Information Processing Letters

Monochromatic geometric k-factors for bicolored point sets with auxiliary points ☆

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A B S T R A C T

Given a bicolored point set S, it is not always possible to construct a monochromatic geometric planar k-factor of S. We consider the problem of finding such a k-factor of S by using auxiliary points. Two types are considered: white points whose position is fixed, and Steiner points which have no fixed position. Our approach provides algorithms for constructing those k-factors, and gives bounds on the number of auxiliary points needed to draw a monochromatic geometric planar k-factor of S.

Keywords:
Computational geometry
Red–blue point sets
k-factors
Steiner points

1. Introduction

Let S be a set of n points in the plane such that each point is colored red or blue. Let R be the set of red points and B the set of blue points. Assume that the points of S are in general position, that is, no three of them lie in the same line. We say that a planar k-regular graph is a *planar k-factor* of S if its vertices are the points of S. If the edges are straight-line segments then we have a *geometric planar k-factor* of S. Observe that a *geometric perfect matching* of S is a geometric planar 1-factor. Note also that since no 6-regular graph is planar, then geometric planar k-factors can only exist for $1 \leqslant k \leqslant 5$. A geometric planar k-factor of $S = R \cup B$ is *monochromatic* if its edges join points of the same color. For simplicity, we shall write k-factor instead of monochromatic geometric planar k-factor when no confusion can arise.

This paper addresses a general study of monochromatic geometric planar k-factors, which as far as we know have not been yet studied for $k > 1$. Indeed, only the case $k = 1$ has been considered in the literature (see [4]). Since it is easy to give configurations of red and blue points for which it is not possible to construct a 1-factor (and in general, a k-factor), then the problem in this case turns to find a largest non-crossing geometric matching of pairs of points of the same color (see [1] for a recent paper). Dumitrescu and Kaye [4] provided an $O(n^2)$-time algorithm that computes a matching of at least 85.71% of the points. Within this context, it seems natural to ask whether the situation can be modified by using auxiliary points, that is, points that do not belong to S. In this paper, we consider two types of auxiliary points: (1) *white points*, which are given as part of the input, their position is fixed; (2) *Steiner points*, which have no fixed position and they are inserted only when they have to be used, not being part of the input. Both Steiner and white points have no color assigned until they are matched with a red or blue point, inheriting its color.

☆ A preliminary version of this paper appeared in [6].

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Once we have studied the case $k = 1$, we go further providing a first approach to the problem of computing monochromatic geometric planar k-factors with $2 \leqslant k \leqslant 5$, for bicolored point sets, also using both types of auxiliary points.

Our main goal is to minimize the number of auxiliary points, and in case of combining both types we shall give preference to white points since, unlike Steiner points, they are part of the input and the use of all of them is not necessary to construct a k-factor. As an analogy, suppose that the red and blue data are two types of installations that have to be joined giving rise to a k-factor. White points play the role of installations already constructed which can easily be adapted to a red or blue use, and Steiner points correspond to new installations that have to be built. From this point of view, Steiner points are much more expensive than white ones, and they should be avoided if it is possible.

Steiner points have been used in several problems in computational geometry [5,7], but white points have not been considered in general (see [12] where they are called Steiner points with fixed position). Note that if no restriction is imposed on the number of Steiner points, it suffices to add a large enough number around every point of S to construct a k-factor. Nevertheless, we prove that there are configurations of points that do not admit a planar k-factor independently of the number of white points used.

In all the figures, red, blue, and white points are depicted as black, gray, and white points, respectively; Steiner points are represented by squares.

2. Monochromatic perfect matchings (1-factors)

Given a bicolored point set S, it is not always possible to obtain a monochromatic perfect matching of S, so it is necessary to use auxiliary points; either Steiner points, or white points, or both. The aim of this section is to provide bounds on the number of auxiliary points that are sufficient to guarantee the existence of a perfect matching of S. A key result within this context is the following theorem.

Theorem 2.1. *(See [4,9].)*

(i) *There exists a monochromatic matching which covers at least $\frac{6}{7}n + O(1)$ points of $S = R \cup B$, and such a matching can be found in $O(n^2)$ time.*

(ii) *There exists a point configuration $R \cup B$ for which every monochromatic geometric matching covers at most $\frac{94}{95}n + O(1)$ points.*

Remark 2.2. The $O(n^2)$-time algorithm provided in [4] can be slightly modified in order to obtain a monochromatic perfect matching of S, by adding a Steiner point close to every point of S that has not been matched. Thus, $n/7$ Steiner points suffice to construct the matching. This modified process will be called Steiner-Matching.

We now recall a result that plays an important role throughout this paper.

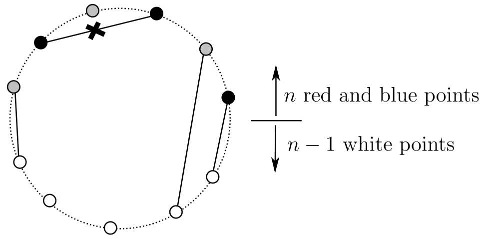

Fig. 1. Any segment joining a pair of, say, red points isolates an odd number of colored points and so $n-1$ white points do not suffice to obtain a monochromatic perfect matching.

Theorem 2.3 *(Equitable Subdivision Theorem). (See [2,8,11].) Given integers $a \geqslant 1$, $b \geqslant 1$ and $g \geqslant 2$, if R contains ag red points and B contains bg blue points, then there exists a subdivision $X_1 \cup X_2 \cup \cdots \cup X_g$ of the plane into g disjoint convex polygons such that every X_i contains exactly a red points and b blue points.*

An equitable subdivision of the plane can be found in $O(n^{4/3} \log^3 n \log g)$ time [2], where $n = (a + b)g$. Consider now a set of n white points, a set of n colored points (red plus blue), and take $a = 1$, $b = 1$ and $g = n$. By Theorem 2.3 (where R and B are the sets of n white points and n colored points, respectively) there are only one white point and one colored point inside every convex polygon. These points can be matched giving rise to a planar monochromatic matching, once the white points take the proper color. This procedure will be called Bichro-Matching. Moreover, Fig. 1 illustrates a configuration of points that requires to add n white points to obtain a monochromatic perfect matching. This implies the following theorem.

Theorem 2.4. *Let $S = R \cup B$. A set of n white points are always sufficient and sometimes necessary to obtain a monochromatic perfect matching of S, which can be found in $O(n^{4/3} \log^4 n)$ time.*

A way of constructing a perfect matching when there are fewer than n white points is to add Steiner points. This can be done as follows.

Let S_W be a set of n red and blue points and $n - m$ white points, and consider its convex hull $\mathrm{CH}(S_W) = \{p_0, \ldots, p_h\}$ sorted clockwise. First, check whether there exist two consecutive points $p_i, p_{i+1} \in \mathrm{CH}(S_W)$, either with the same color or a colored point and a white point. If so, match them, update $\mathrm{CH}(S_W)$ to $\mathrm{CH}(S_W \setminus \{p_i, p_{i+1}\})$, and search for another pair. Once all pairs p_i, p_{i+1} have been matched, we obtain a set S'_W that contains the remaining n' colored points and s white points, and such that $\mathrm{CH}(S'_W)$ has only white points or consists of alternating red and blue points (note that it might be $S_W = S'_W$).

Now, if $\mathrm{CH}(S'_W)$ has only white points, rotate clockwise a line anchored on a point $p'_i \in \mathrm{CH}(S'_W)$ starting at p'_{i+1} in order to compute a set D with the same number of white and colored points. Observe that if $s > n'$, our set D might not exist: there could be a set containing the n' colored points and a number $\ell > n'$ of white points; to obtain D it suffices to color an adequate number of white points.

Then, procedure Bichro-Matching gives a matching of D. Finally, update $CH(S'_w)$ by removing the set D.

When $CH(S'_w)$ consists of alternating red and blue points, rotate clockwise the line anchored on p'_i starting at p'_{i+1} until finding a point $q \in S'_w$. If q and p'_{i+1} have the same color or q is white, match them and update $CH(S'_w)$ to $CH(S'_w \setminus \{p'_{i+1}, q\})$. Otherwise, q is matched with p'_i and p'_{i+1} with a Steiner point; update $CH(S'_w)$ to $CH(S'_w \setminus \{p'_i, p'_{i+1}, q\})$.

The process concludes when either there are no white points left and we can apply procedure Steiner-Matching (here we include the case in which there are only white points left; they can simply be colored), or there is the same number of white and colored points and procedure Bichro-Matching can be applied.

The above described procedure, called White-Steiner-Matching, implies the following theorem.

Theorem 2.5. *Given a set $S = R \cup B$ of n colored points, and a set of $n - m$ white points, at most $m/3$ Steiner points suffice to obtain a monochromatic perfect matching of S. It is possible to construct such a matching in $O(m^2) + O(n^{\frac{4}{3}} \log^4 n)$ time.*

Proof. Consider the set S_w that is the union of S and the $n - m$ white points. By Remark 2.2 and Theorem 2.4, procedure White-Steiner-Matching gives a monochromatic perfect matching of S using white and Steiner points. Observe that at each step, any matched pair can be separated from the remaining points of S_w by a line, avoiding crossings with the segments added in the next steps of the process.

After removing all pairs p_i, p_{i+1} from $CH(S_w)$ and obtained the set S'_w, Steiner points are used in two situations: either $CH(S'_w)$ consists of alternating red and blue points, or there are no white points left. In the former case, each Steiner point is used to match three colored points. Since $n - m$ colored points are matched using white points, then $m/3$ Steiner points might be needed. In the latter case, the process calls procedure Steiner-Matching, using one Steiner point for every seven colored points.

Analysis of procedure White-Steiner-Matching: the process searches through the convex hull for a suitable pair, matches, removes it from $CH(S_w)$, and updates the convex hull. Updating $CH(S_w)$ can be done in $O(\log n)$ time [3]. If $CH(S_w)$ has only white points, the process calls procedure Bichro-Matching on subsets D_i of n_i points, and $\sum_i n_i \leqslant 2n$. By Theorem 2.4, this can be done in $\sum_i O(n_i^{4/3} \log^4 n_i)$ which is $O(n^{4/3} \log^4 n)$. Finally, procedure Steiner-Matching is applied at most once and takes $O(m^2)$ time. □

3. Monochromatic *k*-factors

In light of the previous discussion, we now provide a first approach to the problem of finding a k-factor, $2 \leqslant k \leqslant 5$, using auxiliary points. The main tool to construct k-factors (for every k) using Steiner points is given by the following theorem.

Theorem 3.1. *Let $S = R \cup B$. It is possible to construct two monochromatic non-crossing Hamiltonian cycles on R and B, respectively, in $O(n \log n)$ time using at most $\lfloor \frac{n}{2} \rfloor$ Steiner points. Moreover, the bound on the number of Steiner points is tight.*

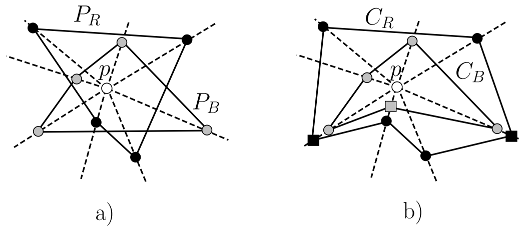

Fig. 2. (a) Polygons P_R and R_B, (b) cycles C_R and C_B.

Proof. If the given sets R and B of red and blue points are line separable, then it is trivial to obtain two monochromatic cycles. Otherwise, we proceed as follows: Find a point $p \in CH(R) \cap CH(B)$ and let P_R (resp. P_B) be the polygon resulting from joining the points of R (resp. B) following the angular order given from p (see Fig. 2(a)). Denote by $I_R O_B$ the set of blue points inside P_R plus the red points outside P_B. Analogously, let $I_B O_R$ be the set of red points inside P_B plus the blue points outside P_R.

Consider now the polar coordinate system centered at p. If $|I_B O_R| \leqslant |I_R O_B|$ (analogous for $|I_R O_B| \leqslant |I_B O_R|$), then for every blue point $b = (\theta, \rho)$ (θ being the angle and ρ the distance from p) in $I_B O_R$ add a red Steiner point $s_r = (\theta, \rho + \varepsilon)$, and for every red point $r = (\theta', \rho')$ in $I_B O_R$ add a blue Steiner point $s_b = (\theta', \rho' - \varepsilon)$. Joining the red points and the blue points in the angular order from p, we obtain two monochromatic non-crossing cycles C_R and C_B (Fig. 2(b)). This procedure will be called Steiner-cycles.

Since $I_R O_B$ and $I_B O_R$ are disjoint and $|I_R O_B \cup I_B O_R| = n$, at most $\lfloor \frac{n}{2} \rfloor$ Steiner points are needed to construct C_R and C_B.

We now prove that the bound is tight. To do this, we show that a set $S = R \cup B$ of n alternating red and blue points in convex position always uses at least $\lfloor \frac{n}{2} \rfloor$ Steiner points. Since the two monochromatic non-crossing cycles can always be constructed using at most $\lfloor \frac{n}{2} \rfloor$ Steiner points, then the result follows.

Let C_B be a non-crossing Hamiltonian cycle on B. Clearly, every edge e of C_B splits the set R into two independent subsets, i.e., it is not possible to match two red points located on opposite sides of e. Thus, at least one Steiner point is required to connect both subsets in order to construct a monochromatic Hamiltonian cycle C_R on R. Note that since every point in the resulting graph must have degree two, each Steiner point can be used to join only two such subsets.

Now, take an edge $b_i b_{i+1} \in E(C_B)$ and assume $b_i \in B$. Then, the point b_{i+1} is either a Steiner point or belongs to B. The latter case implies, by the argument above, the use of at least one red Steiner point to construct C_R. Therefore, we are using at least one Steiner point (either red or blue) per each edge of C_B which implies at least $\lfloor \frac{n}{2} \rfloor$ Steiner points in total.

Procedure Steiner-cycles takes $O(n)$ time to check whether R and B are line separable [10]. Then, it computes the convex hulls and sorts the points to construct P_R and P_B in $O(n \log n)$ time. Also, in $O(n \log n)$ time, the procedure determines the blue (resp. red) points that are exterior and interior to P_R (res. P_B). Finally, it traverses both polygons adding Steiner points obtaining C_R and C_B in $O(n)$ time. □

The output of procedure Steiner-cycles is a 2-factor of S consisting of two monochromatic non-crossing cycles containing at most $\lfloor \frac{n}{2} \rfloor$ Steiner points. Nevertheless, it is possible to reduce the number of Steiner points used by decreasing the connectivity of the resulting graph. This can be done as follows.

Split S into subsets of m points, for m odd and $5 \leqslant m < n$, sorting by x-coordinate (assuming without loss of generality, that no two points have the same x-coordinate) and apply procedure Steiner-cycles to each group. Since consecutive subsets of m points are in disjoint vertical strips, and Steiner points can be added arbitrarily close to the points of S, there are no intersections between the 2-factors constructed in different steps of the procedure. Thus, procedure Steiner-cycles uses at most $\lfloor \frac{m}{2} \rfloor$ Steiner points for each group of m points of S. Since m is odd, then for every group of m points we use $\frac{m-1}{2}$ Steiner points and so the total number is $\frac{m-1}{2}\frac{n}{m}$.

Observe that the conditions $m \geqslant 5$ and m odd are necessary since for groups of $m < 5$ points, the number of Steiner points needed to construct the cycles is much higher, and m even gives rise to $\frac{n}{2}$ Steiner points.

It is easy to check that $m = 5$ is the value that gives the best performance, obtaining $\frac{2n}{5}$ Steiner points. When the cardinality of S is not divisible by five, we have to add at most four Steiner points to the quantity above to construct a 2-factor with the last points of the set.

Thus, from the discussion above we have the following theorem.

Theorem 3.2. $\frac{2}{5}n + 4$ *Steiner points suffice to construct a* 2*-factor of* S*, which can be found in* $O(n \log n)$ *time.*

The ideas of Theorems 3.1 and 3.2 are now extended to 3-, 4-, and 5-factors.

Theorem 3.3. *Let* $S = R \cup B$*. Then* $n + 4$*,* $2n$ *and* $5n$ *Steiner points suffice to construct two monochromatic non-crossing* 3-*,* 4- *and* 5-*regular graphs, respectively. Moreover, there exists an algorithm which finds such graphs in* $O(n \log n)$ *time.*

Proof. Suppose first that R and B are not line-separable and apply procedure Steiner-cycles obtaining two nested monochromatic cycles C_R and C_B. Recall that the cycles are constructed by sorting the points of R and B angularly from a point $p \in \mathrm{CH}(R) \cap \mathrm{CH}(B)$, adding Steiner points when necessary. Assume that C_R is the inner cycle (the process is analogous for the outer cycle C_B).

To obtain a 3-factor, split the points of C_R into consecutive triplets with respect to the order given by p. Add a Steiner point close to the midpoint of each triplet on the line that joins it with p, and also join all the points of every triplet to the corresponding Steiner point (see Fig. 3(a)). If $|C_R|$ is not divisible by 3, then there are one

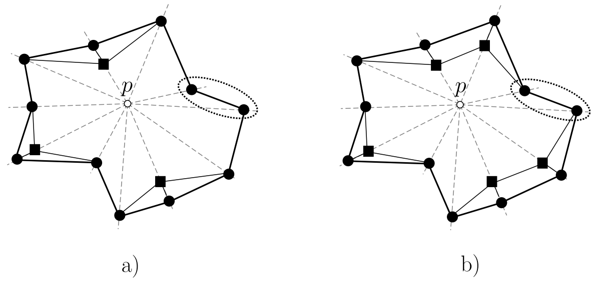

Fig. 3. (a) Three triplets with their respective Steiner points, and two points left. (b) Adding two Steiner points and new adjacencies in the graph of (a) to obtain a 3-factor.

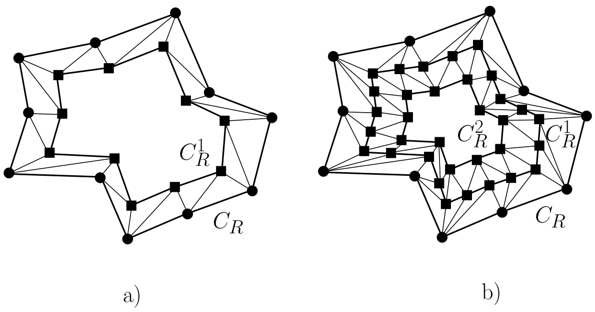

Fig. 4. (a) A 4-factor, (b) A 5-factor.

or two points left (Fig. 3(a)) for which extra Steiner points can be added as follows.

Suppose that we have two points left (similar for one) and consider the previous and next points, in the angular order, to our consecutive points. Add a Steiner point for each of them on the line that joins them with p and also add the adjacencies shown in Fig. 3(b) to obtain degree three. Note that since the points are angularly sorted around a point inside our resulting graph, there are no crossings. Observe also that for the outer cycle, the points are added close to the midpoint of each triplet *outside* the cycle. Thus, we use at most $\frac{|C_R|+|C_B|}{3} + 4 \leqslant \frac{1}{3}(n + \lfloor \frac{n}{2} \rfloor) + 4$ Steiner points plus the at most $\lfloor \frac{n}{2} \rfloor$ Steiner points used to construct the monochromatic cycles C_R and C_B. Hence, the total number is at most $n + 4$.

A 4-factor is obtained by placing a copy of the inner cycle C_R (analogous for the outer cycle C_B), say C_R^1, inside (resp. outside) it and joining every point of C_R to two consecutive points of C_R^1 as it is illustrated in Fig. 4(a). Due to the angular construction of the cycles, every edge can be drawn without intersection, obtaining a 4-factor. This process adds one new Steiner point for every point of C_R and C_B since the copies C_R^1 and C_B^1 are formed only by Steiner points, i.e., at most $n + \lfloor \frac{n}{2} \rfloor$, plus at most $\lfloor \frac{n}{2} \rfloor$ to construct the cycles. Thus, the total number of Steiner points is at most $2n$.

Finally, a 5-factor can be obtained from the previous 4-factor as follows (see Fig. 4(b)). Add one Steiner point s_i in each edge e_i of C_R^1, draw a copy of C_R inside C_R^1, and call it C_R^2. Join s_i to three vertices: the common neighbor of the endpoints of e_i in C_R and the endpoints of the copy of e_i in C_R^2. Also join the two endpoints of e_i to the endpoints of the copy of e_i in C_R^2. Proceed analogously for

Table 1
Summary of the obtained bounds.

	Steiner points	White points	Both
1-factors	$n/7$ [4,9]	n (tight)	$n-m$ white and $m/3$ Steiner
2-factors	$2n/5+O(1)$	$2n$ (tight)	$2n-m$ white and m Steiner
3-factors	$n+4$	N/A	N/A
4-factors	$9n/5+10$	N/A	N/A
5-factors	$23n/5+24$	N/A	N/A

the cycle C_B. Since the copies of C_R and C_B are formed by Steiner points, the total number of Steiner points used is $5n$.

Suppose now that R and B are line-separable. Then, we proceed as above independently for C_R and C_B considering the angular order given by points inside $\mathrm{CH}(R)$ and $\mathrm{CH}(B)$, respectively, as procedure STEINER-CYCLES describes. □

As in the case of 2-factors, we can reduce the number of Steiner points used to obtain 4- and 5-factors by decreasing the connectivity of the resulting graph. The process is analogous to the one described for 2-factors, obtaining again the best performance for $m=5$. The case $k=3$ cannot be improved using this method since the bound on the number of Steiner points is the same when taking the points in groups of m and when considering just the whole set S. Thus, the following bounds are obtained for 4- and 5-factors, where the constants come from the remaining points of S when S is not divisible by five.

Theorem 3.4. *Let $S=R\cup B$. Then $\frac{9}{5}n+10$ Steiner points suffice to construct a 4-factor of S, and $\frac{23}{5}n+24$ Steiner points suffice to construct a 5-factor of S. Moreover, there exists an algorithm which finds such factors in $O(n\log n)$ time.*

Another approach to the problem of constructing k-factors is to use white points (considered also in Section 2). We establish a tight bound for $k=2$, and show that this result cannot be extended to $k>2$ (Proposition 3.6) since there are point configurations for which it is not possible to obtain a k-factor independently of the color of the points.

It can be easily shown that $2n$ white points can be necessary to construct a 2-factor when no Steiner points are used. It suffices to consider the points on a circle, locating all the white points on one hemisphere and the colored points alternating in color in the other hemisphere. Any segment matching, say two red points, will eventually leave a blue point isolated. Hence, a 2-factor must be constructed by matching each colored point with two white ones. Note that the final 2-factor is a collection of triangles.

On the other hand, Theorem 2.3 guarantees that a k-factor can always be constructed by using $2n$ white points. Considering $a=1$, $b=2$, and $g=n$, we obtain n convex sets containing just one colored point and two white points. The union of these triangles is a 2-factor. Thus, we have the following theorem.

Theorem 3.5. *$2n$ white points are always sufficient and sometimes necessary to obtain a 2-factor of $S=R\cup B$. Such a 2-factor can be constructed in $O(n^{4/3}\log^4 n)$ time.*

When the number of white points is smaller that $2n$, say $2n-m$, it might be necessary to replacing the missing white points with m Steiner points. Observe that if $m>\frac{2}{5}n$, Theorem 3.2 implies that it is better to construct the 2-factor using only Steiner points.

Proposition 3.6. *For $k>2$ it is not possible to construct a k-factor of a point set in convex position.*

Proof. Let Ω be a set of points in convex position. Suppose on the contrary that there exists a k-factor $\mathcal{F}$ of the set Ω for $k>2$, and consider its dual graph $\mathcal{F}^*$. Let u be the vertex in $\mathcal{F}^*$ corresponding to the unbounded face of $\mathcal{F}$, and G the graph obtained by deleting vertex u in $\mathcal{F}^*$.

Since the points of Ω are in convex position, then every diagonal splits the point set into two parts. This implies that G has no cycles and so is a forest. A leaf in G corresponds to a face f in $\mathcal{F}$ that has a unique adjacent bounded face. Hence, all vertices in f but two have degree 2; a contradiction since $\mathcal{F}$ is k-regular and $k>2$. □

Remark 3.7. As mentioned, Theorem 3.5 cannot be extended to k-factors when $k>2$. Indeed, the above result says that given a set S of red and blue points and any number of white points, all of them in convex position, it is not possible to construct a k-factor and so we have configurations of points for which no k-factor can be obtained independently of the number of white points used.

One might try to use similar arguments as those used in Theorem 3.5 to construct 3-, 4-, and 5-factors using white points and also Steiner points: split the plane into convex regions containing one colored point and two white points and add 1, 3, or 9 Steiner points inside the triangle to obtain a 3-, 4-, or 5-factor, respectively. Nevertheless, the number of Steiner points that this procedure requires is greater than the number of Steiner points needed to construct the k-factor without considering white points (see Theorem 3.3).

4. Conclusion

Table 1 summarizes some of the bounds obtained in this paper.

As a consequence of Proposition 3.6 the study for 3-, 4- and 5-factors cannot be extended to white points or combinations of both types of auxiliary points. It would be interesting to find better strategies for combining Steiner and white points, and also to study if the bounds on Steiner points are tight.

Acknowledgements

We thank to two anonymous referees for their many useful suggestions and comments, which helped improve the paper substantially.

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