|\nabla\varphi|^2+|\nabla\widehat{\varphi}|^2+\nabla\varphi\cdot\nabla\widehat{\varphi}-1\right)\nabla\widetilde{\varphi}. \tag{17}$$

Consequently, by using that $\nabla\widetilde{\varphi}\in L^\infty(\boldsymbol{L}^4)$, the Hölder and Young's inequalities, the last term on the right hand side of (14) can be bounded as follows

$$\begin{aligned}|(\boldsymbol{f}(\nabla\varphi)-\boldsymbol{f}(\nabla\widehat{\varphi}),\nabla\widehat{\varphi})| &\leq \frac{1}{\varepsilon^2}||\nabla\varphi|^2+|\nabla\widehat{\varphi}|^2+\nabla\varphi\cdot\nabla\widehat{\varphi}-1|_2|\nabla\widetilde{\varphi}|_4|\nabla\widehat{\varphi}|_4\\ &\leq \frac{C}{\varepsilon^2}\Big(|\nabla\varphi|_4^2+|\nabla\widehat{\varphi}|_4^2+1\Big)|\nabla\widehat{\varphi}|_4\\ &\leq \frac{C}{\varepsilon^2}\Big(|\nabla\widehat{\varphi}|_4^2+1\Big)|\nabla\widehat{\varphi}|_4\leq\frac{1}{6\varepsilon^2}|\nabla\widehat{\varphi}|_4^4+\frac{C}{\varepsilon^2}.\end{aligned} \tag{18}$$

where C depends on $|\Omega|$ and $\|\nabla\widetilde{\varphi}\|_{L^\infty(L^4)}$.

Finally, from (14)-(18), the inequality (11) is deduced. ■

Lemma 4 *The following inequality holds:*

$$\int_\Omega F(\nabla\varphi)\leq\frac{1}{2\varepsilon^2}|\nabla\widehat{\varphi}|_4^4+\frac{C_2}{\varepsilon^2} \tag{19}$$

where C_2 *depends on* $|\Omega|$ *and* $\|\nabla\widetilde{\varphi}\|_{L^\infty(L^4)}$

Proof. Since $F(\nabla\varphi) = \dfrac{1}{4\varepsilon^2}\left(|\nabla\varphi|^2 - 1\right)^2$, one has

$$\begin{aligned}\int_\Omega F(\nabla\varphi) &= \frac{1}{4\varepsilon^2}\int_\Omega |\nabla\varphi|^4 + \frac{1}{2\varepsilon^2}\int_\Omega |\nabla\varphi|^2 + \frac{1}{4\varepsilon^2}|\Omega| \\ &\le \frac{1}{2\varepsilon^2}\int_\Omega |\nabla\varphi|^4 + \frac{C}{\varepsilon^2} \le \frac{1}{2\varepsilon^2}\int_\Omega |\nabla\widehat{\varphi}|^4 + \frac{C_2}{\varepsilon^2}\end{aligned}$$

where C depends on $|\Omega|$ and C_2 depends, moreover, on $\|\nabla\widetilde{\varphi}\|_{L^4(L^4)}$. Therefore, (19) holds. ■

2.3 Energy Inequality

Lemma 5 (Energy equality) *If $(\boldsymbol{u},\varphi)$ is a regular enough solution of (10), the following energy equality holds:*

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}|\boldsymbol{u}|_2^2 + \frac{1}{2}|\Delta\widehat{\varphi}|_2^2 + \int_\Omega F(\nabla\varphi)\right) + |\nabla\varphi^T D(\boldsymbol{u})\nabla\varphi|_2^2 + |D(\boldsymbol{u})\nabla\varphi|_2^2 \\ &\quad + \nu|\nabla\boldsymbol{u}|_2^2 + |\omega|_2^2 = (\partial_t\widetilde{\varphi},\omega) + (\partial_t\nabla\widetilde{\varphi},\boldsymbol{f}(\nabla\varphi)).\end{aligned} \tag{20}$$

where $\omega = \Delta^2\widehat{\varphi} - \nabla\cdot\boldsymbol{f}(\nabla\varphi)$.

Proof. Taking $\boldsymbol{u}$ as test function in the $\boldsymbol{u}$-system of (10), one has

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}|_2^2 + \nu|\nabla\boldsymbol{u}|_2^2 + (\sigma_{nl}^d,\nabla\boldsymbol{u}) - (\omega\cdot\nabla\varphi,\boldsymbol{u}) = 0. \tag{21}$$

The nonlinear dissipative tensor σ_{nl}^d verifies:

$$(\sigma_{nl}^d,\nabla\boldsymbol{u}) = (\sigma_{nl}^d, D(\boldsymbol{u})) = |\nabla\varphi^T D(\boldsymbol{u})\nabla\varphi|_2^2 + |D(\boldsymbol{u})\nabla\varphi|_2^2, \tag{22}$$

since

$$(\mathbf{n}\otimes\boldsymbol{n}) : D(\boldsymbol{u}) = \boldsymbol{n}_i\boldsymbol{n}_j D(\boldsymbol{u})_{ij} = \boldsymbol{n}^T D(\boldsymbol{u})\boldsymbol{n}$$

and

$$(D(\boldsymbol{u})\boldsymbol{n}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes D(\boldsymbol{u})\boldsymbol{n}) : D(\boldsymbol{u}) = 2(D(\boldsymbol{u})\boldsymbol{n}\otimes\boldsymbol{n}) : D(\boldsymbol{u}) = D(\boldsymbol{u})_{ik}\boldsymbol{n}_k\boldsymbol{n}_j D(\boldsymbol{u})_{ij} = |D(\boldsymbol{u})\boldsymbol{n}|^2.$$

Therefore, from (21) we obtain

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}|_2^2 + \nu|\nabla\boldsymbol{u}|_2^2 + |\nabla\varphi^T D(\boldsymbol{u})\nabla\varphi|_2^2 + |D(\boldsymbol{u})\nabla\varphi|_2^2 + |\omega|_2^2 - (\omega\cdot\nabla\varphi,\boldsymbol{u}) = 0. \tag{23}$$

On the other hand, by taking ω as test function in the φ-equation of (10), one has

$$\frac{1}{2}\frac{d}{dt}|\Delta\widehat{\varphi}|_2^2 - (\partial_t\widehat{\varphi},\nabla\cdot\boldsymbol{f}(\nabla\varphi)) + (\boldsymbol{u}\cdot\nabla\varphi,\omega) + |\omega|_2^2 = (\partial_t\widetilde{\varphi},\omega). \tag{24}$$

The second term on the left hand side of (24) can be written as

$$-(\partial_t\widehat{\varphi},\nabla\cdot\boldsymbol{f}(\nabla\varphi)) = (\partial_t\nabla\varphi,\boldsymbol{f}(\nabla\varphi)) - (\partial_t\nabla\widetilde{\varphi},\boldsymbol{f}(\nabla\varphi)) = \frac{d}{dt}\int_\Omega F(\nabla\varphi) - (\partial_t\nabla\widetilde{\varphi},\boldsymbol{f}(\nabla\varphi)). \tag{25}$$

By adding (23) and (24) and into account (25) we obtain (20). ■

Corollary 6 (Energy inequality) *Under hypothesis of Lemma 5, the following energy inequality holds:*

$$\frac{d}{dt}\left(|\boldsymbol{u}|_2^2 + |\Delta\widehat{\varphi}|_2^2 + 2\int_\Omega F(\nabla\varphi)\right) + 2\nu|\nabla\boldsymbol{u}|_2^2 + |\Delta^2\widehat{\varphi} - \nabla\cdot\boldsymbol{f}(\nabla\varphi)|_2^2 \le \frac{1}{4\varepsilon^2}|\nabla\widehat{\varphi}|_4^4 + \frac{C_3}{\varepsilon^2} \quad (26)$$

where C_3 depends on $\|\nabla\widetilde{\varphi}\|_{L^\infty(L^4)}$ and $\|\partial_t\nabla\widetilde{\varphi}\|_{L^\infty(L^4)}$.

Proof. The first term on the right hand side of (20) can be bounded as follows:

$$(\partial_t\widetilde{\varphi},\omega) \le \frac{1}{2}|\partial_t\widetilde{\varphi}|_2^2 + \frac{1}{2}|\omega|_2^2 \le \frac{1}{2}|\omega|_2^2 + C.$$

Into account the expression of $\boldsymbol{f}$ and that $\|\partial_t\nabla\widetilde{\varphi}\|_{L^\infty(L^4)} \le C$, the second term on the right hand side of energy equality (20) can be written as

$$\begin{aligned}
(\partial_t\nabla\widetilde{\varphi},\boldsymbol{f}(\nabla\varphi)) &\le \frac{1}{\varepsilon^2}||\nabla\varphi|^2-1|_2|\nabla\varphi|_4|\partial_t\nabla\widetilde{\varphi}|_4 \le \frac{C}{\varepsilon^2}\Big(|\nabla\varphi|_4^2+1\Big)|\nabla\varphi|_4\\
&\le \frac{C}{\varepsilon^2}\Big(|\nabla\widehat{\varphi}|_4^2+|\nabla\widetilde{\varphi}|_4^2+1\Big)\Big(|\nabla\widehat{\varphi}|_4+|\nabla\widetilde{\varphi}|_4\Big)\\
&\le \frac{C}{\varepsilon^2}\Big(|\nabla\widehat{\varphi}|_4^2+1\Big)\Big(|\nabla\widehat{\varphi}|_4+1\Big) \le \frac{1}{8\varepsilon^2}|\nabla\widehat{\varphi}|_4^4+\frac{C}{\varepsilon^2}.
\end{aligned}$$

Hence, as $|\nabla\varphi^T D(\boldsymbol{u})\nabla\varphi|_2^2 + |D(\boldsymbol{u})\nabla\varphi|_2^2$ is positive, (26) is obtained. ∎

Corollary 7 *The following inequality holds:*

$$\begin{aligned}
\frac{d}{dt}\left(|\boldsymbol{u}|_2^2 + |\Delta\widehat{\varphi}|_2^2 + 2\int_\Omega F(\nabla\varphi)\right) + 2\nu|\nabla\boldsymbol{u}|_2^2 + |\Delta\widehat{\varphi}|_2^2\\
+\frac{1}{4\varepsilon^2}|\nabla\widehat{\varphi}|_4^4 + \frac{1}{2}|\Delta^2\widehat{\varphi} - \nabla\cdot\boldsymbol{f}(\nabla\varphi)|_2^2 \le C_4
\end{aligned} \quad (27)$$

where C_4 depends on ε, $|\Omega|$, $\|\nabla\widetilde{\varphi}\|_{L^\infty(L^4)}$ and $\|\partial_t\nabla\widetilde{\varphi}\|_{L^\infty(L^4)}$.

Proof. Recalling (11), by adding C_1 to both terms of inequality (26), we have

$$\begin{aligned}
\frac{d}{dt}\left(|\boldsymbol{u}|_2^2 + |\Delta\widehat{\varphi}|_2^2 + 2\int_\Omega F(\nabla\varphi)\right) + 2\nu|\nabla\boldsymbol{u}|_2^2 + |\Delta\widehat{\varphi}|_2^2 + \frac{1}{2\varepsilon^2}|\nabla\widehat{\varphi}|_4^4\\
+\frac{1}{2}|\Delta^2\widehat{\varphi} - \nabla\cdot\boldsymbol{f}(\nabla\varphi)|_2^2 \le \frac{1}{4\varepsilon^2}|\nabla\widehat{\varphi}|_4^4 + C_4
\end{aligned}$$

and therefore (27) holds. ∎

3 Weak/strong uniqueness

The aims of this section is to check that the classic argument to prove uniqueness of weak/strong solutions of the Navier-Stokes model (see for instance [Lions'96]) is valid now for the smectic-A model (1)-(3) even taking into account the high nonlinear character of the dissipative stress tensor σ^d. The sketch of the proof given in [Liu'00] is very short and, in our opinion, the most important nonlinear terms are not clearly bounded. Then, we will do a formal proof, see [Lions'96] for a rigorous justification in the Navier-Stokes case. We will need this result of weak/strong uniqueness twice later.

Theorem 8 *If* $(\boldsymbol{u}_1, \varphi_1)$ *and* $(\boldsymbol{u}_2, \varphi_2)$ *are respectively a weak and a strong solution of* (1)-(3) *in* $[0, T]$, *then* $\boldsymbol{u}_1 = \boldsymbol{u}_2$, *and* $\varphi_1 = \varphi_2$ *a.e. in* $\Omega \times [0, T]$.

Proof. We denote $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\varphi = \varphi_1 - \varphi_2$ (notice that $\varphi = \widehat{\varphi}$). Making the difference between (1) for $(\boldsymbol{u}_1, \varphi_1)$ and $(\boldsymbol{u}_2, \varphi_2)$, considering $\boldsymbol{u}$ and $\Delta^2\varphi$ as test functions, the following equalities holds

$$\left\{\begin{array}{l} \displaystyle\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}|_2^2 + \nu|\nabla \boldsymbol{u}|_2^2 + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_2, \boldsymbol{u}) + (\sigma^d_{nl}(D\boldsymbol{u}_1, \nabla\varphi_1) - \sigma^d_{nl}(D\boldsymbol{u}_2, \nabla\varphi_2), D(\boldsymbol{u})) \\ \qquad -((\Delta^2\varphi - \nabla\cdot(\boldsymbol{f}(\nabla\varphi_1) - \boldsymbol{f}(\nabla\varphi_2))\nabla\varphi_1, \boldsymbol{u}) + ((\Delta^2\varphi_2 - \nabla\cdot\boldsymbol{f}(\nabla\varphi_2))\nabla\varphi, \boldsymbol{u}) = 0, \\ \displaystyle\frac{1}{2}\frac{d}{dt}|\Delta\varphi|_2^2 + |\Delta^2\varphi|_2^2 + (\boldsymbol{u}\cdot\nabla\varphi_1, \Delta^2\varphi) + (\boldsymbol{u}_2\cdot\nabla\varphi, \Delta^2\varphi) \\ \qquad -(\nabla\cdot(\boldsymbol{f}(\nabla\varphi_1) - \boldsymbol{f}(\nabla\varphi_2)), \Delta^2\varphi) = 0. \end{array}\right. \tag{28}$$

From (28), cancelling the term $(\boldsymbol{u}\cdot\nabla\varphi_1, \Delta^2\varphi)$ in both equations, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|\boldsymbol{u}|_2^2 + |\Delta\varphi|_2^2) + \nu|\nabla\boldsymbol{u}|_2^2 + |\Delta^2\varphi|_2^2 = -((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}_2, \boldsymbol{u}) \\ &-(\sigma^d_{nl}(D\boldsymbol{u}_1, \nabla\varphi_1) - \sigma^d_{nl}(D\boldsymbol{u}_2, \nabla\varphi_2), D(\boldsymbol{u})) + (\Delta^2\varphi_2\nabla\varphi, \boldsymbol{u}) \\ &-(\nabla\cdot\boldsymbol{f}(\nabla\varphi_2)\nabla\varphi, \boldsymbol{u}) - (\nabla\cdot(\boldsymbol{f}(\nabla\varphi_1) - \boldsymbol{f}(\nabla\varphi_2))\nabla\varphi_1, \boldsymbol{u}) \\ &-(\boldsymbol{u}_2\cdot\nabla\varphi, \Delta^2\varphi) - (\nabla\cdot(\boldsymbol{f}(\nabla\varphi_1) - \boldsymbol{f}(\nabla\varphi_2)), \Delta^2\varphi) := \sum_{i=1}^{7} I_i. \end{aligned} \tag{29}$$

The first, third, fourth and six terms on the right hand side of (29) are bounded, respectively, by

$$I_1 \le \delta\nu|\nabla\boldsymbol{u}|_2^2 + C|\nabla\boldsymbol{u}_2|_2^4|\boldsymbol{u}|_2^2, \qquad I_3 \le \delta\nu|\nabla\boldsymbol{u}|_2^2 + C|\Delta^2\varphi_2|_2^2|\Delta\varphi|_2^2$$

$$I_4 \le \delta\nu|\nabla\boldsymbol{u}|_2^2 + C\frac{1}{\varepsilon^2}\left(3|\nabla\varphi_2|_4^2|\Delta\varphi_2|_4 + |\Delta\varphi_2|_2\right)|\Delta\varphi|_2^2, \qquad I_6 \le \delta|\Delta^2\varphi|_2^2 + C|\Delta\varphi|_2^2$$

for any $\delta > 0$ a small enough constant. Taking into account the equality

$$\nabla\cdot\boldsymbol{f}(\nabla\varphi_1) - \nabla\cdot\boldsymbol{f}(\nabla\varphi_2) = \frac{1}{\varepsilon^2}\Big((3|\nabla\varphi_1|^2 - 1)\Delta\varphi + 3(\nabla\varphi_1 + \nabla\varphi_2)\Delta\varphi_2\nabla\varphi\Big)$$

the fifth term on the right hand side of (29) can be written as

$$I_5 = \frac{1}{\varepsilon^2}\left((3|\nabla\varphi_1|^2-1)\nabla\varphi_1\Delta\varphi, \boldsymbol{u}\right) + \frac{1}{\varepsilon^2}\left(3(\nabla\varphi_1+\nabla\varphi_2)\nabla\varphi_1\Delta\varphi_2\nabla\varphi, \boldsymbol{u}\right)$$

and it is bounded by

$$\begin{aligned} I_5 &\le \frac{C}{\varepsilon^2}\Big(|(3|\nabla\varphi_1|^2-1)\nabla\varphi_1|_3|\Delta\varphi|_2 + |3(\nabla\varphi_1+\nabla\varphi_2)\nabla\varphi_1\Delta\varphi_2|_{3/2}|\nabla\varphi|_6\Big)|\boldsymbol{u}|_6 \\ &\le \delta\nu|\nabla\boldsymbol{u}|_2^2 + \frac{C}{\varepsilon^2}\left(|(3|\nabla\varphi_1|^2-1)\nabla\varphi_1|_3^2 + |3(\nabla\varphi_1+\nabla\varphi_2)\nabla\varphi_1\Delta\varphi_2|_2^2\right)|\Delta\varphi|_2^2. \end{aligned}$$

Analogously, the seventh one is bounded as follows:

$$\begin{aligned} I_7 &\le \frac{C}{\varepsilon^2}\Big(|3|\nabla\varphi_1|^2-1|_\infty|\Delta\varphi|_2 + |3(\nabla\varphi_1+\nabla\varphi_2)\Delta\varphi_2|_3|\nabla\varphi|_6\Big)|\Delta^2\varphi|_2 \\ &\le \delta|\Delta^2\varphi|_2^2 + \frac{C}{\varepsilon^2}\left(|3|\nabla\varphi_1|^2-1|_\infty^2 + |3(\nabla\varphi_1+\nabla\varphi_2)\Delta\varphi_2|_3^2\right)|\Delta^2\varphi|_2^2. \end{aligned}$$

With regard to the second term (recall that $\boldsymbol{n}_1 = \nabla\varphi_1$ and $\boldsymbol{n} = \nabla\varphi$),

$$(\sigma_{nl}^d(D\boldsymbol{u}_1, n_1) - \sigma_{nl}^d(D\boldsymbol{u}_2, n_2), D(\boldsymbol{u})) = ((\boldsymbol{n}_1^t D(\boldsymbol{u})\boldsymbol{n}_1)\boldsymbol{n}_1\otimes\boldsymbol{n}_1, D(\boldsymbol{u})) + (\text{the rest of terms}, D(\boldsymbol{u})),$$

taking into account that

$$((\boldsymbol{n}_1^t D(\boldsymbol{u})\boldsymbol{n}_1)\boldsymbol{n}_1\otimes\boldsymbol{n}_1, D(\boldsymbol{u})) \ge 0,$$

the problem is to born appropriately $|\text{the rest of terms}|_2^2$. Concretely, the more nonlinear term can be bounded as follows

$$\begin{aligned} |(\boldsymbol{n}_1^t D(\boldsymbol{u}_2)\boldsymbol{n}_1)\boldsymbol{n}_1\otimes\boldsymbol{n}|_2^2 &\le |D(\boldsymbol{u}_2)|_2^2|\boldsymbol{n}_1|_\infty^6|\boldsymbol{n}|_\infty^2 \le C\|\boldsymbol{u}_2\|_1^2\|\varphi_1\|_2^3\|\varphi_1\|_3^3\|\varphi\|_2\|\varphi\|_3 \\ &\le C\,\|\varphi_1\|_4^{3/2}\|\varphi\|_2^{3/2}\|\varphi\|_4^{1/2} \le \delta\|\varphi\|_4^2 + C\|\varphi_1\|_4^2\|\varphi\|_2^2 \end{aligned}$$

Here, the interpolation inequalities $|\varphi|_\infty^2 \le C\,\|\varphi\|_2\|\varphi\|_3$ and $\|\varphi\|_3^2 \le C\,\|\varphi\|_2\|\varphi\|_4$ have been used jointly with the L^∞ in time estimates $\|\boldsymbol{u}_2\|_1 \le C$ and $\|\varphi_1\|_2 \le C$.

Therefore, one arrives at

$$\begin{cases} \dfrac{d}{dt}\left(|\boldsymbol{u}|_2^2 + |\Delta\varphi|_2^2\right) \le a(t)(|\boldsymbol{u}|_2^2 + |\Delta\varphi|_2^2) \\ |\boldsymbol{u}(0)|_2^2 + |\Delta\varphi(0)|_2^2 = 0, \end{cases}$$

where $a(t)$ is bounded in $L^1(0,T)$. Applying Gronwall's Lemma, one has $\boldsymbol{u} = 0$ and $\Delta\varphi = 0$. Finally, since $\varphi = 0$ on $\partial\Omega$, then $\varphi = 0$ in Q_T. Therefore, uniqueness of Galerkin approximate solution for the initial-boundary problem (42) is proved. ■

4 Global weak solution of the initial-value problem.

Definition 9 *We say that* $(\boldsymbol{u},\varphi)$ *is a weak solution of* (1)-(3) *in* $[0,+\infty)$ *if*

$$\nabla\cdot\boldsymbol{u}=0 \text{ in } Q,\quad \boldsymbol{u}|_\Sigma=0,\quad \varphi|_\Sigma=\varphi_1,\quad \partial_{\rm n}\varphi|_\Sigma=\varphi_2$$

$$\|(\boldsymbol{u}(t),\varphi(t))\|_{0\times 2}\le C_1\quad \forall t\ge 0 \tag{30}$$

$$\forall\gamma>0,\quad e^{-\gamma t}\int_0^t e^{\gamma s}\|(\boldsymbol{u}(s),\varphi(s))\|^2_{1\times 4}\,ds\le C_2\left(1+\frac{1}{\nu}\right),\quad \forall t\ge 0, \tag{31}$$

where $C_1, C_2>0$ *are constants independent of* ν*, verifying*

$$\langle\partial_t\boldsymbol{u},\boldsymbol{v}\rangle+((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{v})+\nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v})+(\sigma^d_{nl},\nabla\boldsymbol{v})-((\Delta^2\varphi-\nabla\cdot\boldsymbol{f}(\nabla\varphi))\nabla\varphi,\boldsymbol{v})=0\quad\forall\,\boldsymbol{v}\in\boldsymbol{V},$$
$$\partial_t\varphi+(\boldsymbol{u}\cdot\nabla)\varphi+\Delta^2\varphi-\nabla\cdot\boldsymbol{f}(\nabla\varphi)=0,\quad a.e.\ in\ Q$$
$$\boldsymbol{u}(0)=\boldsymbol{u}_0,\quad \varphi(0)=\varphi_0\quad in\ \Omega.$$

In the finite time case ($T<\infty$)*,* (31) *holds even when* $\gamma=0$*, i.e.* $(\boldsymbol{u},\varphi)\in L^2(0,T;\boldsymbol{H}^1\times H^4)$.

Remark: (30) and (31) imply that $\partial_t\boldsymbol{u}\in L^{4/3}_{loc}([0,\infty);\boldsymbol{V}')$ and $\partial_t\varphi\in L^2_{loc}([0,\infty);\boldsymbol{L}^2)$. In particular, $\langle\partial_t\boldsymbol{u},\boldsymbol{v}\rangle$ denotes the duality product between $\boldsymbol{V}'$ and $\boldsymbol{V}$.

Theorem 10 (Existence of weak solutions of the initial-valued problem) *Let* $\boldsymbol{u}_0\in\boldsymbol{H}$ *and* $\varphi_0\in H^2$*. Let* Ω*,* φ_1 *and* φ_2 *be regular enough, verifying the compatibility conditions* $\varphi_0|_{\partial\Omega}=\varphi_1(0)$*,* $\partial_{\rm n}\varphi_0|_{\partial\Omega}=\varphi_2(0)$ *and such that the lifting function* $\widetilde{\varphi}$ *defined in* (9) *satisfies*

$$\widetilde{\varphi}\in L^\infty(0,+\infty;\boldsymbol{H}^4(\Omega))\quad and\quad \partial_t\widetilde{\varphi}\in L^\infty(0,+\infty;\boldsymbol{W}^{1,4}(\Omega)).$$

Then, there exists a weak solution $(\boldsymbol{u},\varphi)$ *of* (1)-(3) *in* $[0,+\infty)$.

Proof. The proof is based on a semi-Galerkin method as in [Liu'00]. The novelty respect to [Liu'00] is that now we will find a weak solution with regularity up to infinity time, even for the time-dependent boundary conditions for the layer variable φ.

Let $\{\boldsymbol{w}_i\}_i\ge 1$ a "special" basis of $\boldsymbol{V}$ formed by eigenfunctions of the Stokes problem

$$(\nabla\boldsymbol{w}_i,\nabla\boldsymbol{v})=\lambda_i(\boldsymbol{w}_i,\boldsymbol{v})\qquad\forall\,\boldsymbol{v}\in\boldsymbol{V},\ \boldsymbol{w}_i\in\boldsymbol{V},\qquad \text{with } \|\boldsymbol{w}_i\|_{L^2}=1,\quad \lambda_i\nearrow+\infty.$$

Let $\boldsymbol{V}^m$ be the finite-dimensional subspace spanned by $\{\boldsymbol{w}_1,\boldsymbol{w}_2,\dots,\boldsymbol{w}_n\}$.

For each $m\ge 1$, we say that $(\boldsymbol{u}_m,\varphi_m)$ is an approximate solution, if $\boldsymbol{u}_m:[0,+\infty)\mapsto\boldsymbol{V}^m$ and $\varphi_m:[0,+\infty)\mapsto\boldsymbol{H}^4$ with $\widehat{\varphi}_m=\varphi_m-\widetilde{\varphi}$ ($\widetilde{\varphi}$ being the lifting function defined in (9)) and the following variational formulation holds:

$$\begin{cases}(\partial_t\boldsymbol{u}_m(t),\boldsymbol{v}_m)+((\boldsymbol{u}_m(t)\cdot\nabla)\boldsymbol{u}_m(t),\boldsymbol{v}_m)+\nu(\nabla\boldsymbol{u}_m(t),\nabla\boldsymbol{v}_m)+(\sigma^{d,m}_{nl},D\boldsymbol{v}_m)\\ \quad-((\Delta^2\widehat{\varphi}_m(t)-\nabla\cdot\boldsymbol{f}(\nabla\varphi_m(t)))\nabla\varphi_m(t),\boldsymbol{v}_m)=0\quad\forall\,\boldsymbol{v}_m\in\boldsymbol{V}^m,\quad\text{a.e. } t\ge 0,\\ \partial_t\widehat{\varphi}_m(t)+(\boldsymbol{u}_m(t)\cdot\nabla)\varphi_m(t)+\Delta^2\widehat{\varphi}_m(t)-\nabla\cdot\boldsymbol{f}(\varphi_m(t))=\partial_t\widetilde{\varphi}(t),\quad\text{a.e. } t\in Q,\\ \boldsymbol{u}_m(0)=\boldsymbol{u}_{0m}:=P_m(\boldsymbol{u}_0),\quad\varphi_m(0)=\varphi_0\qquad\text{in }\Omega.\end{cases} \tag{32}$$

Here, $P_m : \boldsymbol{H} \mapsto \boldsymbol{V}^m$ denotes the usual orthogonal projector from $\boldsymbol{H}$ onto $\boldsymbol{V}^m$. In particular, $\boldsymbol{u}_{0m} \to \boldsymbol{u}_0$ in $\boldsymbol{L}^2$. Moreover, we have denoted $\sigma_{nl}^{d,m} = \sigma_{nl}^d(D(\boldsymbol{u}_m), \nabla\varphi_m)$.

The existence and uniqueness of local in time solution of (32) is proved in [Liu'00]. Moreover, one has the following estimates (independent of m): $\boldsymbol{u}_m$ is bounded in $L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V})$ and φ_m is bounded in $L^\infty(0,T;\boldsymbol{H}^2) \cap L^2(0,T;\boldsymbol{H}^4)$. This suffices to control the nonlinear terms and to pass to the limit in (32), obtaining a weak solution of the initial-valued problem (1)-(3) in $[0,T]$, with $T>0$ a finite fixed final time. Next, to extend the solution to the whole time interval $[0,+\infty)$, we will prove that the approximate solutions $(\boldsymbol{u}_m(t), \varphi_m(t))$ of (32) are bounded in $[0,+\infty)$.

From (11) and (26), one has

$$\frac{d}{dt}\left(|\boldsymbol{u}_m|_2^2 + |\Delta\widehat{\varphi}_m|_2^2 + 2\int_\Omega F(\nabla\varphi_m)\right) + 2\nu|\nabla\boldsymbol{u}_m|_2^2 + 2|\Delta\widehat{\varphi}_m|_2^2 + \frac{3}{4\varepsilon^2}|\nabla\widehat{\varphi}_m|_4^4 \le C \tag{33}$$

(since the φ_m-equation is verified pointwise in Q and $(\boldsymbol{u}_m, \varphi_m)$ are regular functions, it is easy to justify the computations of Lemma 3 and Corollary 6, in order to arrive at (33)). Applying inequality (19) and the Poincaré inequality $P|\boldsymbol{u}| \le |\nabla\boldsymbol{u}|$ (with $P>0$ a constant) to (33) one obtains

$$\frac{d}{dt}\left(|\boldsymbol{u}_m|_2^2 + |\Delta\widehat{\varphi}_m|_2^2 + 2\int_\Omega F(\nabla\varphi_m)\right) + C_0\left(|\boldsymbol{u}_m|_2^2 + |\Delta\widehat{\varphi}_m|_2^2 + 2\int_\Omega F(\nabla\varphi_m)\right) \le C, \tag{34}$$

where $C_0 = \min\{2\nu P, 3/4\}$. Multiplying by $e^{C_0 s}$ and integrating in $s \in [0,t]$ we have

$$\begin{aligned} |\boldsymbol{u}_m(t)|_2^2 + |\Delta\widehat{\varphi}_m(t)|_2^2 + 2\int_\Omega F(\nabla\varphi_m) &\le e^{-C_0 t}\left(|\boldsymbol{u}_{0m}|_2^2 + |\Delta\varphi_0|_2^2 + 2\int_\Omega F(\nabla\varphi_0)\right) \\ &+C(1-e^{-C_0 t}) \le |\boldsymbol{u}_0|_2^2 + |\Delta\varphi_0|_2^2 + 2\int_\Omega F(\nabla\varphi_0) + C \end{aligned} \tag{35}$$

for all $t \ge 0$. Therefore, one deduces the following estimates independently of m: $\boldsymbol{u}_m$ is bounded in $L^\infty(0,+\infty;\boldsymbol{H})$, φ_m is bounded in $L^\infty(0,+\infty;\boldsymbol{H}^2(\Omega))$, and $\int_\Omega F(\nabla\varphi_m)$ is bounded in $L^\infty(0,+\infty)$.

Now, using (27), multiplying by $e^{\gamma t}$ for any $\gamma > 0$, we get

$$\begin{aligned} &\frac{d}{dt}\left(e^{\gamma t}(|\boldsymbol{u}_m|_2^2 + |\Delta\widehat{\varphi}_m|_2^2 + 2\int_\Omega F(\nabla\varphi_m))\right) \\ &+e^{\gamma t}\Big(2\nu|\nabla\boldsymbol{u}_m|_2^2 + |\Delta\widehat{\varphi}_m|_2^2 + \frac{1}{4\varepsilon^2}|\nabla\widehat{\varphi}_m|_4^4 + \frac{1}{2}|\Delta^2\widehat{\varphi}_m - \nabla\cdot\boldsymbol{f}(\nabla\varphi_m)|_2^2\Big) \le Ce^{\gamma t}, \end{aligned} \tag{36}$$

hence it is easy to deduce the estimates

$$\forall\gamma>0, \quad e^{-\gamma t}\int_0^t e^{\gamma s}\|\boldsymbol{u}_m(s)\|_1^2\,ds \le \frac{C_2}{\nu}, \quad \forall t \ge 0, \tag{37}$$

and

$$\forall\gamma>0, \quad e^{-\gamma t}\int_0^t e^{\gamma s}|\Delta^2\widehat{\varphi}_m(s) - \nabla\cdot\boldsymbol{f}(\nabla\varphi_m(s))|_2^2\,ds \le C_2, \quad \forall t \ge 0. \tag{38}$$

From the equality

$$\nabla \cdot \boldsymbol{f}(\nabla\varphi_m) = \frac{3}{\varepsilon^2}|\nabla\varphi_m|^2\Delta\varphi_m - \frac{1}{\varepsilon^2}\Delta\varphi_m \tag{39}$$

and previous estimates $\Delta\varphi_m$ is bounded in $L^\infty(0,+\infty;\boldsymbol{L}^2(\Omega))$ and $|\nabla\varphi_m|^2$ is bounded in $L^\infty(0,+\infty;\boldsymbol{L}^3(\Omega))$, one has that $\nabla\cdot\boldsymbol{f}(\nabla\varphi_m)$ is bounded in $L^\infty(0,+\infty;\boldsymbol{L}^{6/5}(\Omega))$ (recall that $\widetilde{\varphi}$ is in $L^\infty(0,+\infty;\boldsymbol{H}^4(\Omega))$). In particular,

$$\forall\gamma>0,\quad e^{-\gamma t}\int_0^t e^{\gamma s}|\nabla\cdot\boldsymbol{f}(\nabla\varphi_m)|_{6/5}^2\,ds \le C_2,\quad \forall t\ge 0.$$

Therefore, into account (38), we obtain

$$\forall\gamma>0,\quad e^{-\gamma t}\int_0^t e^{\gamma s}|\Delta^2\widehat{\varphi}_m(s)|_{6/5}^2\,ds \le C_2,\quad \forall t\ge 0, \tag{40}$$

Applying to the imbedding of $\boldsymbol{W}^{3,6/5}(\Omega)$ into $\boldsymbol{H}^2(\Omega)$, to the sequence $\nabla\widehat{\varphi}_m$, one has

$$\forall\gamma>0,\quad e^{-\gamma t}\int_0^t e^{\gamma s}\|\nabla\widehat{\varphi}_m(s)\|_2^2\,ds \le C_2,\quad \forall t\ge 0, \tag{41}$$

From (41), the bound of $\nabla\widehat{\varphi}_m$ in $L^\infty(0,+\infty;\boldsymbol{H}^1(\Omega))$ and the interpolation inequality $|g|_\infty \le \|g\|_1^{1/2}\|g\|_2^{1/2}$, we obtain

$$\forall\gamma>0,\quad e^{-\gamma t}\int_0^t e^{\gamma s}|\nabla\widehat{\varphi}_m(s)|_\infty^4\,ds \le C_2,\quad \forall t\ge 0.$$

By using (39) and the bound of φ_m in $L^\infty(0,+\infty;\boldsymbol{H}^2)$, one has

$$|\nabla\cdot\boldsymbol{f}(\nabla\varphi_m)|_2^2 \le C\left(|\nabla\varphi_m|_\infty^4|\Delta\varphi_m|_2^2 + |\Delta\varphi_m|_2^2\right) \le C\left(|\nabla\varphi_m|_\infty^4 + 1\right).$$

Therefore,

$$\forall\gamma>0,\quad e^{-\gamma t}\int_0^t e^{\gamma s}|\nabla\cdot\boldsymbol{f}(\widehat{\varphi}_m(s))|_2^2\,ds \le C_2,\quad \forall t\ge 0.$$

From this last inequality and (38) one has

$$\forall\gamma>0,\quad e^{-\gamma t}\int_0^t e^{\gamma s}|\Delta^2\widehat{\varphi}_m(s)|_2^2\,ds \le C_2,\quad \forall t\ge 0,$$

hence the regularity (31) for φ can be deduced.

At this point, the existence of weak solutions of (1)-(3) in $(0,+\infty)$ can be proved by means of a rather standard pass to the limit argument (see [Liu'00] for the finite time case). ■

5 Weak time-periodic solutions

In this section, let $T > 0$ a finite fixed number which states the time period.

Definition 11 *We say that* $(\boldsymbol{u}, \varphi)$ *is a weak time-periodic solution of* (1), (2) *and* (4) *if*

$$\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{H}) \cap L^2(0, T; \boldsymbol{H}^1), \quad \varphi \in L^\infty(0, T; \boldsymbol{H}^2) \cap L^2(0, T; \boldsymbol{H}^4)$$

satisfying (1) *and boundary conditions* (2) *as in Definition 9 and time-periodic conditions* $\boldsymbol{u}(0) = \boldsymbol{u}(T)$, $\varphi(0) = \varphi(T)$ *in the sense of spaces* $\boldsymbol{L}^2$ *and* $\boldsymbol{H}^2$ *respectively.*

Theorem 12 (Existence of weak time-periodic solutions) *Let* Ω, φ_1 *and* φ_2 *be regular enough with* $\varphi_1(0) = \varphi_1(T)$, $\varphi_2(0) = \varphi_2(T)$, *and such that the lifting function* $\widetilde{\varphi}$ *defined in* (9) *satisfies*

$$\widetilde{\varphi} \in L^\infty(0, T; \boldsymbol{H}^4(\Omega)), \quad \partial_t \widetilde{\varphi} \in L^\infty(0, T; \boldsymbol{W}^{1,4}(\Omega)).$$

Then, there exists a weak time-periodic solution of (1), (2) *and* (4).

Proof. In the proof of this theorem, a fully Galerkin method (approximating in finite dimension both variables $\boldsymbol{u}$ and φ) will be used. The reason is that this finite-dimensional Galerkin problem let us to find time-periodic approximate solutions via a fixed-point argument for the operator mapping the initial and final time values. Firstly, we consider the initial-boundary Galerkin problem associated to any arbitrary finite-dimensional initial data. Afterwards, the key is to find an initial data at $t = 0$ which will be "reproduced" at final time $t = T$. Finally, by means of a pass to the limit procedure, a weak time-periodic solution will be found.

We divide the proof in several steps.

Step 0: Existence of local in time Galerkin solution.

Let $\{\boldsymbol{w}_i\}_n \geq 1$ and $\{\phi_i\}_n \geq 1$ "special" basis of $\boldsymbol{V}$ and $H_0^2(\Omega)$, respectively, formed by eigenfunctions of the Stokes problem

$$(\nabla \boldsymbol{w}_i, \nabla \boldsymbol{v}) = \lambda_i(\boldsymbol{w}_i, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{V},\ \boldsymbol{w}_i \in \boldsymbol{V}, \quad \text{con } \|\boldsymbol{w}_i\|_{L^2} = 1, \quad \lambda_i \nearrow +\infty$$

and of the bilaplacian problem

$$(\Delta \phi_i, \Delta e) = \mu_i(\phi_i, e) \qquad \forall\, e \in H_0^2,\ \phi_i \in H_0^2, \quad \text{con } \|\phi_i\|_{L^2} = 1, \quad \mu_i \nearrow +\infty.$$

Let $\boldsymbol{V}^m$ and W^m be the finite-dimensional subspaces spanned by $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}$ and $\{\phi_1, \phi_2, \ldots, \phi_n\}$ respectively.

Given $\boldsymbol{u}_0 \in \boldsymbol{H}$ and $\varphi_0 \in H_0^2$, for each $m \geq 1$, we seek an approximate solution $(\boldsymbol{u}_m, \varphi_m)$, with $\boldsymbol{u}_m : [0, T] \mapsto \boldsymbol{V}^m$ and $\varphi_m = \widehat{\varphi}_m + \widetilde{\varphi}$, with $\widehat{\varphi}_m : [0, T] \mapsto W^m$, verifying the following variational formulation a.e. $t \in (0, T)$:

$$
\begin{cases}
(\partial_t \boldsymbol{u}_m(t), \boldsymbol{v}_m) + ((\boldsymbol{u}_m(t)\cdot\nabla)\boldsymbol{u}_m(t), \boldsymbol{v}_m) + \nu(\nabla \boldsymbol{u}_m(t), \nabla \boldsymbol{v}_m) + (\sigma_{nl}^{d,m}(t), D\boldsymbol{v}_m) \\
\quad -(Q_m\left(\Delta^2\widehat{\varphi}_m(t) - \nabla\cdot \boldsymbol{f}(\nabla\varphi_m(t))\right)\nabla\varphi_m(t), \boldsymbol{v}_m) = 0 \quad \forall\, \boldsymbol{v}_m \in \boldsymbol{V}^m, \\
(\partial_t\widehat{\varphi}_m(t), e_m) + ((\boldsymbol{u}_m(t)\cdot\nabla)\varphi_m(t), e_m) + (\Delta^2\widehat{\varphi}_m(t) - \nabla\cdot\boldsymbol{f}(\nabla\varphi_m(t)), e_m) \\
\quad = (\partial_t\widetilde{\varphi}(t), e_m), \quad \forall\, e_m \in W^m, \\
\boldsymbol{u}_m(0) = \boldsymbol{u}_{0m} = P_m(\boldsymbol{u}_0), \quad \varphi_m(0) = \varphi_{0m} = P_m(\varphi_0) \quad \text{in } \Omega.
\end{cases}
\tag{42}
$$

Here, $P_m : \boldsymbol{H} \mapsto \boldsymbol{V}^m$ denotes the usual orthogonal projector from $\boldsymbol{H}$ onto $\boldsymbol{V}^m$, and $Q_m : L^2 \mapsto W^m$ the orthogonal projector from L^2 onto W^m. In particular, $\boldsymbol{u}_{0m} \to \boldsymbol{u}_0$ in $\boldsymbol{L}^2$ and $\varphi_{0m} \to \varphi_0$ in H^2 (as $m \to 0$).

If we write

$$
\boldsymbol{u}_m(t) = \sum_{j=1}^m \xi_{i,m}(t)\phi_i \quad \text{and} \quad \widehat{\varphi}_m(t) = \sum_{j=1}^m \zeta_{i,m}(t)\varphi_i,
$$

(42) can be rewritten as a first order ordinary differential system (in normal form) associated to the unknowns $(\xi_{i,m}(t), \zeta_{i,m}(t))$. Then, one has existence of a maximal solution (defined in some interval $[0,\tau_m) \subset [0,T]$) of the related Cauchy problem. Moreover, from a priori estimates (independent on m) which will be obtained below, in particular one has that $\tau_m = T$.

Step 1: Energy estimates.

By taking in (42) $\boldsymbol{v}_m = \boldsymbol{u}_m \in \boldsymbol{V}^m$ and $e_m = Q_m(\Delta^2\widehat{\varphi}_m - \nabla\cdot\boldsymbol{f}(\varphi_m(t))) \in W^m$ as test functions in (42), one can arrives at a similar inequality to (26) changing $(\boldsymbol{u}, \widehat{\varphi})$ by $(\boldsymbol{u}_m, \widehat{\varphi}_m)$ and $|\Delta^2\widehat{\varphi}_m - \nabla\cdot\boldsymbol{f}(\varphi_m(t))|_2^2$ by $|Q_m(\Delta^2\widehat{\varphi}_m - \nabla\cdot\boldsymbol{f}(\varphi_m(t)))|_2^2$. That is, one has

$$
\begin{aligned}
\frac{d}{dt}\left(|\boldsymbol{u}_m|_2^2 + |\Delta\widehat{\varphi}_m|_2^2 + 2\int_\Omega F(\nabla\varphi_m)\right) + 2\nu|\nabla\boldsymbol{u}_m|_2^2 + |Q_m(\Delta^2\widehat{\varphi}_m - \nabla\cdot\boldsymbol{f}(\nabla\varphi_m))|_2^2 \\
\le \frac{1}{4\varepsilon^2}|\nabla\widehat{\varphi}_m|_4^4 + C.
\end{aligned}
\tag{43}
$$

On the other hand, the proof of Lemma 3 can be mimic for the case of Galerkin solutions, obtaining the following inequality (similar to (11))

$$
|\Delta\widehat{\varphi}_m|_2^2 + \frac{1}{2\varepsilon^2}|\nabla\widehat{\varphi}_m|_4^4 \le \frac{1}{2}|Q_m(\Delta^2\widehat{\varphi}_m - \nabla\cdot\boldsymbol{f}(\nabla\varphi_m))|_2^2 + C_1. \tag{44}
$$

Following the same argument of the proof of Theorem 10, from (43) and (44) one has (35) and (36). Since now the final time $T > 0$ is finite, in particular, the following estimates holds:

$$
\boldsymbol{u}_m \text{ is bounded in } L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V}),
$$

$$\varphi_m \text{ is bounded in } L^\infty(0,T;\boldsymbol{H}^2)$$

and

$$\omega_m := Q_m(\Delta^2\widehat{\varphi}_m - \nabla\cdot\boldsymbol{f}(\nabla\varphi_m)) \text{ is bounded in } L^2(0,T;\boldsymbol{L}^2).$$

Step 2: φ_m is bounded in $L^2(0,T;\boldsymbol{H}^4)$.

We have defined $\omega_m = Q_m(\Delta^2\widehat{\varphi}_m - \nabla\cdot\boldsymbol{f}(\nabla\varphi_m))$, namely,

$$\omega_m \in W^m, \qquad (\omega_m, e_m) = \big(\Delta^2\widehat{\varphi}_m - \nabla\cdot\boldsymbol{f}(\nabla\varphi_m), e_m\big) \quad \forall\, e_m \in W^m. \tag{45}$$

By taking $e_m = \Delta^2\widehat{\varphi}_m \in W^m$ as test function in (45) (that is possible because a spectral basis of the eigenfunctions of the bilaplacian has been considered), one obtains

$$|\Delta^2\widehat{\varphi}_m|_2^2 \le |\nabla\cdot\boldsymbol{f}(\nabla\varphi_m))|_2|\Delta^2\widehat{\varphi}_m|_2 + |\omega_m|_2|\Delta^2\widehat{\varphi}_m|_2,$$

hence

$$\|\widehat{\varphi}_m\|_4 \le C\Big(|\nabla\cdot\boldsymbol{f}(\nabla\varphi_m)|_2 + |\omega_m|_2\Big). \tag{46}$$

From (39) and by using the bound of φ_m in $L^\infty(0,+\infty;\boldsymbol{H}^2)$ and the interpolation inequality $\|\varphi\|_3 \le C\|\varphi\|_2^{1/2}\|\varphi\|_4^{1/2}$, one has

$$\begin{aligned} |\nabla\cdot\boldsymbol{f}(\nabla\varphi_m))|_2^2 &\le C\left(|\nabla\varphi_m|_6^4|\nabla\nabla\varphi_m|_6^2 + |\Delta\varphi_m|_2^2\right) \le C\left(\|\varphi_m\|_2^4\|\varphi_m\|_3^2 + \|\varphi_m\|_2^2\right) \\ &\le C\left(\|\varphi_m\|_4\|\varphi_m\|_2 + 1\right) \le \delta\|\widehat{\varphi}_m\|_4^2 + \delta\|\widetilde{\varphi}\|_4^2 + C \le \delta\|\widehat{\varphi}_m\|_4^2 + C. \end{aligned} \tag{47}$$

By using this last inequality for δ small enough in (46) we obtain $\|\widehat{\varphi}_m\|_4^2 \le C + C|\omega_m|_2^2$. As ω_m is bounded in $L^2(Q)$, integrating in $[0,T]$ we have that φ_m is bounded in $L^2(0,T;H^4)$.

Step 3: Uniqueness of Galerkin solution

By applying the arguments given in Theorem 8 to $(\boldsymbol{u}_m, \varphi_m)$, we can obtain the uniqueness of Galerkin solution. Notice that this is possible because $\Delta^2\varphi_m \in W^m$ (and $\boldsymbol{u}_m \in \boldsymbol{V}^m$).

Step 4: Existence of time-periodic Galerkin solution

Given $(\boldsymbol{u}_0^m, \varphi_0^m) \in V^m \times W^m$, we define the map

$$\begin{aligned} L^m : [0,T] &\mapsto \mathbb{R}^m \times \mathbb{R}^m \\ t &\mapsto (\xi_{1m}(t), ..., \xi_{mm}(t), \zeta_{1m}(t), ..., \zeta_{mm}(t)) \end{aligned}$$

where $(\xi_{1m}(t), ..., \xi_{mm}(t))$ and $(\zeta_{1m}(t), ..., \zeta_{mm}(t))$ are the coefficients of $\boldsymbol{u}_m(t)$ and $\widehat{\varphi}_m(t)$ respect to $\boldsymbol{V}^m$ and $\boldsymbol{W}^m$ respectively, being $(\boldsymbol{u}_m(t), \widehat{\varphi}_m(t))$ the (unique) approximate solution of (42) corresponding to the initial data $(\boldsymbol{u}_0^m, \varphi_0^m)$.

Now, varying the initial data $(\boldsymbol{u}_0^m, \varphi_0^m)$, we are going to define a new map $\Phi^m : \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}^m \times \mathbb{R}^m$ as follows: given $L_0^m \in \mathbb{R}^m \times \mathbb{R}^m$, we define $\Phi^m(L_0^m) = L^m(T)$, where $L^m(t)$ is related to the solution of problem (42) with initial data $L_0^m (= L^m(0))$.

By uniqueness of approximate solution of (42), this map is well-defined. Moreover, using regularity of the corresponding ordinary differential system (equivalent to (42)), this map is continuous.

In order to prove existence of fixed point of Φ^m, we will use Leray-Schauder's Theorem. Consequently, we have to prove that for all $\lambda \in [0,1]$, solutions $L_0^m(\lambda)$ of

$$L_0^m(\lambda) = \lambda \Phi^m(L_0^m(\lambda))$$

are uniformly bounded (independent of λ). Since $L_0^m(0) = \{0\}$, it is suffices to analyze $\lambda \in (0,1]$ and the equation

$$\frac{1}{\lambda} L_0^m(\lambda) = \Phi^m(L_0^m(\lambda)).$$

Since we have considered the eigenfunctions of Δ^2 to furnish W^m and (45), it is easy to justify the computations of lemma 3, lemma 4 and Corollary 6, in order to arrive at (35). Considering the norm $\|L^m(t)\|_{\mathbf{R}^m \times \mathbf{R}^m} = \big(\|\boldsymbol{u}_m(t)\|_{L^2}^2 + \|\Delta\widehat{\varphi}_m(t)\|_{L^2}^2\big)^{1/2}$ in $\mathbb{R}^m \times \mathbb{R}^m$, inequality (35) yields

$$\|\frac{1}{\lambda} L_0^m(\lambda)\|_{\mathbf{R}^m \times \mathbf{R}^m}^2 \leq e^{-C_0 T} \|L_0^m(\lambda)\|_{\mathbf{R}^m \times \mathbf{R}^m}^2 + C(1 - e^{C_0 T}).$$

Since $\lambda \in (0,1]$, one has

$$\|L_0^m(\lambda)\|_{\mathbf{R}^m \times \mathbf{R}^m}^2 \leq \frac{C(1 - e^{C_0 T})}{e^{C_0 T} - 1}$$

which is a bound independent of λ (and m). Consequently, Leray-Schauder Theorem implies the existence of fixed point of Φ^m, and therefore the existence of time-periodic Galerkin solutions.

Moreover, for each time-periodic Galerkin solution $(\boldsymbol{u}_m, \varphi_m)$, their corresponding initial-end data $(\boldsymbol{u}_m(0), \varphi_m(0)) = (\boldsymbol{u}_m(T), \varphi_m(T))$ is bounded in the $L^2 \times H^2$-norm, i.e

$$\|(\boldsymbol{u}_m, \widehat{\varphi}_m)(0)\|_{L^2 \times H^2} \leq C \quad (C \text{ independent of } m).$$

Step 5: Pass to the limit in time-periodic Galerkin solutions

The pass to the limit in variational formulation (42) can be done using estimations (independents of m) and compactness obtained in order to control nonlinear terms. Consequently, here we will only write the pass to the limit in time-periodic conditions.

From estimations of (φ_m) in $L^\infty(H^2)$ and $(\partial_t \varphi_m)$ in $L^2(L^{3/2})$ and using the triplet of spaces $H^2 \hookrightarrow H^1 \hookrightarrow L^{3/2}$, one has that (φ_m) is relatively compact in $C([0,T]; H^1)$, hence

$\varphi_m(T) \to \varphi_m(T)$ and $\varphi_m(0) \to \varphi(0)$ in $H^1(\Omega)$. Since $\varphi_m(T) = \varphi_m(0)$, then $\varphi(T) = \varphi(0)$ in $H^1(\Omega)$. Moreover, it is easy to see that $\varphi \in C_w([0,T]; H^2)$ (i.e. φ is continuous from $[0,T]$ onto H^2, respect to the weak topology in H^2), therefore $\varphi(T) = \varphi(0)$ in $H^2(\Omega)$. The argument for $\boldsymbol{u}$ is similar.

Consequently, we have found a weak time-periodic solution of problem (1)-(2), (4) and the proof of Theorem 12 is finished. ■

6 Regularity for the initial-value problem

The idea now is to obtain regularity for the weak solutions of the initial-value problem (1)-(3) (see [Lin,Liu'95], [Lin,Liu'00] for a nematic liquid crystal case and [Liu'00] for the smectic-A case, imposing time-independent boundary data in all these previous cases). In this sense, we will see that a global regularity result hold but only for the case of dominant viscosity, that is for ν big enough.

In our opinion, the global regularity imposing constraints of initial data near of special equilibrium solutions is an interesting problem, which up to our knowledge remains as an open problem.

Definition 13 *We say that a weak solution $(\boldsymbol{u}, \varphi)$ of (1)-(3) is a strong solution if*

$$\|(\boldsymbol{u}(t), \varphi(t))\|_{1\times 4} \le C_3 \quad \forall t \ge 0, \tag{48}$$

$$\forall \gamma > 0, \quad e^{-\gamma t} \int_0^t e^{\gamma s} \|(\boldsymbol{u}(s), \varphi(s))\|^2_{2\times 6}\, ds \le C_4, \quad \forall t \ge 0 \tag{49}$$

and verifying point-wise the fully differential system (1).

Theorem 14 *In the conditions of theorem 10, if moreover $(\boldsymbol{u}_0, \varphi_0) \in \boldsymbol{H}^1 \times H^4$ with $\|\boldsymbol{u}_0\|_1 \le R_1$, $\|\varphi_0\|_4 \le R_2$,*

$$\partial_t \widetilde{\varphi} \in L^\infty(0, +\infty; \boldsymbol{W}^{1,4}(\Omega)) \quad \text{and} \quad \partial_{tt} \widetilde{\varphi} \in L^\infty(0, +\infty; \boldsymbol{L}^2(\Omega)),$$

then for each $\nu \ge \nu_0$, with $\nu_0 = \nu_0(R_1, R_2, \partial_t \widetilde{\varphi}, \partial_{tt} \widetilde{\varphi})$, there exists a unique strong solution of (1)-(3) in $[0, +\infty)$, which verifies (48) and (49) with constants C_3 and C_4 depending on ν_0 (but independent of ν).

Proof. We define

$$\widehat{\omega} = -\partial_t \widehat{\varphi} - (\boldsymbol{u} \cdot \nabla) \widehat{\varphi}. \tag{50}$$

By owing to $\widehat{\varphi}|_\Sigma = 0$, $\nabla \widehat{\varphi}|_\Sigma = 0$ and $\boldsymbol{u}|_\Sigma = 0$, we have

$$\widehat{\omega}|_\Sigma = 0, \quad \nabla \widehat{\omega}|_\Sigma = 0. \tag{51}$$

On the other hand, we are going to obtain the following inequalities:

$$\|\varphi\|_4 \le C(|\widehat{\omega}|_2 + |\boldsymbol{u}|_2 + 1), \qquad \|\varphi\|_6 \le C(\|\widehat{\omega}\|_2 + \|\boldsymbol{u}\|_2 + 1). \tag{52}$$

Indeed, as $\omega = \Delta^2\varphi - \nabla\cdot\boldsymbol{f}(\nabla\varphi) = -\partial_t\varphi - (\boldsymbol{u}\cdot\nabla)\varphi = \widehat{\omega} - \partial_t\widetilde{\varphi} - (\boldsymbol{u}\cdot\nabla)\widetilde{\varphi}$, one has

$$\Delta^2\widehat{\varphi} = \Delta^2\varphi = \omega + \nabla\cdot\boldsymbol{f}(\nabla\varphi) = \widehat{\omega} - \boldsymbol{u}\cdot\nabla\widetilde{\varphi} - \partial_t\widetilde{\varphi} + \nabla\cdot\boldsymbol{f}(\nabla\varphi). \tag{53}$$

Hence

$$\|\widehat{\varphi}\|_4 \le |\widehat{\omega}|_2 + |\nabla\widetilde{\varphi}|_\infty|\boldsymbol{u}|_2 + |\partial_t\widetilde{\varphi}|_2 + |\nabla\cdot\boldsymbol{f}(\nabla\varphi)|_2.$$

Proceeding in the analogous way that in (47) to bound the term $|\nabla\cdot\boldsymbol{f}(\nabla\varphi)|_2$ and using the regularity of $\widetilde{\varphi}$ one arrives at the bound of $\|\widehat{\varphi}\|_4$ given in (52). The bound for $\|\widehat{\varphi}\|_6$ given in (52) can be obtained in a similar way.

Notice that,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\widehat{\omega}|_2^2 &= (\widehat{\omega}, \partial_t\widehat{\omega}) = (\widehat{\omega}, \partial_t(\omega + \boldsymbol{u}\cdot\nabla\widetilde{\varphi} + \partial_t\widetilde{\varphi}))\\
&= (\widehat{\omega}, \partial_t(\Delta^2\widehat{\varphi} - \nabla\cdot\boldsymbol{f}(\nabla\varphi) + \boldsymbol{u}\cdot\nabla\widetilde{\varphi} + \partial_t\widetilde{\varphi}))\\
&= (\Delta\widehat{\omega}, \partial_t\Delta\widehat{\varphi}) + (\nabla\widehat{\omega}, \partial_t\boldsymbol{f}(\nabla\varphi)) + (\widehat{\omega}, \partial_t\boldsymbol{u}\cdot\nabla\widetilde{\varphi} + \boldsymbol{u}\cdot\partial_t\nabla\widetilde{\varphi} + \partial_{tt}\widetilde{\varphi}).
\end{aligned}$$

By using (50), one has

$$\partial_t\Delta\widehat{\varphi} = -\Delta\widehat{\omega} - \Delta((\boldsymbol{u}\cdot\nabla)\widehat{\varphi}) = -\Delta\widehat{\omega} - \nabla^2\boldsymbol{u}\nabla\widehat{\varphi} - \nabla\boldsymbol{u}\nabla^2\widehat{\varphi} - (\boldsymbol{u}\cdot\nabla)\Delta\widehat{\varphi}$$

and

$$\begin{aligned}
\partial_t\boldsymbol{f}(\nabla\varphi) = (3|\nabla\varphi|^2 - 1)\partial_t\nabla\varphi &= (3|\nabla\varphi|^2 - 1)(-\nabla\widehat{\omega} - \nabla((\boldsymbol{u}\cdot\nabla)\widehat{\varphi}) + \nabla\partial_t\widetilde{\varphi})\\
&= (3|\nabla\varphi|^2 - 1)(-\nabla\widehat{\omega} - \nabla\boldsymbol{u}\nabla\widehat{\varphi} - (\boldsymbol{u}\cdot\nabla)\nabla\widehat{\varphi} + \nabla\partial_t\widetilde{\varphi}),
\end{aligned}.$$

Therefore, we obtain that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\widehat{\omega}|_2^2 + |\Delta\widehat{\omega}|_2^2 &= -(\Delta\widehat{\omega}, \nabla^2\boldsymbol{u}\nabla\widehat{\varphi}) - (\Delta\widehat{\omega}, \nabla\boldsymbol{u}\nabla^2\widehat{\varphi}) - (\Delta\widehat{\omega}, (\boldsymbol{u}\cdot\nabla)\Delta\widehat{\varphi}) - (\nabla\widehat{\omega}, 3|\nabla\varphi|^2\nabla\widehat{\omega})\\
&-(\nabla\widehat{\omega}, 3|\nabla\varphi|^2\nabla\boldsymbol{u}\nabla\widehat{\varphi}) - (\nabla\widehat{\omega}, 3|\nabla\varphi|^2(\boldsymbol{u}\cdot\nabla)\nabla\widehat{\varphi})) + (\nabla\widehat{\omega}, \nabla\widehat{\omega}) + (\nabla\widehat{\omega}, \nabla\boldsymbol{u}\nabla\widehat{\varphi})\\
&+(\nabla\widehat{\omega}, (\boldsymbol{u}\cdot\nabla)\nabla\widehat{\varphi})) + (\widehat{\omega}, \partial_t\boldsymbol{u}\cdot\nabla\widetilde{\varphi}) + (\widehat{\omega}, \boldsymbol{u}\cdot\partial_t\nabla\widetilde{\varphi}) + (\widehat{\omega}, \partial_{tt}\widetilde{\varphi}) + (\nabla\widehat{\omega}, (3|\nabla\varphi|^2 - 1)\partial_t\nabla\widetilde{\varphi}).
\end{aligned}$$

By bounding the terms on the right hand side of previous equality one arrives at

$$\frac{d}{dt}|\widehat{\omega}|_2^2 + \|\widehat{\omega}\|_2^2 \le \frac{\nu}{2}\|\boldsymbol{u}\|_2^2 + \frac{1}{2}|\partial_t\boldsymbol{u}|_2^2 + \frac{C}{\nu}\|\widehat{\omega}\|_2^2(1 + \|\widehat{\varphi}\|_3 + \|\widehat{\varphi}\|_3^2) + C|\widehat{\omega}|_2^2 + C, \tag{54}$$

where $C > 0$ may denote different constants, always independent of ν.

On the other hand, taking $A\boldsymbol{u}+\partial_t\boldsymbol{u}$ as test functions in the $\boldsymbol{u}$-system (A being the Stokes operator, i.e. $A=-P\Delta$ with P the Leray projector onto $\boldsymbol{H}$) it is easy to obtain

$$\frac{d}{dt}((\nu+1)\|\boldsymbol{u}\|_1^2)+\nu\|\boldsymbol{u}\|_2^2+|\partial_t\boldsymbol{u}|_2^2\le\frac{1}{2}\|\widehat{\omega}\|_2^2+\left(C+\frac{C}{\nu}\right)\|\boldsymbol{u}\|_1\|\boldsymbol{u}\|_2^2$$
$$+\left(C+\frac{C}{\nu}\right)(|\widehat{\omega}|_2^2+\|\boldsymbol{u}\|_1)+\frac{C}{\nu}\left(\|\widehat{\varphi}\|_3^4+\|\widehat{\varphi}\|_3^2+1\right)\|\boldsymbol{u}\|_2^2,$$

(the last term on the right hand side of previous inequality is a bound of $|\nabla\cdot\sigma_{nl}^d|_2^2$). Since we want to choose ν big enough, for instance we assume $\nu_0\ge 1$. Then, for each $\nu>\nu_0\ge 1$, we get

$$\frac{d}{dt}((\nu+1)\|\boldsymbol{u}\|_1^2)+\nu\|\boldsymbol{u}\|_2^2+|\partial_t\boldsymbol{u}|_2^2\le\frac{1}{2}\|\widehat{\omega}\|_2^2+C\left(\|\boldsymbol{u}\|_1\|\boldsymbol{u}\|_2^2+|\widehat{\omega}|_2^2+\|\boldsymbol{u}\|_1\right)$$
$$+\frac{C}{\nu}\left(\|\widehat{\varphi}\|_3^4+\|\widehat{\varphi}\|_3^2+1\right)\|\boldsymbol{u}\|_2^2. \tag{55}$$

Adding (54) and (55) we have

$$\frac{d}{dt}((\nu+1)\|\boldsymbol{u}\|_1^2+|\widehat{\omega}|_2^2)+\frac{\nu}{2}\|\boldsymbol{u}\|_2^2+\frac{1}{2}|\partial_t\boldsymbol{u}|_2^2+\frac{1}{2}\|\widehat{\omega}\|_2^2\le\frac{C}{\nu}\|\widehat{\omega}\|_2^2(1+\|\widehat{\varphi}\|_3+\|\widehat{\varphi}\|_3^2)$$
$$+D(\|\boldsymbol{u}\|_1\|\boldsymbol{u}\|_2^2+|\widehat{\omega}|_2^2+\|\boldsymbol{u}\|_1)+\frac{C}{\nu}\left(\|\widehat{\varphi}\|_3^4+\|\widehat{\varphi}\|_3^2+1\right)\|\boldsymbol{u}\|_2^2+E \tag{56}$$

where C, D and E are constants independent of $\nu\ge 1$. On the other hand, using (52), the regularity of $\boldsymbol{u}$, $\widehat{\varphi}$, $\widetilde{\varphi}$ and the interpolation inequality $\|\widehat{\varphi}\|_3\le C\|\widehat{\varphi}\|_2^{1/2}\|\widehat{\varphi}\|_4^{1/2}$ we get $\|\widehat{\varphi}\|_3\le C(1+|\widehat{\omega}|_2^{1/2})$. Hence, from (56) one has the following inequality:

$$\frac{d}{dt}((\nu+1)\|\boldsymbol{u}\|_1^2+|\widehat{\omega}|_2^2)+\frac{\nu}{2}\|\boldsymbol{u}\|_2^2+\frac{1}{2}|\partial_t\boldsymbol{u}|_2^2+\frac{1}{2}\|\widehat{\omega}\|_2^2\le\frac{C}{\nu}\|\widehat{\omega}\|_2^2(1+|\widehat{\omega}|_2^{1/2}+|\widehat{\omega}|_2)$$
$$+D(\|\boldsymbol{u}\|_1\|\boldsymbol{u}\|_2^2+|\widehat{\omega}|_2^2+\|\boldsymbol{u}\|_1)+\frac{C}{\nu}\left(|\widehat{\omega}|_3^2+|\widehat{\omega}|_3+1\right)\|\boldsymbol{u}\|_2^2+E. \tag{57}$$

If we denote

$$\Phi_1(t)=\|\boldsymbol{u}\|_1^2,\qquad \Phi_2(t)=|\widehat{\omega}|_2^2,\qquad \Psi_1(t)=\|\boldsymbol{u}\|_2^2,\qquad \Psi_2(t)=\|\widehat{\omega}\|_2^2,$$

we obtain from (57),

$$\frac{d}{dt}((\nu+1)\Phi_1+\Phi_2)+\left(\frac{\nu}{2}-D\Phi_1^{1/2}-\frac{C}{\nu}\left(\Phi_2+\Phi_2^{1/2}+1\right)\right)\Psi_1$$
$$+\left(\frac{1}{2}-\frac{C}{\nu}(1+\Phi_2^{1/4}+\Phi_2^{1/2})\right)\Psi_2\le D(\Phi_2+\Phi_1^{1/2})+E. \tag{58}$$

Let R_1, R_2, M and $\nu_0\ge 1$ some positive constants that we will specify below, such that if $\Phi_1(0)\le R_1$ and $\Phi_2(0)\le R_2$, we will prove that

$$(\nu+1)\Phi_1(t)+\Phi_2(t)\le M\qquad\forall\, t\in[0,+\infty), \tag{59}$$

for any $\nu \geq \nu_0$. Indeed, by contradiction, let $t^* > 0$ the first value such that $(\nu+1)\Phi_1(t^*) + \Phi_2(t^*) = M$, hence

$$(\nu+1)\Phi_1(t^*) + \Phi_2(t^*) = M \quad \text{and} \quad (\nu+1)\Phi_1(t) + \Phi_2(t) < M \quad \forall\, t \in [0, t^*).$$

Then,

$$\Phi_1(t) \leq \frac{M}{\nu+1} \quad \text{and} \quad \Phi_2(t) \leq M \quad \forall\, t \in [0, t^*].$$

Assume that there exists ν_0 big enough such that, for each $\nu \geq \nu_0$

$$\frac{\nu}{2} - D\left(\frac{M}{\nu+1}\right)^{1/2} - \frac{C}{\nu}(M + M^{1/2} + 1) \geq \frac{\nu+1}{4}$$

and

$$\frac{1}{2} - \frac{C}{\nu}\left(1 + M^{1/4} + M^{1/2}\right) \geq \frac{1}{4}. \tag{60}$$

Then, for each $t \in [0, t^*]$

$$\frac{d}{dt}((\nu+1)\Phi_1 + \Phi_2) + \frac{\nu+1}{4}\Psi_1 + \frac{1}{4}\Psi_2 \leq D(\Phi_2 + \Phi_1^{1/2}) + E. \tag{61}$$

We define $P = \min\{P_1, P_2\}$ where $1/P_1$ and $1/P_2$ are the Poincaré constants that verify $\Phi_1 \leq \frac{1}{P_1}\Psi_1$ and $\Phi_2 \leq \frac{1}{P_2}\Psi_2$ respectively. Therefore,

$$\frac{d}{dt}((\nu+1)\Phi_1 + \Phi_2) + \frac{P}{4}((\nu+1)\Phi_1 + \Phi_2) \leq D(\Phi_2 + \Phi_1^{1/2}) + E. \tag{62}$$

Multiplying (62) by $e^{Pt/4}$ and integrating in $[0, t^*]$ we deduce

$$\begin{aligned}(\nu+1)\Phi_1(t^*) + \Phi_2(t^*) &\leq ((\nu+1)\Phi_1(0) + \Phi_2(0))e^{-Pt^*/4} \\ &+ e^{-Pt^*/4}\int_0^{t^*}(D(\Phi_2(s) + \Phi_1^{1/2}(s)) + E)e^{Ps/4}\,ds.\end{aligned} \tag{63}$$

By (53), $\widehat{\omega} = \Delta^2\widehat{\varphi} + \boldsymbol{u}\cdot\nabla\widetilde{\varphi} + \partial_t\widetilde{\varphi} - \nabla\cdot\boldsymbol{f}(\nabla\varphi)$, hence we get

$$\Phi_2 = |\widehat{\omega}|_2^2 \leq C(\|\widehat{\varphi}\|_4^2 + |\boldsymbol{u}|^2 + 1 + \|\widehat{\varphi}\|_4) \leq C(\|\widehat{\varphi}\|_4^2 + |\boldsymbol{u}|^2 + 1).$$

Therefore, taking into account weak estimates (31), the second term on the right hand side of (63) is bounded by a constant C_w independent of ν (in fact, C_w depends on the constant C_2 given in (31)) and

$$(\nu+1)\Phi_1(t^*) + \Phi_2(t^*) \leq ((\nu+1)\Phi_1(0) + \Phi_2(0)) + C_w\left(\frac{1}{\nu} + 1\right) \leq ((\nu+1)R_1 + R_2) + 2C_w.$$

Hence, if we choose

$$M > (\nu+1)R_1 + R_2 + 2C_w, \tag{64}$$

then we arrives at a contradiction. Therefore, we could get the estimate (59) whether there exists big enough constants M and ν_0 such that (60) and (64) hold, for each $\nu \geq \nu_0$. Indeed, if we choose $M = \lambda\,\nu$ then (64) holds for any $\lambda > 2R_1 + R_2 + 2C_w$. If we fix λ with this condition, then the two conditions given in (60) hold if

$$\frac{\lambda^{1/2}}{\nu} \leq \varepsilon \quad \text{and} \quad \frac{1}{\nu}(1 + \lambda^{1/4}\nu^{1/4} + \lambda^{1/2}\nu^{1/2}) \leq \varepsilon$$

for $\varepsilon > 0$ small enough. But these conditions hold for each $\nu \geq \nu_0$ with ν_0 big enough respect to λ. Therefore, we get estimates (59).

From (59), we obtain $\boldsymbol{u} \in L^\infty(0, +\infty; \boldsymbol{H}^1)$ and $\widehat{\omega} \in L^\infty(0, +\infty; \boldsymbol{L}^2)$. Recalling (52) we also obtain $\varphi \in L^\infty(0, +\infty; \boldsymbol{H}^4)$. By going back to (61), multiplying by $e^{\gamma t}$ for any $\gamma > 0$ and integrating in $[0, t]$ we deduce

$$\forall\,\gamma > 0, \quad e^{-\gamma t} \int_0^t e^{\gamma s} \|(\boldsymbol{u}(s), \widehat{\omega}(s))\|^2_{2\times 2}\, ds \leq C_2, \quad \forall\, t \geq 0.$$

Again, by applying (52) we get (49). ∎

7 Regularity for the time-periodic problem

The results obtained up to now allow us to obtain, for big enough ν, the regularity given in Definition 13 also for the time-periodic problem. Indeed, arguing as in [Climent et al.] for a nematic crystal model, to prove that weak time-periodic solution is regular it suffices to use the following three results:

1. the existence of the weak time-periodic solution (proved in Section 4),
2. the weak/strong uniqueness of the initial-valued problem (proved in Section 3),
3. the existence of global strong solution for big enough viscosity of the initial-valued problem (proved in Section 6).

Consequently, the regularity for time-periodic solutions can be deduced.

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