Discrete Applied Mathematics

On the metric dimension, the upper dimension and the resolving number of graphs

Delia Garijo *, Antonio González, Alberto Márquez

Dept. de Matemática Aplicada I, Universidad de Sevilla, Spain

A B S T R A C T

This paper deals with three resolving parameters: the metric dimension, the upper dimension and the resolving number. We first answer a question raised by Chartrand and Zhang asking for a characterization of the graphs with equal metric dimension and resolving number. We also solve in the affirmative a conjecture posed by Chartrand, Poisson and Zhang about the realization of the metric dimension and the upper dimension. Finally, we prove that no integer $a \geq 4$ is realizable as the resolving number of an infinite family of graphs.

Keywords:
Resolving set
Metric dimension
Upper dimension
Resolving number

1. Introduction

In this paper, we study *resolving sets* for finite simple connected graphs. They were introduced in the 1970s independently by Slater [9], and Harary and Melter [5]. The usefulness of these sets comes from their multiple applications in several areas, among them: coin weighing problems, network discovery and verification, robot navigation, strategies for Mastermind game and pharmaceutical chemistry (we refer the reader to [1] for a number of references on this topic). Resolving sets are formally defined as follows.

Let $G = (V(G), E(G))$ be a finite simple connected graph of order $n = |V(G)|$. The *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest u–v path in G. A vertex $u \in V(G)$ *resolves* a pair $\{x, y\} \subset V(G)$ if $d(u, x) \neq d(u, y)$. A set of vertices $S \subseteq V(G)$ is a *resolving set* of G if every pair of vertices of G is resolved by some vertex of S. A resolving set S of minimum size is a *metric basis*, and $|S|$ is the *metric dimension* of G, denoted by $\dim(G)$.

Our aim is not only to deal with metric bases and metric dimension but also with two other resolving parameters defined by Chartrand et al. [3], namely *the upper dimension* and *the resolving number*, that give an insight of how large the set of resolving sets of a graph is.

A resolving set S of G is *minimal* if no proper subset of S is a resolving set. An *upper basis* is a minimal resolving set containing the maximum number of vertices. The *upper dimension* $\dim^+(G)$ is the size of an upper basis. The *resolving number* $\mathrm{res}(G)$ is the minimum k such that every k-subset of $V(G)$ is a resolving set of G. For instance, $\dim^+(P_n) = \mathrm{res}(P_n) = 2$, $\dim^+(C_n) = 2$ and $\mathrm{res}(C_n) = 3$ where P_n and C_n denote, respectively, a path of order $n \geq 4$ and a cycle of even order $n \geq 4$ [3].

Clearly, every $(n-1)$-subset of $V(G)$ is a resolving set and every resolving set contains a minimal resolving set. Hence,

$$1 \leq \dim(G) \leq \dim^+(G) \leq \mathrm{res}(G) \leq n - 1.$$

* Corresponding author. Tel.: +34 954552795; fax: +34 954556683.
E-mail addresses: dgarijo@us.es (D. Garijo), gonzalezh@us.es (A. González), almar@us.es (A. Márquez).

When $\dim(G) = \mathrm{res}(G) = k$, the graph G is called *randomly k-dimensional*, i.e., every k-subset of $V(G)$ is a metric basis and so G has the maximum number of metric bases of a fixed size k.

Chartrand and Zhang [4] posed the problem of characterizing the randomly k-dimensional graphs. They solved the case $k \leq 2$, obtaining the complete graphs K_1 and K_2 (for $k = 1$) and odd cycles (for $k = 2$), and leaving open the main following question.

Problem 1.1 (*[4]*)**.** Are there randomly k-dimensional graphs other than complete graphs and odd cycles?

Concerning the three parameters, Chartrand et al. [3] investigated different questions related to graph realization. In particular, they proved that every pair a, b of integers with $2 \leq a \leq b$ is realizable as the metric dimension and the resolving number, respectively, of some connected graph G. It was also shown the analogous result for $\dim(G) = \dim^+(G) = a$ and $\mathrm{res}(G) = b$. Moreover, the authors proved that every pair among the three parameters can differ by an arbitrarily large number. Finally, it was remarked that there were reasons to believe that every pair a, b of integers with $2 \leq a \leq b$ is realizable as the metric dimension and the upper dimension, respectively, of some connected graph. Thus, they proposed the following conjecture.

Conjecture 1.2 (*[3]*)**.** *For every pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G with $\dim(G) = a$ and $\dim^+(G) = b$.*

In this paper, we solve Problem 1.1 (Theorem 2.5) and settle in the affirmative Conjecture 1.2 (Theorem 3.5). We also show that, unlike the metric dimension and the upper dimension, no integer $a \geq 4$ is realizable as the resolving number of an infinite family of graphs (Theorem 3.7).

2. Randomly k-dimensional graphs

In this section, we characterize the randomly k-dimensional graphs. While preparing this paper, we have learnt of [6], where the authors prove the same result. Here we present an alternative, shorter proof.

We start with some technical lemmas needed for the case $k = 3$. Denote by $\mathcal{P}_\lambda(G)$ the λ-subsets of $V(G)$ and let $N_i(u)$ be the set of vertices at distance i from $u \in V(G)$. For $\{u, v\}, \{x, y\} \in \mathcal{P}_2(G)$ we say that *the pair* $\{u, v\}$ *resolves the pair* $\{x, y\}$ if either u or v resolves it. In general, $T \in \mathcal{P}_\lambda(G)$ *resolves a pair* $\{x, y\}$ if there is a vertex in T that resolves it.

Lemma 2.1. *Let G be a randomly 3-dimensional graph. Then, the following statements hold.*

(a) *For every pair $\{u, v\} \in \mathcal{P}_2(G)$ there exist unique pairs $\{x, y\}, \{r, s\} \in \mathcal{P}_2(G)$ such that $\{x, y\}$ is not resolved by $\{u, v\}$, and $\{u, v\}$ is not resolved by $\{r, s\}$.*

(b) *Every vertex $u \in V(G)$ satisfies*

$$\sum_{1 \leq i \leq \mathrm{ecc}(u)} \binom{|N_i(u)|}{2} = n - 1 \tag{1}$$

where by convention $\binom{1}{2} = 0$, and $\mathrm{ecc}(u)$ denotes the eccentricity of u, i.e., the maximum distance from u to any other vertex.

Proof. To prove Statement (a), consider a graph G verifying that $\dim(G) = \mathrm{res}(G) = 3$. For every pair $P = \{u, v\} \in \mathcal{P}_2(G)$, consider the set

$$S_P = \{\{x, y\} \mid P \text{ does not resolve } \{x, y\}\} \subset \mathcal{P}_2(G).$$

Since $\dim(G) = 3$ then S_P is non-empty. Moreover, $S_P \cap S_{P'} = \emptyset$ whenever $P \neq P'$. Indeed, suppose on the contrary that there is a pair $\{x, y\}$ resolved by neither $P = \{u, v\}$ nor $P' = \{u', v'\}$, and assume that $u, v \neq u'$. Then the set $\{u, v, u'\}$ is not a metric basis, which is a contradiction. Therefore, $S_P \cap S_{P'} = \emptyset$. Thus, $|S_P| = 1$ for every $P \in \mathcal{P}_2(G)$. This proves that for every pair $\{u, v\}$ there is a unique pair $\{x, y\}$ such that $\{u, v\}$ does not resolve $\{x, y\}$. Further, the map $\varphi : \mathcal{P}_2(G) \to \mathcal{P}_2(G)$ given by $\varphi(P) = S_P$ is well-defined and injective (where by abuse of notation, we consider $S_P \in \mathcal{P}_2(G)$). Then φ is a bijection and so there is a unique pair $\{r, s\}$ which does not resolve $\{u, v\}$. Hence, Statement (a) follows.

Consider now the set S_u of non-resolved pairs by a vertex $u \in V(G)$. As a consequence of Statement (a) we have $|S_u| = n-1$ (it suffices to consider the $n - 1$ distinct pairs $\{u, v\}$ with $v \in V(G) \setminus \{u\}$). On the other hand, a pair $\{x, y\} \in S_u$ if and only if $x, y \in N_i(u)$ for $i = d(x, u) = d(y, u)$. Hence,

$$|S_u| = \sum_{1 \leq i \leq \mathrm{ecc}(u)} \binom{|N_i(u)|}{2}$$

which proves Statement (b). $\square$

Given $u \in V(G)$, consider the partition $\mathcal{P}(u) = \{N_i(u) \mid 0 \leq i \leq \mathrm{ecc}(u)\}$ of $V(G)$ into classes (where $N_0(u) = \{u\}$). Lemma 2.1(b) says that there is a compensation between vertices of $V(G) \setminus \{u\}$ and pairs of vertices located in the same class of $\mathcal{P}(u)$. For instance, classes of size at least 4 always contribute to Eq. (1) with more pairs than vertices (6 pairs and

4 vertices in case of size 4) and so they have to be compensated with classes of size at most 2 whose contribution is bigger in terms of vertices than in pairs. Note that classes of size 3 which contribute with 3 pairs, are self-compensated. Therefore, we have the following lemma.

Lemma 2.2. *Let $u \in V(G)$ and let $\mathcal{P}(u) = \{N_i(u) \mid 0 \le i \le \mathrm{ecc}(u)\}$ be a partition of $V(G)$ into classes with $N_0(u) = \{u\}$. Then, the following statements hold.*

(a) *If $\mathcal{P}(u)$ contains a class of size at least 4 then it contains at least two classes of size at most 2 different than $N_0(u)$.*
(b) *If $\mathcal{P}(u)$ contains a class of size at most 2 different than $N_0(u)$ then it contains a class of size at least 4.*

The following straightforward observation will be useful for the proofs of this paper.

Observation 2.3 ([8])**.** *Let $u, v, w \in V(G)$ such that $\{v, w\} \in E(G)$ and $d(u, v) = d$. Then, $d(u, w) \in \{d - 1, d, d + 1\}$.*

Lemmas 2.1 and 2.2 are the key tools to avoid the case analysis in the characterization of the randomly 3-dimensional graphs.

Proposition 2.4. *If a graph G is randomly 3-dimensional then G is the complete graph on 4 vertices.*

Proof. First, observe that G does not contain vertices of degree 1 (Lemma 1 of [7]). Indeed, if a vertex u has a unique neighbour v, then the pair $\{u, v\}$ is resolved by every vertex of G, which contradicts Lemma 2.1(a).

Claim 1. *The degree of every vertex of G is at most 3.*

Proof of Claim 1. Suppose on the contrary that there is a vertex $u \in V(G)$ of degree at least 4, and let $u_1, u_2, u_3, u_4 \in N_1(u)$. By Lemma 2.1(a), each set $\mathcal{A}_{ij} = \{v \in V(G) \mid d(v, u_i) = d(v, u_j)\}$ with $1 \le i < j \le 4$ contains exactly two vertices of G. Let $v \in V(G)$ such that $d(u, v) = d$. By Observation 2.3, $d(v, u_i) \in \{d - 1, d, d + 1\}$ and so the set $\{d(v, u_i) \mid 1 \le i \le 4\}$ has at most 3 elements which implies that v belongs to at least one of the six sets $\mathcal{A}_{ij}$. Hence, $n \le 7$ since $u \in \mathcal{A}_{ij}$ for all i, j.

Consider now the partition $\mathcal{P}(u)$ in which $N_1(u)$ is a class of size at least 4. By Lemma 2.2(a), there are at least two classes of size at most 2 different than $N_0(u)$. Further, $n \le 7$ and so there are exactly three classes of size 1 in $\mathcal{P}(u)$ (one being $N_0(u)$) which implies that the furthest vertex from u has degree 1; a contradiction. Therefore, every vertex of G has degree at most 3. □

Claim 2. $n \in \{4, 7, 10\}$.

Proof of Claim 2. Since $\dim(G) = 3$, then G is neither a path nor a cycle and so there is a vertex $u \in V(G)$ of degree 3 with neighbours, say u_1, u_2, u_3. Arguing as in the proof of Claim 1, defining the analogous sets $\mathcal{A}_{ij}$ but for the vertices u, u_1, u_2, u_3, we have $n \le 10$ since every set contains exactly two vertices of G, u belongs to three of them, and every vertex of G belongs to at least one of the six sets.

The sets $\{u\}$ and $\{u_1, u_2, u_3\}$ are the classes $N_0(u)$ and $N_1(u)$, respectively, in the partition $\mathcal{P}(u)$. If this partition does not contain more classes, then $n = 4$. Otherwise, by Lemma 2.2, there are three possibilities for $\mathcal{P}(u)$: (1) one class of size 4 and two classes of size 1 (plus $N_0(u)$ and $N_1(u)$); (2) $N_0(u)$ and three classes of size 3 (one being $N_1(u)$); (2) $N_0(u)$ and two classes of size 3 (one being $N_1(u)$). This gives $n = 10$ or $n = 7$. □

Claim 3. *There is no vertex of degree 2.*

Proof of Claim 3. Suppose on the contrary that there is a vertex $u \in V(G)$ of degree 2. Then, $|N_1(u)| = 2$. By Lemma 2.2, $\mathcal{P}(u)$ contains a class of size at least 4 and another class of size at most 2 different than $N_0(u)$. Since $n \le 10$, we have the following two possibilities for $\mathcal{P}(u)$: (1) one class of size 4, two classes of size 1 (one being $N_0(u)$) and one class of size 2; (2) one class of size 4, one class of size 1 (being $N_0(u)$) and two classes of size 2. This gives, respectively, $n = 8$ and $n = 9$ contradicting Claim 2. □

The three previous claims prove that a graph G of order n satisfying $\dim(G) = \mathrm{res}(G) = 3$ is 3-regular and $n \in \{4, 7, 10\}$. Clearly, $n \ne 7$ since there is no 3-regular graph with 7 vertices.

Consider now two vertices $u, v \in V(G)$ such that $d(u, v) = d(G)$, where $d(G)$ denotes the diameter of G, and let $N_1(u) = \{u_1, u_2, u_3\}$. By Observation 2.3, the distance from v to every vertex of the set $\{u, u_1, u_2, u_3\}$ is either $d(G)$ or $d(G) - 1$. Hence, v belongs to at least two of the six sets $\mathcal{A}_{ij}$ as defined in the proof of Claim 2. By Lemma 2.1(a), each set contains exactly two vertices of G and u belongs to three of them. This gives $n < 10$ and so $n = 4$, which implies that G is isomorphic to the complete graph K_4 (the only 3-regular graph on 4 vertices). □

Now, we reach the desired characterization that solves Problem 1.1.

Theorem 2.5. *A graph G is randomly k-dimensional if and only if G is a complete graph or an odd cycle.*

Proof. If G is isomorphic to a complete graph or an odd cycle, it is straightforward to prove that G is randomly k-dimensional.

Suppose now that G is a graph of order n satisfying $\dim(G) = \operatorname{res}(G) = k$. As was said before the case $k \leq 2$ was proved in [4], obtaining the complete graphs K_1 and K_2 (for $k = 1$) and odd cycles (for $k = 2$). Moreover, Proposition 2.4 proves the result for $k = 3$ and so we can assume $k \geq 4$.

Arguing as in the proof of Lemma 2.1(a) we have that for every $T \in \mathcal{P}_{k-1}(G)$, the non-empty set

$$S_T = \{\{x, y\} \mid T \text{ does not resolve } \{x, y\}\} \subset \mathcal{P}_2(G)$$

verifies that $S_T \cap S_{T'} = \emptyset$ whenever $T \neq T'$. Therefore $|\mathcal{P}_{k-1}(G)| \leq |\mathcal{P}_2(G)|$, i.e.,

$$\binom{n}{k-1} \leq \binom{n}{2} \Longrightarrow k \in \{1, 2, 3, n-1, n, n+1\}.$$

Hence, $k = n - 1$ since $4 \leq k = \dim(G) \leq n - 1$. This implies that G is isomorphic to the complete graph K_n, which is the only graph of order n with metric dimension $n - 1$ [2]. □

3. Realization

3.1. The metric dimension and the upper dimension

This subsection is devoted to settle in the affirmative Conjecture 1.2. In order to do this, we compute the upper dimension of two families of graphs for which the metric dimension is easily obtained. These graphs are constructed from the grid graphs attaching at the origin either a triangle or a number of pendant vertices. We start with some notation and technical lemmas.

Let G_ℓ be the grid graph of order $\ell \times \ell$ with $\ell \geq 2$, whose vertex set is the Cartesian product $[0, \ell - 1] \times [0, \ell - 1]$ and distances given by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

We shall use (x_1, x_2) to indicate the coordinates of a vertex $x \in V(G_\ell)$ (analogously, $y = (y_1, y_2)$, $z = (z_1, z_2)$, etc.). The following sets of vertices are called *quadrants of* $x \in V(G_\ell)$:

$$Q_1(x) = \{y \in V(G_\ell) \mid y_1 \geq x_1,\ y_2 \geq x_2\}, \qquad Q_2(x) = \{y \in V(G_\ell) \mid y_1 \leq x_1,\ y_2 \geq x_2\},$$
$$Q_3(x) = \{y \in V(G_\ell) \mid y_1 \leq x_1,\ y_2 \leq x_2\}, \qquad Q_4(x) = \{y \in V(G_\ell) \mid y_1 \geq x_1,\ y_2 \leq x_2\},$$

and the sets $D_i = \{x \in V(G_\ell) \mid x_1 + x_2 = i\}$ with $0 \leq i \leq 2\ell - 2$ are the *diagonals* of G_ℓ (see Fig. 1(a)). A pair of vertices $\{x, y\}$ is said to be a *diagonal pair* if $x, y \in D_i$ for some i. Note that a quadrant $Q_i(x)$ might be equal to $\{x\}$ and there is a total order $<_i$ in each diagonal D_i (or simply "$<$" when no confusion can arise) given by

$$x <_i y \Longleftrightarrow x_1 < y_1.$$

In the sequel we shall assume, without loss of generality, that the order of the two elements of a diagonal pair $\{x, y\}$ is $x < y$ (analogously, $r < s$ for $\{r, s\}$ or $t < z$ for $\{t, z\}$).

Let $R(x, y)$ be the set of vertices of G_ℓ that resolve the pair $\{x, y\} \subset V(G_\ell)$, and let S be a resolving set of G_ℓ. Note that the set $R(x, y) \cap S$ is non-empty for every pair $\{x, y\}$.

Lemma 3.1. *Let $\{x, y\}$ be a diagonal pair such that $d(x, y) = 2$. Then,*

$$R(x, y) = Q_2(x) \cup Q_4(y).$$

Proof. For every vertex $u \in Q_2(x)$, there is a shortest $u - y$ path going through x and so $d(u, y) = d(u, x) + d(x, y) = d(u, x) + 2$. Thus, u resolves $\{x, y\}$ (analogous for $u \in Q_4(y)$).

Let $u \in V(G_\ell) \setminus (Q_2(x) \cup Q_4(y))$, $z = (x_1, y_2)$ and $\tilde{z} = (y_1, x_2)$. Clearly, there are two shortest paths P_1 and P_2 joining u to x and u to y, respectively, such that either $z \in P_1, P_2$ or $\tilde{z} \in P_1, P_2$ (see Fig. 1(b)). Since $z, \tilde{z}$ do not resolve the pair $\{x, y\}$ then $u \notin R(x, y)$. □

Given a resolving set S of G_ℓ, a pair $\{x, y\}$ is said to be *S-unique* if there is a unique vertex $u \in S$ resolving $\{x, y\}$, i.e., $R(x, y) \cap S = \{u\}$. The vertex u and the pair $\{x, y\}$ are said to be *associated to each other*. The following observation is straightforward.

Observation 3.2. *Let S be a resolving set of G_ℓ, and let $\{x, y\}$ be an S-unique pair with associated vertex u. If there is a pair $\{r, s\}$ such that $R(r, s) \subseteq R(x, y)$ then $\{r, s\}$ is S-unique with associated vertex u. Note that necessarily $u \in R(r, s)$.*

Lemma 3.3. *Let S be a resolving set of G_ℓ, and let $\{x, y\} = \{(x_1, x_2), (y_1, y_2)\}$ be an S-unique diagonal pair with associated vertex u such that $d(x, y) > 2$. Then, there exist exactly $y_1 - x_1$ S-unique diagonal pairs $\{r, s\}$ with associated vertex u and $d(r, s) = 2$.*

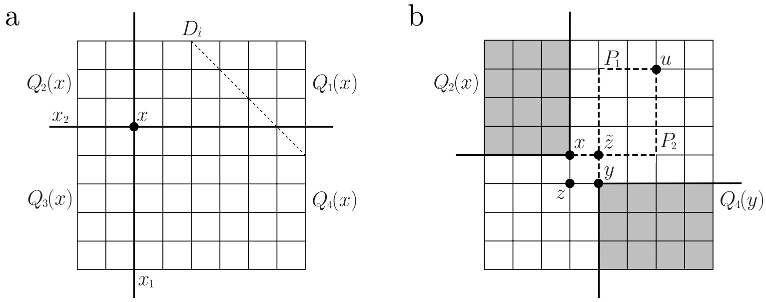

Fig. 1. (a) Quadrants of x and a diagonal D_i, (b) the shadowed region illustrates $Q_2(x) \cup Q_4(y)$, and the dotted edges form the paths P_1 and P_2.

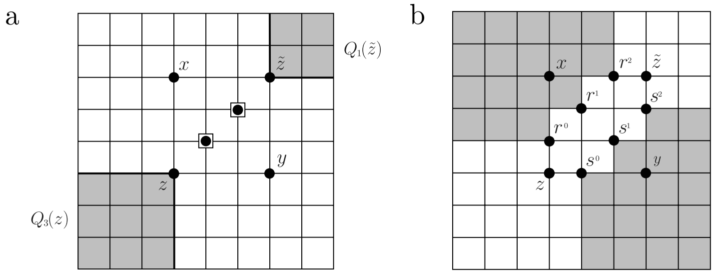

Fig. 2. (a) All the vertices in the shadowed region plus the two squared vertices do not resolve the pair $\{x, y\}$, (b) the shadowed region illustrates $R(x, y)$.

Proof. A similar argument as in the proof of Lemma 3.1, considering $z = (x_1, y_2)$ and $\tilde{z} = (y_1, x_2)$, gives that every vertex $u \in Q_3(z) \cup Q_1(\tilde{z})$ does not resolve the pair $\{x, y\}$. Clearly, the vertices $(x_1 + j,\ y_2 + j)$ with $0 < j < y_1 - x_1$ do not resolve the pair $\{x, y\}$ either (see Fig. 2(a)). Thus, the expression of $R(x, y)$ in this case is

$$R(x, y) = V(G_\ell) \setminus (Q_3(z) \cup Q_1(\tilde{z}) \cup \{(x_1 + j, y_2 + j) \mid 0 < j < y_1 - x_1\}).$$

This set can also be expressed as follows:

$$R(x, y) = \bigcup_{0 \le j < y_1 - x_1} R(r^j, s^j)$$

where $r^j = (x_1 + j, y_2 + j + 1)$, $s^j = (x_1 + j + 1, y_2 + j)$ and $d(r^j, s^j) = 2$ (see Fig. 2(b)). Since $R(r^j, s^j) \subseteq R(x, y)$, Observation 3.2 proves the result. $\square$

Two diagonal pairs $\{x, y\}$, $\{r, s\}$ such that $d(x, y) = d(r, s) = 2$ are said to be *in the same row* if $x_2 = r_2$ and $y_2 = s_2$. Analogously, they are *in the same column* if $x_1 = r_1$ and $y_1 = s_1$.

Lemma 3.4. *Let S be a resolving set of G_ℓ, and let $\{x, y\}$ be an S-unique diagonal pair with associated vertex u such that $d(x, y) = 2$. If there exist two S-unique diagonal pairs $\{r, s\}$, $\{t, z\}$ in the same row (column) than $\{x, y\}$ with associated vertices, respectively, v and w and $u \neq v, w$, then $v = w$.*

Proof. Suppose that the pairs $\{r, s\}$, $\{t, z\}$ are in the same row (analogous for columns) than $\{x, y\}$, i.e., $x_2 = r_2 = t_2$ and $y_2 = s_2 = z_2$. Assume also that $x_1 < r_1 < t_1$. Clearly, $R(r, s) \subset (R(x, y) \cup R(t, z))$ and so $v = w$ since $v \neq u$. $\square$

Now, we reach our main result in this subsection which settles in the affirmative Conjecture 1.2.

Theorem 3.5. *For every pair a, b of integers with $2 \le a \le b$, there exists a connected graph G with $\dim(G) = a$ and $\dim^+(G) = b$.*

Proof. Let H_ℓ be the graph obtained from G_ℓ with $\ell \ge 2$, by attaching a triangle at vertex $(0, 0)$, i.e., $V(H_\ell) = V(G_\ell) \cup \{\alpha, \beta\}$ and $E(H_\ell) = E(G_\ell) \cup \{\{\alpha, \beta\}, \{\alpha, (0, 0)\}, \{\beta, (0, 0)\}\}$ (see Fig. 3(a)). Observe that distances in H_ℓ behave as in G_ℓ, except for the new vertices α and β for which $d(\alpha, x) = d(\beta, x) = x_1 + x_2 + 1$ for every $x = (x_1, x_2) \in V(G_\ell)$. Thus, the previous lemmas can be applied to the graph H_ℓ.

Claim A. $\dim(H_\ell) = 2$ *and* $\dim^+(H_\ell) = 2\ell - 2$.

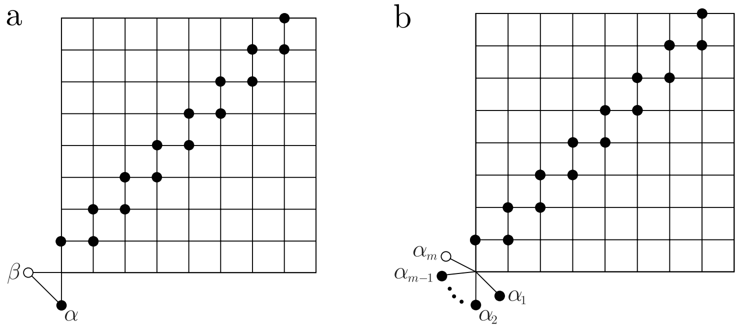

Fig. 3. (a) A minimal resolving set of H_ℓ of size $2\ell - 2$, (b) a minimal resolving set of $H_{\ell,m}$ of size $m + 2\ell - 4$.

Proof of Claim A. It is well-known that $\dim(G_\ell) = 2$ the set $\{(0, 0), (\ell - 1, 0)\}$ being a metric basis (see for instance [8]). This set can be adapted to a metric basis of H_ℓ by considering $\{\alpha, (\ell - 1, 0)\}$. Hence, $\dim(H_\ell) = 2$.

To prove that $\dim^+(H_\ell) \geq 2\ell - 2$ one can easily check that the set

$$S = \{(x_1, x_2) \mid 1 \leq x_1 \leq \ell - 2,\ x_2 \in \{x_1, x_1 + 1\}\} \cup \{(0, 1), \alpha\}$$

is a resolving set of H_ℓ of size $2\ell - 2$. Moreover, S is minimal because removing either a vertex (x_1, x_1) or $(x_1, x_1 + 1)$ from S gives that either the pair $\{(x_1, x_1), (x_1 - 1, x_1 + 1)\}$ or the pair $\{(x_1, x_1 + 1), (x_1 + 1, x_1)\}$ is not resolved by any element of S. Clearly, $(0, 1)$ and α cannot be removed from S. Fig. 3(a) illustrates this minimal resolving set.

We next prove that $\dim^+(H_\ell) \leq 2\ell - 2$. Let S be a minimal resolving set of H_ℓ. Consider the pair $\{\alpha, \beta\}$ which is only resolved by either α or β. Without loss of generality, we assume that $\alpha \in S$ and $\beta \notin S$.

Since S is minimal, every vertex $u \in S$ has an associated S-unique pair, say $p(u)$. Observe that $\{\beta, (0, 0)\}$ is not an S-unique pair (every vertex of G_ℓ resolves it) and so there is no vertex $u \in S$ so that $p(u) = \{\beta, (0, 0)\}$. Also note that α resolves all the non-diagonal pairs of G_ℓ. Hence, every vertex $u \in S \setminus \{\alpha\}$ has an associated S-unique diagonal pair $p(u)$. Moreover, by Lemma 3.3, we can assume that the elements of $p(u)$ are at distance 2 from each other.

Let us consider all these S-unique pairs. By Lemma 3.4, for all pairs of the same row (or column), there are at most two distinct vertices associated to these pairs. Moreover, we claim that there is at most one such vertex in the first row (and the first column). Indeed, by Lemma 3.1 we have $R((0, 1), (1, 0)) = Q_2((0, 1)) \cup Q_4((1, 0)) = \{(0, x_2) \mid 1 \leq x_2 \leq \ell - 1\} \cup \{(x_1, 0) \mid 1 \leq x_1 \leq \ell - 1\}$. Suppose that there is a vertex $v \in S \cap Q_2((0, 1))$ (analogous for $v \in S \cap Q_4((1, 0))$ by symmetry). Since all the pairs in the same row as $\{(0, 1), (1, 0)\}$ are resolved by v and S is minimal, then there is no other vertex of S associated to pairs in such row.

Hence, in total, since there are $\ell - 1$ rows (and columns), there are at most $2(\ell - 2) + 1$ S-unique pairs that can be associated to the vertices of $S \setminus \{\alpha\}$, and thus $|S \setminus \{\alpha\}| = |S| - 1 \leq 2(\ell - 2) + 1$. $\square$

Consider now the graph $H_{\ell,m}$ obtained from G_ℓ with $\ell \geq 3$ by attaching a set of $m \geq 2$ pendant vertices $\{\alpha_1, \dots, \alpha_m\}$ at $(0, 0)$ (see Fig. 3(b)).

Claim B. $\dim(H_{\ell,m}) = m + 1$ *and* $\dim^+(H_{\ell,m}) = m + 2\ell - 4$.

Proof of Claim B. As was said before, the set $\{(0, 0), (\ell - 1, 0)\}$ is a metric basis of G_ℓ [8]. Thus, it can be easily checked that the set $\{\alpha_1, \dots, \alpha_m, (\ell - 1, 0)\}$ is a resolving set of $H_{\ell,m}$, which gives $\dim(H_{\ell,m}) \leq m + 1$. To prove that $\dim(H_{\ell,m}) \geq m + 1$, it suffices to show that $|S| \geq m + 1$ for every metric basis S.

A metric basis S must contain all the pendant vertices but at most one. Suppose that $\{\alpha_1, \dots, \alpha_{m-1}\} \subset S$ and $\alpha_m \notin S$ (if $\{\alpha_1, \dots, \alpha_m\} \subset S$, the result clearly follows). Since no pendant vertex resolves the pair $\{(0, 1), (1, 0)\}$, then there is a vertex, say $u \in R((0, 1), (1, 0)) = Q_2((0, 1)) \cup Q_4((1, 0))$. Without loss of generality, suppose that $u \in Q_2((0, 1))$. Then the pair $\{\alpha_m, (1, 0)\}$ is not resolved by any vertex in the set $\{\alpha_1, \dots, \alpha_{m-1}, u\}$ and so $|S| \geq m + 1$. Therefore, $\dim(H_{\ell,m}) = m + 1$.

Mimicking the proof of Claim A, only replacing α by $\alpha_1, \dots, \alpha_{m-1}$ and β by α_m (compare Fig. 3(a) and (b)) it is proved that $\dim^+(H_{\ell,m}) = m + 2\ell - 4$. $\square$

Claims A and B give a connected graph G with $\dim(G) = a$ and $\dim^+(G) = b$ whenever $a = 2$ and b is even ($G \cong H_\ell$ for $\ell = (b + 2)/2$) or $a > 2$ and $b - a$ is odd ($G \cong H_{\ell,m}$ for $\ell = 2 + (b - a + 1)/2$ and $m = a - 1$).

In order to obtain the graph G in the remaining cases, we modify slightly the graphs H_ℓ and $H_{\ell,m}$ by removing the set of vertices $\{(x_1, x_2) \mid x_1 = \ell - 1\}$. Denote by $\tilde{H}_\ell$ and $\tilde{H}_{\ell,m}$ the resulting graphs. Note that a $(\ell - 1) \times \ell$ grid, say $\tilde{G}_\ell$, plays now the role of G_ℓ but all the tools developed above can also be applied in this case. Hence, one can follow the proofs of Claims A and B to compute the metric dimension and the upper dimension of $\tilde{H}_\ell$ and $\tilde{H}_{\ell,m}$. There are only three changes:

1. Take the set $\{(0, 0), (\ell - 2, 0)\}$ as a metric basis of $\tilde{G}_\ell$.
2. Remove the vertex $(\ell - 2, \ell - 1)$ from S obtaining a minimal resolving set of size $2\ell - 3$ (for $\tilde{H}_\ell$) or $2\ell + m - 5$ (for $\tilde{H}_{\ell,m}$).

3. Apply the column version of Lemma 3.4 to get $|S \setminus \{\alpha\}| \leq 2(\ell-2)$ or $|S \setminus \{\alpha_1, \ldots, \alpha_{m-1}\}| \leq 2(\ell-2)$ which directly gives $|S| \leq 2\ell-3$ (for $\tilde{H}_\ell$) or $|S| \leq 2\ell+m-5$ (for $\tilde{H}_{\ell,m}$).

Thus, we have that $\dim(\tilde{H}_\ell) = 2$, $\dim^+(\tilde{H}_\ell) = 2\ell-3$, $\dim(\tilde{H}_{\ell,m}) = m+1$ and $\dim^+(\tilde{H}_{\ell,m}) = m+2\ell-5$. Therefore, we obtain a graph G with $\dim(G) = a$ and $\dim^+(G) = b$ whenever $a = 2$ and b is odd ($G \cong \tilde{H}_\ell$ for $\ell = 2+(b-1)/2$) or $a > 2$ and $b-a$ is even ($G \cong \tilde{H}_{\ell,m}$ for $\ell = 3+(b-a)/2$ and $m = a-1$). □

3.2. The resolving number

In Section 3.1, we have proved that every pair a, b of integers with $2 \leq a \leq b$ is realizable as the metric dimension and the upper dimension, respectively, of a certain graph. Modifying slightly the above constructions, one can easily prove that every pair a, b is realizable as the metric dimension and the upper dimension, respectively, of an infinite family of graphs. It suffices to replace the vertex $(0, 0)$ in G_ℓ by a path of arbitrary length. If the resulting graph plays the role of G_ℓ in the study developed in the previous subsection, then the values of the metric dimension and the upper dimension are preserved.

Theorem 3.7 below says that no integer $a \geq 4$ is realizable as the resolving number of an infinite family of graphs (note that the path P_2 is the only graph with resolving number 1 but there are infinite families of graphs with resolving number 2 and 3, concretely, odd cycles and paths for $a = 2$ and even cycles for $a = 3$). In order to prove this result, we first relate the resolving number to the diameter of a graph, which is of independent interest.

Proposition 3.6. *Let G be a graph with diameter $d(G)$ and resolving number $\mathrm{res}(G) \geq 3$. If G is not an even cycle, then $d(G) \leq 3\mathrm{res}(G)-5$.*

Proof. Let us denote $r = \mathrm{res}(G)$. Suppose on the contrary that $d(G) > 3r-5$. Then, we can assume that there are two vertices $u, v \in V(G)$ such that $d(u, v) = 3r-4 = 3(r-1)-1$. Consider a shortest u–v path $P = \{u = u_1, u_2, \ldots, u_{3(r-1)} = v \mid u_i$ is adjacent to $u_{i+1}\}$, and suppose that there is a vertex $w \notin P$ adjacent to some vertex u_i with $i \neq 1, 3(r-1)$ (otherwise it can be easily checked that $\{u_1, \ldots, u_r\}$ is not a resolving set).

Clearly, every vertex $u_j \in P$ does not resolve either $\{w, u_{i-1}\}$ or $\{w, u_i\}$ or $\{w, u_{i+1}\}$. Indeed, assume $i \leq j$ (analogous for $i > j$). By Observation 2.3, u_j does not resolve at least one pair among those formed by the vertices u_{i-1}, u_i, u_{i+1}, w. Moreover, the pairs $\{u_{i-1}, u_i\}$, $\{u_{i-1}, u_{i+1}\}$ and $\{u_i, u_{i+1}\}$ are all resolved by u_j, since P is a shortest path. Thus, one pair among $\{w, u_{i-1}\}$, $\{w, u_i\}$, $\{w, u_{i+1}\}$ is not resolved by u_j.

Consider now the sets $A = \{u_j \in P \mid d(u_j, w) = d(u_j, u_{i-1})\}$, $B = \{u_j \in P \mid d(u_j, w) = d(u_j, u_i)\}$ and $C = \{u_j \in P \mid d(u_j, w) = d(u_j, u_{i+1})\}$. By the argument above, $A \cup B \cup C = P$. Furthermore, $|P| = 3(r-1)$ and $|A|, |B|, |C| \leq r-1$ (since these sets are not resolving sets of G) and so $|A| = |B| = |C| = r-1$. This implies that A, B and C are pairwise disjoint but $u_i \in A \cap C$; a contradiction. □

Observe that when $\mathrm{res}(G) \leq 2$ or G is an even cycle, the bound of Proposition 3.6 does not hold. It suffices to consider the path P_2 (for $\mathrm{res}(G) = 1$), an odd cycle of length at least 5 (for $\mathrm{res}(G) = 2$) and an even cycle of length at least 6 (for $\mathrm{res}(G) = 3$).

Theorem 3.7. *For every integer $a \geq 4$, the set of graphs with resolving number a is finite.*

Proof. A graph G of order n, diameter $d(G)$ and metric dimension $\dim(G)$ satisfies the following relation [8]:

$$n \leq d(G)^{\dim(G)} + \dim(G).$$

Since $\dim(G) \leq \mathrm{res}(G)$ then

$$n \leq d(G)^{\mathrm{res}(G)} + \mathrm{res}(G).$$

By Proposition 3.6, we obtain

$$n \leq (3\mathrm{res}(G)-5)^{\mathrm{res}(G)} + \mathrm{res}(G) = (3a-5)^a + a.$$

This upper bound for n depends only on the value of a and so the result follows. □

4. Concluding remarks and open questions

In this paper, we have settled in the affirmative a conjecture posed by Chartrand et al. [3] claiming that every pair a, b of integers with $2 \leq a \leq b$ is realizable as the metric dimension and the upper dimension, respectively, of some connected graph. We have also shown that, surprisingly, the set of graphs with given resolving number $a \geq 4$ is always finite, and we have characterized the randomly k-dimensional graphs, avoiding the brute force case analysis.

It would be interesting to study the realization of triples a, b, c of integers as the metric dimension, the upper dimension and the resolving number, respectively, of some connected graph. Also, the question of bounding the size of the set of graphs (may be restricting to specific families) with given resolving number a remains open. It would be also interesting to provide a polynomial upper bound on n in terms of the resolving number since we believe that the exponential upper bound given in the proof of Theorem 3.7 is not tight.

Acknowledgements

We thank two anonymous referees for their many useful suggestions and comments, which helped to improve the paper substantially.

The first and second authors were partially supported by JA-FQM164. The third author was partially supported by the ESF EUROCORES programme EuroGIGA ComPoSe IP04 MICINN Project EUI-EURC-2011-4306, and JA-FQM164.

References

[1] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, C. Seara, D.R. Wood, On the metric dimension of cartesian products of graphs, SIAM J. Discrete Math. 21 (2) (2007) 423–441.
[2] G. Chartrand, Linda Eroh, Mark A. Johnson, Ortrud R. Ollermann, Resolvavility in graphs and the metric dimension of a graph, Discrete Appl. Math. 105 (2000) 99–113.
[3] G. Chartrand, C. Poisson, P. Zhang, Resolvability and the upper dimension of graphs, Comput. Math. Appl. 39 (2000) 19–28.
[4] G. Chartrand, P. Zhang, On the chromatic dimension of a graph, Congr. Numer. 145 (2000) 97–108.
[5] F. Harary, R.A. Melter, On the metric dimension of a graph, Ars Combin. 2 (1976) 191–195.
[6] M. Jannesari, B. Omoomi, Characterization of randomly k-dimensional graphs, 2011. arxiv.org/abs/1103.3570v1.
[7] M. Jannesari, B. Omoomi, On randomly k-dimensional graphs, Appl. Math. Lett. 24 (2011) 1625–1629.
[8] S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, Discrete Appl. Math. 70 (1996) 217–229.
[9] P.J. Slater, Leaves of trees, Congr. Numer. 14 (1975) 549–559.