
On higher dimensional cocyclic Hadamard matrices

V. Álvarez · J. A. Armario · M. D. Frau · P. Real

Abstract Provided that a cohomological model for G is known, we describe a method for constructing a basis for n -cocycles over G , from which the whole set of n -dimensional n -cocyclic matrices over G may be straightforwardly calculated. Focusing in the case $n = 2$ (which is of special interest, e.g. for looking for cocyclic Hadamard matrices), this method provides a basis for 2-cocycles in such a way that representative 2-cocycles are calculated all at once, so that there is no need to distinguish between inflation and transgression 2-cocycles (as it has traditionally been the case until now). When $n > 2$, this method provides an uniform way of looking for higher dimensional n -cocyclic Hadamard matrices for the first time. We illustrate the method with some examples, for $n = 2, 3$. In particular, we give some examples of improper 3-dimensional 3-cocyclic Hadamard matrices.

Keywords (Co)homological model · Cocyclic matrix · Proper/improper higher dimensional Hadamard matrix

All authors are partially supported by FEDER funds via the research Projects FQM-296 and FQM-016 from JJAA.

V. Álvarez · J. A. Armario (✉) · M. D. Frau · P. Real
Dpto. Matemática Aplicada I, University of Seville, Avda.
Reina Mercedes s/n, 41012 Seville, Spain
e-mail: armario@us.es

V. Álvarez
e-mail: valvarez@us.es

M. D. Frau
e-mail: mdfrau@us.es

P. Real
e-mail: real@us.es

1 Introduction

Hadamard matrices are square matrices with entries ± 1 such that their rows are pairwise orthogonal. They were first noticed by Sylvester in problems related to tessellated pavements, and later on by Hadamard related to the maximal determinant problem, which asks for the largest determinant for all matrices with entries ± 1 . Recommended references on Hadamard matrices and their applications are [22] and more recently [23].

It is easy to prove that the size of Hadamard matrices must be 1, 2 or a multiple of 4. Nevertheless, it is an open question whether Hadamard matrices exist for every size $4t$. This is known as the *Hadamard Conjecture*. A lot of work has been made concerning this conjecture, and the ways in which Hadamard matrices might be constructed. All known construction methods such as Sylvester Hadamard matrices, Paley Hadamard matrices, Williamson Hadamard matrices or Ito Hadamard matrices fail to yield Hadamard matrices for every order which is a multiple of 4. The most promising methods which may lead to a constructive proof of the Hadamard Conjecture are the two-circulant core construction [26] and the cocyclic approach [25].

A matrix M is said to be *cocyclic* if there exist a group G of order $|G| = 4t$ and a 2-cocycle $\psi : G \times G \rightarrow \{\pm 1\}$ such that $M = (\psi(g_i, g_j))$. The main advantages of the cocyclic framework concerning Hadamard matrices may be summarized in the following facts:

- Determining if a ± 1 matrix M of order $4t$ is Hadamard would consist of checking whether the rows of M are pairwise orthogonal, which would require in total $O(t^3)$ operations. In fact, since these matrices give the solution to the problem of the maximal determinant problem of matrices of size $4t \times 4t$ [20], it suffices to check whether $|H| = (4t)^{2t}$, which actually can be done in less than $O(t^3)$ time, using some fast matrix multiplication techniques. Nevertheless, a cocyclic matrix $M = (\psi(g_i, g_j))$ is Hadamard if and only if the summation of each row but the first is zero (see the cocyclic Hadamard test of [25]), which requires at most $O(t^2)$ operations.
- The search space is reduced to the set of cocyclic matrices over a given group, instead of the whole set of matrices with entries in $\{-1, 1\}$. In spite of this fact, the proportion of cocyclic Hadamard matrices among the full set of cocyclic matrices seems to be not significantly different from that of usual Hadamard matrices among ± 1 matrices, as calculations in [1] suggest.

In this paper we are concerned with higher dimensional Hadamard matrices, and the way in which the cocyclic framework may be used in this context. For commodity, in what follows any 3D-matrix $A(a_{i_1, i_2, i_3})_{1 \leq i_j \leq v, 1 \leq j \leq 3}$ will be described by listing its 2-dimensional horizontal sections $A(a_{i_1, i_2, i})$, for $1 \leq i \leq v$. As usual, negative entries will be denoted simply by $-$.

As introduced by Shlichta in [27, 28], an *improper* n -dimensional Hadamard matrix of order v is a (± 1) array $A = (a_{i_1, i_2, \dots, i_n})_{1 \leq i_j \leq v, 1 \leq j \leq n}$ such that all its parallel $(n-1)$ -dimensional sections are mutually orthogonal; that is, for each $1 \leq l \leq n$, and for all indices x and y in dimension l ,

$$\sum_{j \neq l} \sum_{1 \leq i_j \leq v} a_{i_1, \dots, x, \dots, i_j, \dots, i_n} \cdot a_{i_1, \dots, y, \dots, i_j, \dots, i_n} = v^{n-1} \delta_{xy}.$$

For instance, the following improper 3-dimensional Hadamard matrix $A = (a_{i_1, i_2, i_3})_{1 \leq i_j \leq 2}$ may be found in [28], where $A(i_1, i_2, 1) = \begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}$ and $A(i_1, i_2, 2) = \begin{pmatrix} - & - \\ - & 1 \end{pmatrix}$. Notice that the section $A(a_{i_1, i_2, i_3}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is not Hadamard, in the usual sense.

Notice that an improper n -dimensional Hadamard matrix may have stronger orthogonality properties in some dimensions. For instance, an n -dimensional Hadamard matrix A is termed *proper* (see [23, 31] for details) if any parallel rows in A are orthogonal; that is, for each pair of dimensions j, l , for all indices x and y in dimension l , and for each set of fixed indices in the other $n - 2$ dimensions,

$$\sum_{1 \leq i_j \leq v} a_{i_1, \dots, x, \dots, i_j, \dots, i_n} \cdot a_{i_1, \dots, y, \dots, i_j, \dots, i_n} = v \delta_{xy}.$$

The matrix of the example above is an improper not proper 3-dimensional Hadamard matrix. Nevertheless, in [28] one may find some proper 3-dimensional Hadamard matrices, such as $A = (a_{i_1, i_2, i_3})_{1 \leq i_j \leq 2}$, consisting of the horizontal sections $A(i_1, i_2, 1) = \begin{pmatrix} 1 & - \\ 1 & 1 \end{pmatrix}$ and $A(i_1, i_2, 2) = \begin{pmatrix} 1 & 1 \\ - & 1 \end{pmatrix}$.

There are some well-known methods for constructing both improper and proper higher-dimensional Hadamard matrices. The interested reader is referred to [14, 23, 31] and the references there cited.

It follows that every planar section of a proper n -dimensional Hadamard matrix of order v is a Hadamard matrix of order v itself, so v must be 2 or a multiple of 4. Actually, it is known that proper n -dimensional Hadamard matrices of order $v = 4t$ exist if and only if usual Hadamard matrices of order $v = 4t$ do exist (see [12, 30] for details).

It is worthwhile that although planar (and therefore proper n -dimensional) Hadamard matrices can only exist on orders $v > 2$ multiple of 4, improper higher-dimensional Hadamard matrices may exist on even orders not multiple of 4 (see [31]). It is easy to prove, though, that the order of an improper higher-dimensional Hadamard matrix must be even, in any case.

Few is known about the existence of improper n -dimensional Hadamard matrices of order v . There are well known examples of improper 3, 4-dimensional Hadamard matrices of order 6 (see [31]). Anyway, it is not known whether improper n -dimensional Hadamard matrices exist for all even orders $v = 2t, n \geq 3$, for odd t . Furthermore, only 3-dimensional improper Hadamard matrices of order $2 \cdot 3^b$ are known, $b \geq 1$. The interested reader is referred to [31] for more details.

However, despite its potential, the subject of higher-dimensional Hadamard matrices remains seriously under-developed. The aim of this paper is to establish a basis for the study of higher-dimensional Hadamard matrices from the cocyclic point of view.

As introduced in [27, 28], the notion of higher dimensional Hadamard matrix is a natural generalization of usual Hadamard matrices to the case of n -dimensional

matrices, for $n \geq 2$. One could wonder whether the cocyclic framework could also be taken into account for higher dimensional Hadamard matrices.

A precedent in the literature on this subject may be found in de Launey’s work in [13] (later extended in [14, 15]). Here, a way to construct a proper n -dimensional Hadamard matrix $A = (a_{i_1, \dots, i_n})$ from a 2-cocyclic planar Hadamard matrix $H = (\psi(g_i, g_j))$ is described, so that

$$a_{i_1, \dots, i_n} = \prod_{k=2}^n \psi \left(\prod_{j=1}^{k-1} g_{i_j}, g_{i_k} \right), \tag{1}$$

for $\psi : G \times G \rightarrow \{-1, 1\}$ being a 2-cocycle over G .

We wonder if one could go farther and use n -cocycles $\phi : G \times \dots \times G \rightarrow \{1, -1\}$ in an attempt to find n -dimensional Hadamard matrices $A = (\phi(i_1, \dots, i_n))$, for $n > 2$. If so, this would extend both the works of Shlichta and de Launey described before. Notice that given a 2-cocycle $\psi : G \times G \rightarrow \{1, -1\}$, a function $f_\psi : G \times \dots \times G \rightarrow \{1, -1\}$ defined as in (1), so that $f_\psi(i_1, \dots, i_n) = \prod_{k=2}^n \psi \left(\prod_{j=1}^{k-1} g_{i_j}, g_{i_k} \right)$, does not define a n -cocycle ϕ in general.

Recall that for a map $\phi : G \times \dots \times G \rightarrow \{1, -1\}$ being a n -cocycle over G , it is necessary and sufficient that for all $g_{i_1}, \dots, g_{i_{n+1}} \in G$,

$$\phi(g_{i_2}, \dots, g_{i_{n+1}}) \cdot \phi(g_{i_1}, \dots, g_{i_n}) \cdot \prod_{k=1}^n \phi(g_{i_1}, \dots, g_{i_k} g_{i_{k+1}}, \dots, g_{i_{n+1}}) = 1. \tag{2}$$

In particular, consider $G = \mathbf{Z}_2$ and the 2-cocycle $\psi : \mathbf{Z}_2 \times \mathbf{Z}_2 \rightarrow \{1, -1\}$, so that $\psi(0, 0) = \psi(0, 1) = \psi(1, 0) = 1, \psi(1, 1) = -1$. By (1), the matrix $A = (f_\psi(i_1, i_2, i_3))$ consisting of $f_\psi(i_1, i_2, i_3) = \psi(i_1, i_2) \cdot \psi(i_1 i_2, i_3)$ defines a proper 3-dimensional Hadamard matrix, whose horizontal sections are $A(i_1, i_2, 0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $A(i_1, i_2, 1) = \begin{pmatrix} 1 & - \\ - & - \end{pmatrix}$. However, f_ψ defines by no means a 3-cocycle over \mathbf{Z}_2 , since substituting $(g_{i_1}, g_{i_2}, g_{i_3}, g_{i_4}) = (0, 0, 1, 1)$ in (2) leads to

$$f_\psi(0, 1, 1) \cdot f_\psi(0, 0, 1) \cdot f_\psi(0, 1, 1) \cdot f_\psi(0, 1, 1) \cdot f_\psi(0, 0, 0) = -1 \neq 1.$$

Thus it makes sense going beyond de Launey’s work about constructing n -dimensional proper Hadamard matrices from orthogonal 2-cocycles, and trying to look for n -dimensional Hadamard matrices coming from n -cocycles, for $n > 2$.

In these circumstances, two main questions should be studied, for $n > 2$:

1. Is there any cocyclic test for n -cocyclic n -dimensional (proper/improper) Hadamard matrices?
2. Is there any effective way to construct n -cocyclic n -dimensional matrices?

Although we have studied the first of these problems, unfortunately we have not been able to isolate any simpler characterization for a n -cocyclic n -dimensional matrix to be (proper/improper) Hadamard for the moment, for $n > 2$.

In order to answer the second question, though, a prerequisite is to determine a basis for n -cocyclic matrices over a given finite group G . Here we provide a method for constructing n -cocyclic n -dimensional matrices over some finite groups G . Nevertheless, the difficult step is not just constructing such matrices, but determining which among them satisfy the proper/improper Hadamard condition.

Focusing in the case $n = 2$ (which is of special interest, e.g. for looking for cocyclic Hadamard matrices), the Universal Coefficient Theorem provides a decomposition of representative 2-cocycles over a group G as the direct sum of the inflation and transgression cocycles over G ,

$$H^2(G, \mathbf{Z}_2) \cong Ext(G/[G, G], \mathbf{Z}_2) \oplus Hom(H_2(G), \mathbf{Z}_2).$$

Until now, all the methods which look for a basis for 2-cocycles uses this decomposition, so that two different processes have to be performed.

In spite of this fact, there is a chance to calculate a basis for $H^2(G, \mathbf{Z}_2)$ all at once in a straightforward manner, provided that a cohomological model for G is known, so that there is no need to distinguish between inflation and transgression 2-cocycles, as it has traditionally been the case. It is the cohomological analog of the homological reduction method described in [8]. An early precedent of this technique is located in [18], focusing on p -groups.

This procedure may be extended in order to construct a basis for n -cocycles as well as n -dimensional n -cocyclic matrices over a group G for which a cohomological model $cohG$ is known. We call this process the ‘‘cohomological reduction method’’.

The term *cohomological model* refers to a special type of homotopy equivalence (termed *contraction* [16])

$$\phi: Hom(\bar{B}(\mathbf{Z}[G]), \mathbf{Z}_2) \underset{K}{\overset{F}{\cong}} cohG$$

from the set of homomorphisms of the *reduced bar construction* (i.e. the reduced complex associated to the standard bar resolution [16]) of the group G onto \mathbf{Z} , to a differential graded comodule of finite type $cohG$. Thus

$$H^*(G) = H^*(Hom(\bar{B}(\mathbf{Z}[G]), \mathbf{Z}_2)) = H^*(cohG)$$

and the n -cohomology of G and its representative n -cocycles may be effectively computed from those of $cohG$, by means of Veblen’s algorithm [29] (involving the Smith’s normal forms of the matrices representing the codifferential operator).

In particular, if $\phi: \bar{B}(\mathbf{Z}[G]) \underset{K}{\overset{F}{\cong}} (hG, d)$ defines a homological model for G , then

$$\phi^*: Hom(\bar{B}(\mathbf{Z}[G]), \mathbf{Z}_2) \underset{F^*}{\overset{K^*}{\cong}} (Hom(hG, \mathbf{Z}_2), d^*)$$

defines a cohomological model for G . This way, the set of groups for which a cohomological model is known includes those groups for which a homological model is known. Consequently, this method extends the homological reduction method described in [8].

The paper is organized as follows. In Sect. 2 we describe the cohomological reduction method itself, that is, how to construct a full basis for n -cocycles over G from a cohomological model $cohG$ for G . Section 3 is devoted to show several 2 and 3 dimensional examples, including the well-known cases of dihedral groups D_{2t} and abelian groups \mathbf{Z}_{2t} , $\mathbf{Z}_{2t} \times \mathbf{Z}_2$ and $\mathbf{Z}_t \times \mathbf{Z}_2^2$ for clarity. All the calculations have been made with aid of some packages in Mathematica provided by the authors in [5–7]. Some conclusions and future work is described in the last section.

2 Describing the cohomological reduction method

The cohomological reduction method provides a computationally efficient way to lift the cohomological information from a cohomological model

$$\phi: Hom(\bar{B}(\mathbf{Z}[G]), \mathbf{Z}_2) \xrightleftharpoons[K]{F} cohG$$

for a group G to the group itself. The injection morphism

$$K : cohG \longrightarrow Hom(\bar{B}(\mathbf{Z}[G]), \mathbf{Z}_2)$$

helps in this task.

Let $\mathcal{B}_{i-1} = \{\mathbf{u}_1, \dots, \mathbf{u}_q\}$, $\mathcal{B}_i = \{\mathbf{e}_1, \dots, \mathbf{e}_r\}$ and $\mathcal{B}_{i+1} = \{\mathbf{v}_1, \dots, \mathbf{v}_s\}$ be the corresponding basis for $cohG$ at dimensions $i - 1$, i and $i + 1$, respectively. Attending to Veblen’s algorithm, since $H^i(G; \mathbf{Z}_2) \cong H^i(cohG; \mathbf{Z}_2) = Ker d^i / Im d^{i-1}$, we need to calculate the binary (i.e. with coefficients in \mathbf{Z}_2) Smith Normal Forms of the matrices representing the codifferential operators d^{i-1} and d^i ,

$$M_{i-1}(d) = \begin{pmatrix} d(\mathbf{u}_1) \\ \vdots \\ d(\mathbf{u}_q) \end{pmatrix}_{q \times r} D_{i-1} = \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}_{q \times r} M_i(d) = \begin{pmatrix} d(\mathbf{e}_1) \\ \vdots \\ d(\mathbf{e}_r) \end{pmatrix}_{r \times s} D_i = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}_{r \times s}$$

so that $H^i(G; \mathbf{Z}_2) \cong H^i(cohG; \mathbf{Z}_2) \cong \mathbf{Z}_2^{r-k-l}$.

Furthermore, some change of basis matrices P_j and Q_j exist, for $j = i - 1, i$, such that

$$\begin{array}{ccc} \mathcal{B}_j & \xrightarrow{M_j(d)} & \mathcal{B}_{j+1} \\ P_j \uparrow & \# & \downarrow Q_j \\ \bar{\mathcal{B}}_j & \xrightarrow{D_j} & \bar{\mathcal{B}}_{j+1} \end{array} \quad D_j = P_j \cdot M_j(d) \cdot Q_j$$

Now we proceed according to the following steps:

1. A basis \bar{C} for $Im d^{i-1}$ with regards to \mathcal{B}_i is obtained from the first l columns of Q_{i-1}^{-1} . Thus \bar{C} is a basis for i -coboundaries over $cohG$.
2. A basis \bar{D} for $Ker d^i$ with regards to \mathcal{B}_i is obtained from the $r - k$ last rows of P_i . Thus \bar{D} is a basis for i -cocycles over $cohG$.

3. Select those $r - k - l$ elements in \bar{D} which are not linear combinations of the elements in \bar{C} . This forms a basis \bar{B} for representative i -cocycles over $\text{coh}G$.
4. Lift \bar{B} to the correspondent basis \mathcal{B} in $H^i(\bar{B}_i(\mathbf{Z}[G]); \mathbf{Z}_2)$ by means of the injection K .

Graphically,

$$\text{Hom}(\bar{B}_i(\mathbf{Z}[G]), \mathbf{Z}_2) \xleftarrow{K} \mathcal{B}_i$$

$$\uparrow P_i, Q_{i-1}^{-1}$$

$$\bar{B}_i$$

Proposition 1 *The scheme above defines a basis \mathcal{B} for representative i -cocycles over G .*

A basis for i -coboundaries may be obtained by Linear Algebra. More concretely, denote $\partial_{[g_1, \dots, g_{i-1}]} : \bar{B}_i(\mathbf{Z}[G]) \rightarrow \mathbf{Z}_2$ the i -coboundary associated to the characteristic map $\delta_{[g_1, \dots, g_{i-1}]}$ of the element $[g_1, \dots, g_{i-1}] \in \bar{B}_{i-1}(\mathbf{Z}[G])$,

$$\partial_{[g_1, \dots, g_{i-1}]}([h_1, \dots, h_i]) = \delta_{[g_1, \dots, g_{i-1}]}([h_2, \dots, h_i]) + \delta_{[g_1, \dots, g_{i-1}]}([h_1, \dots, h_{i-1}])$$

$$+ \sum_{j=1}^{i-1} \delta_{[g_1, \dots, g_{i-1}]}([h_1, \dots, h_j h_{j+1}, \dots, h_i]) \quad \text{mod } 2$$

Take the $4t \times \dots \times 4t$ i -dimensional matrix $M_{\partial_{[g_1, \dots, g_{i-1}]}}$ related to $\partial_{[g_1, \dots, g_{i-1}]}$ as vectors of length $4^i t^i$. Moreover, consider the $4^{i-1} t^{i-1} \times 4^i t^i$ matrix C whose rows are the vectors $M_{\partial_{[g_1, \dots, g_{i-1}]}}$. Then a row reduction on C leads to a basis for i -coboundaries. It suffices to keep track of those coboundaries $\partial_{[g_1, \dots, g_{i-1}]}$ whose transformed rows in $M_{\partial_{[g_1, \dots, g_{i-1}]}}$ after the row reduction are not zero.

Lemma 1 *The morphisms $\partial_{[g_1, \dots, g_{i-1}]}$ above define a basis for i -coboundaries.*

The cohomological reduction method provides then the following algorithm for computing n -dimensional n -cocyclic matrices over G .

Algorithm 1 (*cohomological reduction method*)

INPUT: group with cohomological model $\{G, \text{coh}G, F, K, \phi\}$

Construct a basis \mathcal{E} for i -coboundaries (Lemma 1).

Construct a basis \mathcal{B} for representative i -cocycles (Proposition 1).

OUTPUT: By juxtaposition, a basis $\mathcal{B} \cup \mathcal{E}$ for i -cocycles over G .

3 Examples

We next show how the cohomological reduction method works for constructing basis for 2-cocycles and 3-cocycles over some groups (which have been shown to provide many 2-cocyclic Hadamard matrices, see [9]).

All the executions and examples of this section have been worked out with aid of the *Mathematica 4.0* notebooks [5,6] described in [4,10] (for constructing homological models) and [3] (in order to form a basis for 2-cocycles from which the search for cocyclic Hadamard matrices is then developed), running on a *Intel(R) Core(TM) i3 CPU, M330, 2.13GHz, 4,00GB RAM, 64 bits*.

In the sequel, the elements of a product $A \times B$ are ordered as the rows of a matrix indexed in $|A| \times |B|$. For instance, if $|A| = r$ and $|B| = c$, the ordering is

$$\langle a_1b_1, a_1b_2, \dots, a_1b_c, a_2b_1, a_2b_2, \dots, a_2b_c, \dots, a_rb_1, \dots, a_rb_c \rangle$$

The elements in the group G are labeled from 1 to $|G|$, accordingly to this ordering.

The back negacyclic matrix of order j is denoted by $BN_j = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & & \ddots & -1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & -1 & \dots & -1 \end{pmatrix}_{j \times j}$,

as usual. The square matrix of order n formed all of 1s is denoted by 1_n . The Kronecker product of matrices is denoted by \otimes , so that $A \otimes B$ is the block matrix $\begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix}$. The Hadamard (pointwise) product of matrices is simply denoted

as $A \cdot B$. We use the Kronecker-Iverson notation $[b]$ (see [19]), which evaluates to 1 for Boolean expressions b having value true, and to 0 for those having value false. Finally, the notation $[x]_m$ refers to $x \bmod m$.

Let consider the families of groups below (assume $\mathbf{Z}_k = \{0, 1, \dots, k - 1\}$ with additive law).

1. $G_1^t = \mathbf{Z}_{2t} \times \mathbf{Z}_2$.
2. $G_2^t = \mathbf{Z}_t \times \mathbf{Z}_2^2 = \mathbf{Z}_t \times (\mathbf{Z}_2 \times \mathbf{Z}_2)$. Notice that $G_2^t \simeq G_1^t$ for odd t .
3. $G_3^t = \mathbf{Z}_{2t}$.
4. $G_4^t = D_{2t}$, for odd t .

In this section we will construct a cohomological model for G_i^t from the homological models for G_i^t described in [9], so that if $\phi: \bar{B}(\mathbf{Z}[G_i^t]) \xrightarrow{F} (hG_i^t, d)$ defines a homological model for G_i^t , then

$$\phi^*: Hom(\bar{B}(\mathbf{Z}[G_i^t]), \mathbf{Z}_2) \xrightarrow{F^*} (Hom(hG_i^t, \mathbf{Z}_2), d^*)$$

defines a cohomological model for G_i^t . Here, as usual, we use $-^*$ for noting the dual object for $-$.

More concretely, it suffices to take duals on the basis \mathcal{B}_k for hG_i^t on degree $1 \leq k \leq 4$, and the differential operators $d_j : B_{j+1} \rightarrow B_j$ and the projections $F_j : \bar{B}_j(\mathbf{Z}[G_i^t]) \rightarrow B_j$ for $2 \leq j \leq 3$. Recall that

$$\bar{B}_k(\mathbf{Z}[G]) = \langle [g_1, \dots, g_k] : g_j \in G \rangle.$$

Notice that the matrices P and Q involved in the calculation of the Smith Normal Form, D , for a matrix A (so that $D = P \cdot A \cdot Q$) are not uniquely determined, in general. In the sequel we will use the matrices coming from the SmithNormalForm package programmed in [7].

For illustrating the method, we will display explicitly the computation for the case of $G_1^t = \mathbf{Z}_{2t} \times \mathbf{Z}_2$, giving a basis for 2- and 3-cocycles over G_1^t in detail. On the contrary, just a summary of the results of the analog computations for G_2^t will be presented. We will show that the basis of 2-cocycles obtained so far and the basis for 2-cocycles calculated in [9] are equivalent. Later on, we will use the basis for 3-cocycles in order to look for 3-dimensional Hadamard matrices of order $4t$ over either G_1^t or G_2^t .

In addition, we will calculate some basis for 3-cocycles over G_3^t and G_4^t , from which a search for 3-dimensional improper Hadamard matrices of order $2t$ will be performed. Surprisingly, although D_{4t} is prolific giving rise to many 2-cocyclic Hadamard matrices, $G_4^t = D_{2t}$ will show to be not suitable for looking for 3-dimensional 3-cocyclic improper Hadamard matrices, since there are not any representative 3-cocycles over G_4^t , and hence the set of 3-cocycles over D_{2t} reduces to the set of 3-coboundaries over D_{2t} .

3.1 Basis for 2-, 3-cocycles over $G_1^t = \mathbf{Z}_{2t} \times \mathbf{Z}_2$

Notice that the i -th element of G_1^t corresponds to $(\lfloor \frac{i-1}{2} \rfloor [i-1]_2) \in \mathbf{Z}_{2t} \times \mathbf{Z}_2$. Conversely, the element $(i_1, i_2) \in \mathbf{Z}_{2t} \times \mathbf{Z}_2$ corresponds to the i -th element of G_1^t , for $i = 2i_1 + i_2 + 1$.

A homological model for G_1^t is described in [9] and consists of

$$\begin{aligned} \mathcal{B}_1 &= \{u_1, u_2\}, \quad \mathcal{B}_2 = \{e_1, e_2, e_3\}, \quad \mathcal{B}_3 = \{v_1, v_2, v_3, v_4\}, \\ d_2(e_1) &= 2t \cdot u_1, \quad d_2(e_3) = 2 \cdot u_2, \quad d_3(v_2) = 2t \cdot e_2, \quad d_3(v_3) = -2 \cdot e_2, \\ F_1[i] &= i_1 \cdot u_1 + i_2 \cdot u_2, \quad F_2[i|j] = [i_1 + j_1 \geq 2t] \cdot e_1 + i_1 j_2 \cdot e_2 \\ &\quad + [i_2 + j_2 \geq 2] \cdot e_3. \end{aligned}$$

In these circumstances, it may be checked that

$$\begin{aligned} \mathcal{B}_1^* &= \{u_1^*, u_2^*\}, \quad \mathcal{B}_2^* = \{e_1^*, e_2^*, e_3^*\}, \quad \mathcal{B}_3^* = \{v_1^*, v_2^*, v_3^*, v_4^*\}, \\ d^1 &= d_2^* = 0, \quad d^2 = d_3^* = 0, \\ F^*(e_1^*)([i|j]) &= e_1^*(F([i|j])) = [[i_1 + j_1 \geq 2t]]_2, \quad F^*(e_2^*)([i|j]) = e_2^*(F([i|j])) \\ &= [i_1 j_2]_2, \quad F^*(e_3^*)([i|j]) = e_3^*(F([i|j])) = [[i_2 + j_2 \geq 2]]_2. \end{aligned}$$

From these data, it may be checked that

D_1	Q_1^{-1}	D_2	P_2
$0_{2 \times 3}$	I_3	$0_{3 \times 4}$	I_3

Thus $H^2(\text{coh}G_1^t; \mathbf{Z}_2) = \mathbf{Z}_2^3$ and $\mathcal{B}_2^* = \{e_1^*, e_2^*, e_3^*\}$ is a basis for representative 2-cocycles over $\text{coh}G_1^t$. Accordingly, a basis for representative 2-cocycles over G_1^t is given by

$$\{F^*(e_1^*), F^*(e_2^*), F^*(e_3^*)\} = \left\{ BN_{2t} \otimes 1_2, 1_t \otimes \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & - & 1 & - \end{pmatrix}, 1_{2t} \otimes BN_2 \right\}.$$

This basis may be extended to a full basis for 2-cocycles over G_1^t , by simply juxtaposing a basis for 2-coboundaries over G_1^t (see Lemma 1). This basis is $\langle \partial_2, \dots, \partial_{4t-2} \rangle$ as it was pointed out in [9].

Remark 1 A basis $\{\beta_1, \beta_2, \gamma_1\}$ for representative 2-cocycles over G_1^t was already determined in [9] (notice that G_1^t was denoted G_2^t there). It may be checked that $\beta_1 = F^*(e_3^*)$ and $\gamma_1 = F^*(e_2^*)$. If $2t = 2^r q$, for odd q , then

$$\beta_2 = 1_q \otimes BN_{2^r} \otimes 1_2 = F^*(e_1^*) \cdot \prod_{k=0}^{\lfloor \frac{q}{2} \rfloor - 1} \partial_{k2^{r+2}+2^{r+1}+1} \dots \partial_{k2^{r+2}+2^{r+2}}.$$

Now we compute a basis for 3-cocycles over G_1^t . From calculations in [4, 9], it is easy to derive that the homological model described above may be extended to degree 4, so that $\mathcal{B}_4 = \{w_1, w_2, w_3, w_4, w_5\}$, $d_4(w_1) = 2t \cdot v_1$, $d_4(w_3) = 2t \cdot v_3 + 2 \cdot v_2$, $d_4(w_5) = 2 \cdot v_4$, $F[i|j|k] = [k_1[i_1 + j_1 \geq 2t] \cdot v_1 + k_2[i_1 + j_1 \geq 2t] \cdot v_2 + i_1[j_2 + k_2 \geq 2] \cdot v_3 + k_2[i_2 + j_2 \geq 2] \cdot v_4]_2$.

Consequently, we consider $\mathcal{B}_4^* = \{w_1^*, w_2^*, w_3^*, w_4^*, w_5^*\}$, $d^3 = 0$, $F^*(v_1^*)([i|j|k]) = v_1^*(F([i|j|k])) = [k_1[i_1 + j_1 \geq 2t]]_2$, $F^*(v_2^*)([i|j|k]) = v_2^*(F([i|j|k])) = [k_2[i_1 + j_1 \geq 2t]]_2$, $F^*(v_3^*)([i|j|k]) = v_3^*(F([i|j|k])) = [i_1[j_2 + k_2 \geq 2]]_2$, $F^*(v_4^*)([i|j|k]) = v_4^*(F([i|j|k])) = [k_2[i_2 + j_2 \geq 2]]_2$.

From these data, it may be checked that

D_2	Q_3^{-1}	D_3	P_3
$0_{3 \times 4}$	I_4	$0_{4 \times 5}$	I_4

Thus $H^3(\text{coh}G_1^t; \mathbf{Z}_2) = \mathbf{Z}_2^4$ and $\mathcal{B}_3^* = \{v_1^*, v_2^*, v_3^*, v_4^*\}$ is a basis for representative 3-cocycles over $\text{coh}G_1^t$. Accordingly, a basis for representative 3-cocycles over G_1^t is given by $\{F^*(v_1^*), F^*(v_2^*), F^*(v_3^*), F^*(v_4^*)\}$.

It may be straightforwardly checked that the $F^*(v_i^*)$ are the 3D-matrices whose horizontal sections are given by:

- $F^*(v_1^*) : \{J_{4t}, J_{4t}, BN_{2t} \otimes 1_2, BN_{2t} \otimes 1_2, \dots, J_{4t}, J_{4t}, BN_{2t} \otimes 1_2, BN_{2t} \otimes 1_2\}$.
- $F^*(v_2^*) : \{J_{4t}, BN_{2t} \otimes 1_2, \dots, J_{4t}, BN_{2t} \otimes 1_2\}$.
- $F^*(v_3^*) : \{J_{4t}, 1_t \otimes A, \dots, J_{4t}, 1_t \otimes A\}$, for $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & - & 1 & - \end{pmatrix}$.
- $F^*(v_4^*) : \{J_{4t}, 1_{2t} \otimes BN_2, \dots, J_{4t}, 1_{2t} \otimes BN_2\}$.

This basis may be extended to a full basis for 3-cocycles over G_1^t , by simply juxtaposing a basis for 3-coboundaries over G_1^t (see Lemma 1). This basis is given by $\langle \partial_1, \dots, \partial_{16t^2-4t-2}, \partial_{16t^2-4t+1} \rangle$ by direct inspection.

3.2 Basis for 2-, 3-cocycles over $G_2^t = \mathbf{Z}_t \times \mathbf{Z}_2^2 = \mathbf{Z}_t \times (\mathbf{Z}_2 \times \mathbf{Z}_2)$

Progressing from the homological model for G_2^t described in [9], the cohomological reduction method straightforwardly provides that $H^2(\text{coh}G_2^t; \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2^3 & \text{for } t \text{ odd,} \\ \mathbf{Z}_2^6 & \text{for } t \text{ even.} \end{cases}$ Depending on whether t is odd or even, a basis for representative 2-cocycles over G_2^t is obtained considering just the first three elements or the full set of six matrices of the following set: $\{BN_t \otimes 1_4, 1_{\frac{t}{2}} \otimes K_2, 1_{\frac{t}{2}} \otimes K_3, 1_t \otimes BN_2 \otimes 1_2, 1_t \otimes K_1, 1_{2t} \otimes BN_2\}$ where the matrices K_1, K_2, K_3 are given by

K_1	K_2	K_3
$\begin{pmatrix} 11 & 11 \\ 11 & 11 \\ 1-1 & 1-1 \\ 1-1 & 1-1 \end{pmatrix}$	$\begin{pmatrix} 1111 & 1111 & 1111 & 1 \\ 1111 & 1111 & 1111 & 1 \\ 1111 & 1111 & 1111 & 1 \\ 11-- & --11-- & -- & -- \\ 11-- & --11-- & -- & -- \\ 11-- & --11-- & -- & -- \\ 11-- & --11-- & -- & -- \end{pmatrix}$	$\begin{pmatrix} 11 & 11 & 11 & 11 & 11 & 1 \\ 11 & 11 & 11 & 11 & 11 & 1 \\ 11 & 11 & 11 & 11 & 11 & 1 \\ 11 & 11 & 11 & 11 & 11 & 1 \\ 1- & -1- & -1- & -1- & -1- & - \\ 1- & -1- & -1- & -1- & -1- & - \\ 1- & -1- & -1- & -1- & -1- & - \\ 1- & -1- & -1- & -1- & -1- & - \end{pmatrix}$

This basis may be extended to a full basis for 2-cocycles over G_2^t , by simply juxtaposing a basis for 2-coboundaries over G_2^t . Such a basis was described in [9] (according to Lemma 1),

t	Basis for 2-coboundaries
$[t]_2 = 1$	$\langle \partial_2, \dots, \partial_{4t-2} \rangle$
$[t]_2 = 0$	$\langle \partial_2, \dots, \partial_{4t-3} \rangle$

Remark 2 All the generators above coincide with those of [9] (notice that G_2^t was denoted G_4^t there), excepting $BN_t \otimes 1_4$, which is substituted by $\beta_3 = 1_q \otimes BN_{2^r} \otimes 1_4$, for $t = 2^r q$. It may be checked that

$$\beta_3 = BN_t \otimes 1_4 \cdot \prod_{k=0}^{\lfloor \frac{q}{2} \rfloor - 1} \partial_{k2^{r+3}+2^{r+2}+1} \dots \partial_{k2^{r+3}+2^{r+3}}$$

The cohomological reduction method also computes $H^3(\text{coh}G_2^t; \mathbf{Z}_2)$ = $\begin{cases} \mathbf{Z}_2^4 & \text{for } t \text{ odd} \\ \mathbf{Z}_2^{10} & \text{for } t \text{ even} \end{cases}$ and a basis $\{\psi_1, \dots, \psi_{10}\}$ (just $\{\psi_7, \dots, \psi_{10}\}$ if t is odd) for 3-cocycles over G_2^t is given by:

- $\psi_1 = \{J_{4t}, J_{4t}, J_{4t}, J_{4t}, BN_t \otimes 1_4, BN_t \otimes 1_4, BN_t \otimes 1_4, BN_t \otimes 1_4, \dots, J_{4t}, J_{4t}, J_{4t}, J_{4t}, BN_t \otimes 1_4, BN_t \otimes 1_4, BN_t \otimes 1_4, BN_t \otimes 1_4\}$.
- $\psi_2 = \{J_{4t}, J_{4t}, BN_t \otimes 1_4, BN_t \otimes 1_4, \dots, J_{4t}, J_{4t}, BN_t \otimes 1_4, BN_t \otimes 1_4\}$.
- $\psi_3 = \{J_{4t}, BN_t \otimes 1_4, \dots, J_{4t}, BN_t \otimes 1_4\}$.
- $\psi_4 = \{J_{4t}, J_{4t}, 1_{\frac{t}{2}} \otimes A \otimes 1_2, 1_{\frac{t}{2}} \otimes A \otimes 1_2, \dots, J_{4t}, J_{4t}, 1_{\frac{t}{2}} \otimes A \otimes 1_2, 1_{\frac{t}{2}} \otimes A \otimes 1_2\}$,
for $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & - & 1 & - \end{pmatrix}$.
- $\psi_5 = \{J_{4t}, 1_{\frac{t}{2}} \otimes A \otimes 1_2, \dots, J_{4t}, 1_{\frac{t}{2}} \otimes A \otimes 1_2\}$.
- $\psi_6 = \{J_{4t}, 1_{\frac{t}{2}} \otimes \begin{pmatrix} J_4 & J_4 \\ B & B \end{pmatrix}, \dots, J_{4t}, 1_{\frac{t}{2}} \otimes \begin{pmatrix} J_4 & J_4 \\ B & B \end{pmatrix}\}$, for $B = \begin{pmatrix} 1 & - & 1 & - \\ 1 & - & 1 & - \\ 1 & - & 1 & - \\ 1 & - & 1 & - \end{pmatrix}$.
- $\psi_7 = \{J_{4t}, J_{4t}, 1_t \otimes BN_2 \otimes 1_2, 1_t \otimes BN_2 \otimes 1_2, \dots, J_{4t}, J_{4t}, 1_t \otimes BN_2 \otimes 1_2, 1_t \otimes BN_2 \otimes 1_2\}$.
- $\psi_8 = \{J_{4t}, 1_t \otimes BN_2 \otimes 1_2, \dots, J_{4t}, 1_t \otimes BN_2 \otimes 1_2\}$.
- $\psi_9 = \{J_{4t}, 1_t \otimes A, \dots, J_{4t}, 1_t \otimes A\}$.
- $\psi_{10} = \{J_{4t}, 1_{2t} \otimes BN_2, \dots, J_{4t}, 1_{2t} \otimes BN_2\}$.

This basis may be extended to a full basis for 3-cocycles over G_2^t , by simply juxtaposing a basis for 3-coboundaries over G_3^t . According to Lemma 1, such a basis is given by by direct inspection.

t	Basis for 3-coboundaries
$[t]_2 = 1$	$\langle \partial_1, \dots, \partial_{16t^2-4t-2}, \partial_{16t^2-4t+1} \rangle$
$[t]_2 = 0$	$\langle \partial_1, \dots, \partial_{16t^2-8t-3}, \partial_{16t^2-8t+1}, \dots, \partial_{16t^2-4t-3}, \partial_{16t^2-4t+1}, \dots, \partial_{16t^2-4t+3} \rangle$

3.3 Basis for 3-cocycles over $G_3^t = \mathbf{Z}_{2t}$

Consider the family $G_3^t = \mathbf{Z}_{2t}$. We now apply the cohomological reduction method for calculating a basis for n -cocycles over G . It may be checked (see [16]) that a homological model for G_3^t is given by $\mathcal{B}_2^* = \{e_1^*\}$, $\mathcal{B}_3^* = \{v_1^*\}$, $\mathcal{B}_4^* = \{w_1^*\}$, $d^2 = 0$, $d^3 = 0$, $F[i|j|k] = [k[i + j \geq 2t]]_2 \cdot v_1$.

From these data, it may be checked that

D_2	Q_3^{-1}	D_3	P_3
0	1	0	1

Thus $H^3(\text{coh}G_3^t; \mathbf{Z}_2) = \mathbf{Z}_2$ and $\mathcal{B}_3^* = \{v_1^*\}$ is a basis for representative 3-cocycles over $\text{coh}G_3^t$. Accordingly, a basis for representative 3-cocycles over G_3^t is given by $\{F^*(v_1^*)\}$, whose horizontal sections are given by $\{J_{2t}, BN_{2t}, \dots, J_{4t}, BN_{4t}\}$.

This basis may be extended to a full basis for 3-cocycles over G_3^t , by simply juxtaposing a basis for 3-coboundaries over G_3^t , which is given by $\langle \partial_1, \dots, \partial_{4t^2-2t} \rangle$ by direct inspection.

3.4 Basis for 3-cocycles over $G_4^t = D_{2t}, t$ odd

Unfortunately, it is known that $H^3(D_{2t}) = 0$ for odd t , and therefore a basis for 2-cocycles over G_4^t consists in a basis for 2-coboundaries over G_4^t , such as $\langle \partial_2, \dots, \partial_{2t+1}, \partial_{4t}, \dots, \partial_{2t^2+1}, \partial_{2t^2+3}, \dots, \partial_{2t^2+t}, \partial_{2t^2+t+2}, \dots, \partial_{4t^2} \rangle$.

3.5 Calculating 3-dimensional Hadamard matrices

Using the basis for 3-cocycles over $G_1^1 = G_2^1 = \mathbf{Z}_2^2$ calculated before, we have performed an exhaustive search for 3-dimensional 3-cocyclic Hadamard matrices over \mathbf{Z}_2^2 . This search yields that there are 64 improper such matrices, none of which is in addition proper. For instance, the matrix related to the product of 3-coboundaries $\partial_4 \partial_7 \partial_{10} \partial_{13}$, whose horizontal sections are given by

$$\left\{ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ - & 1 & 1 & - \end{pmatrix}, \begin{pmatrix} 1 & 1 & - & - \\ 1 & 1 & 1 & 1 \\ 1 & - & - & 1 \\ - & 1 & - & 1 \end{pmatrix}, \begin{pmatrix} 1 & - & 1 & - \\ 1 & - & - & 1 \\ 1 & 1 & 1 & 1 \\ - & - & 1 & 1 \end{pmatrix}, \begin{pmatrix} - & 1 & 1 & - \\ - & 1 & - & 1 \\ - & - & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right\}.$$

An exhaustive computer search for 3-dimensional Hadamard matrices 3-cocyclic over $G_3^2 = \mathbf{Z}_4$, yields that there are 32 improper such matrices, none of which is in addition proper. For instance, the matrix related to the product of 3-coboundaries $\partial_4 \partial_7 \partial_8 \partial_9$, whose horizontal sections are given by

$$\left\{ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & - & - & 1 \\ - & 1 & - & 1 \\ 1 & 1 & - & - \end{pmatrix}, \begin{pmatrix} 1 & 1 & - & - \\ 1 & - & 1 & - \\ - & 1 & 1 & - \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & - & 1 & - \\ 1 & 1 & - & - \\ - & - & - & - \\ 1 & - & - & 1 \end{pmatrix}, \begin{pmatrix} - & 1 & 1 & - \\ - & - & - & - \\ 1 & 1 & - & - \\ - & 1 & - & 1 \end{pmatrix} \right\}.$$

Since there is more interest in improper n -dimensional Hadamard matrices of even order $v = 4k + 2$ (in particular, for $n = 3$ and $v \neq 2 \cdot 3^b$), we have also performed a partial heuristic search for 3-dimensional improper Hadamard matrices of order $2t$ over G_3^t and G_4^t , for $t = 3, 5$, from which unfortunately we have not got any 3-dimensional 3-cocyclic improper Hadamard matrix. Contributions in this sense (with these or other groups of order $v = 4k + 2$) would be appreciated, since improper 3-dimensional Hadamard matrices of order different to $2 \cdot 3^b$ are still to be discovered.

4 Conclusions and further work

It is well-known that there exists a proper n -dimensional Hadamard matrix of order $v \equiv 0 \pmod{4}$ for every $n \geq 3$ if and only if there exists a planar Hadamard matrix of order v . However, few is known about the existence of improper n -dimensional Hadamard matrices of order v . It is evident that v must be even, but surprisingly it need not to be a multiple of 4. And it is not known whether improper n -dimensional Hadamard matrices exist for all even orders $v = 2t$, $n \geq 3$, for odd t . Furthermore, only 3-dimensional improper Hadamard matrices of order $2 \cdot 3^b$ are known, $b \geq 1$ (see [31] for more details).

In this paper we have been concerned with higher dimensional Hadamard matrices, and the way in which the cocyclic framework may be introduced in this context, going beyond de Launey's works in [13–15].

In particular, we have provided a method for constructing n -cocyclic n -dimensional matrices over some finite groups G , from which a deeper search for improper/proper Hadamard matrices might be performed. The input data of our process is a group G for which a cohomological model is known, so that the cohomological reduction method may be straightforwardly applied.

Focusing in the case $n = 2$ (which is of special interest, e.g. for looking for cocyclic Hadamard matrices), it follows that there is no need to distinguish between inflation and transgression 2-cocycles, as it has traditionally been the case. Some examples have been given in Sect. 3.

We have also provided some examples of 3-dimensional 3-cocyclic Hadamard matrices, progressing on the basis for 3-cocycles provided by our algorithm.

Finally, we would like to conclude this paper proposing several problems concerning higher dimensional (cocyclic) Hadamard matrices, which should be studied in a near future, such as:

1. Characterize n -cocyclic n -dimensional proper and improper Hadamard matrices. One could think that an n -dimensional n -cocyclic matrix would consist of 2-cocyclic layers. This is not true at all, as it will be discussed elsewhere. Thus the traditional cocyclic test for 2-dimensional 2-cocyclic Hadamard matrices cannot be naturally extended to the n -dimensional case so far. A deeper analysis must be done.
2. Look for n -dimensional proper and improper n -cocyclic Hadamard matrices. This is a very difficult task, since the search space seems to grow drastically in exponential size (e.g., the basis for 3-dimensional 3-cocyclic matrices over G_1^t and G_2^t , t even, consists of $16t^2 - 4t + 3$ and $16t^2 - 4t + 7$ generators, respectively). Maybe one should think of constructing n -dimensional improper Hadamard matrices based on planar cocyclic matrices satisfying some certain constraints, in light of de Launey's fruitful way for constructing n -dimensional proper Hadamard matrices from 2-cocyclic Hadamard matrices, as described in [13–15].
3. Determine whether any of the already known construction methods for generating higher dimensional proper Hadamard matrices from 2-dimensional ones (see [23, 31] for details) involves n -cocyclic matrices.

Furthermore, is it possible to derive a method for constructing n -dimensional proper/improper n -cocyclic Hadamard matrices from proper/improper k -cocyclic Hadamard matrices, $k < n$ (e.g. via the cup product in cohomology)?

4. Do a n -dimensional improper (n -cocyclic) Hadamard conjecture make sense?

Proper higher dimensional Hadamard matrices might exist only for orders 1, 2 and a multiple of 4, and they do exist if and only if a planar Hadamard matrix of the same order do exist. This reduces the problem to the Hadamard Conjecture. Nevertheless, as it has been discussed earlier, only higher dimensional improper Hadamard matrices of order $2 \cdot 3^b$ are known. So the question is widely open for the improper case.

Acknowledgments The authors want to express their gratitude to the anonymous referees for their valuable advices and suggestions, which have helped to improve the readability of the paper for a better understanding.

References

1. Álvarez, V., Armario, J.A., Frau, M.D., Gudiel, F.: The maximal determinant of cocyclic $(-1, 1)$ -matrices over D_{2t} . *Linear Algebra Appl.* **436**, 858–873 (2012)
2. Álvarez, V., Armario, J.A., Frau, M.D., Real, P.: A genetic algorithm for cocyclic Hadamard matrices. *AAECC-16 Proc. LNCS* **3857**, 144–153 (2006)
3. Álvarez, V., Armario, J.A., Frau, M.D., Real, P.: Calculating cocyclic Hadamard matrices in mathematica: exhaustive and heuristic searches. *ICMS-2 Proc. LNCS* **4151**, 422 (2006)
4. Álvarez, V., Armario, J.A., Frau, M.D., Real, P.: A Mathematica notebook for computing the homology of iterated products of groups. *ICMS-2 Proc. LNCS* **4151**, 57 (2006)
5. Álvarez, V., Armario, J.A., Frau, M.D., Real, P.: <http://library.wolfram.com/infocenter/MathSource/6516/>
6. Álvarez, V., Armario, J.A., Frau, M.D., Real, P.: <http://library.wolfram.com/infocenter/MathSource/6384/>
7. Álvarez, V., Armario, J.A., Frau, M.D., Real, P.: <http://library.wolfram.com/infocenter/MathSource/6621/>
8. Álvarez, V., Armario, J.A., Frau, M.D., Real, P.: A system of equations for describing cocyclic Hadamard matrices. *J. Comb. Des.* **16**, 276–290 (2008)
9. Álvarez, V., Armario, J.A., Frau, M.D., Real, P.: The homological reduction method for computing cocyclic Hadamard matrices. *J. Symb. Comput.* **44**, 558–570 (2009)
10. Álvarez, V., Armario, J.A., Frau, M.D., Real, P.: (Co)homology of iterated semidirect products of abelian groups. *Appl. Algebra Eng. Commun. Comput.* **23**, 101–127 (2012)
11. Baliga, A., Horadam, K.J.: Cocyclic Hadamard matrices over $\mathbf{Z}_t \times \mathbf{Z}_2^2$. *Australas. J. Comb.* **11**, 123–134 (1995)
12. de Launey, W.: $(0, G)$ -Designs with applications. Ph.D. Thesis. University of Sydney, Sydney, Australia (1987)
13. de Launey, W.: On the construction of n -dimensional designs from 2-dimensional designs. *Australas. J. Comb.* **1**, 67–81 (1990)
14. de Launey, W., Flannery, D.: *Algebraic Design Theory. Mathematical Surveys and Monographs*, 175. American Mathematical Society, Providence (2011)
15. de Launey, W., Horadam, K.J.: A weak difference set construction for higher dimensional designs. *Des. Codes Cryptogr.* **3**, 75–87 (1993)
16. Eilenberg, S., Mac Lane, S.: On the groups $H(\pi, n)$ II. *Ann. Math.* **66**, 49–139 (1954)
17. Flannery, D.L.: Cocyclic Hadamard matrices and Hadamard groups are equivalent. *J. Algebra* **192**, 749–779 (1997)
18. Grabmeier, J., Lambe, L.A.: Computing resolutions over finite p -groups. In: Betten, A., Kohnert, A., Lave, R., Wassermann, A. (eds.) *Proceedings ALCOMA'99. Springer Lecture Notes in Computational Science and Engineering*. Springer, New York (2000)

19. Graham, R.L., Knuth, D.E., Patashnik, O.: *Concrete Mathematics*. Addison-Wesley, Reading (1989)
20. Hadamard, J.: Résolution d'une question relative aux déterminants. *Bull. Sci. Math.* **17**, 240–246 (1893)
21. Hammer, J., Seberry, J.: Higher dimensional orthogonal designs and applications. *IEEE Trans. Inf. Theory* **27**(6), 772–779 (1981)
22. Hedayat, A., Wallis, W.D.: *Hadamard matrices and their applications*. *Ann. Stat.* **6**, 1184–1238 (1978)
23. Horadam, K.J.: *Hadamard Matrices and Their Applications*. Princeton University Press, Princeton (2007)
24. Horadam, K.J., Lin, C.: Construction of proper higher dimensional Hadamard matrices from perfect binary arrays. *JCMCC* **28**, 237–248 (1998)
25. Horadam, K.J., de Launey, W.: Generation of cocyclic Hadamard matrices. *Computational algebra and number theory (Sydney, 1992)*. volume 325 of *Math. Appl.*, 279–290. Kluwer Acad. Publ., Dordrecht (1995)
26. Kotsireas, I.S.: Structured Hadamard conjecture. In: Borwein, J.M., et al. (eds.) *Number Theory and Related Fields*. In Memory of Alf van der Poorten, *Springer Proceedings in Mathematics and Statistics*, pp. 215–227. Springer, New York (2013)
27. Shlichta, P.J.: Three and four-dimensional Hadamard matrices. *Bull. Am. Phys. Soc.* **16**, 825–826 (1971)
28. Shlichta, P.J.: Higher dimensional Hadamard matrices. *IEEE Trans. Inf. Theory* **IT-25**, 566–572 (1979)
29. Veblen, O.: *Analysis Situs*, 5th edn. A.M.S. Publications, Providence (1931)
30. Yang, Y.X.: The proofs of some conjectures on higher dimensional Hadamard matrices. *Kexue Tongbao (English translation)* **31**, 1662–1667 (1986)
31. Yang, Y.X.: *Theory and Applications of Higher-Dimensional Hadamard Matrices*. *Combinatorics and Computer Science Series*. Science Press/Kluwer, Beijing/Dordrecht (2001)