On the practical global uniform asymptotic stability of stochastic differential equations[*]

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Abstract

The method of Lyapunov functions is one of the most effective ones for the investigation of stability of dynamical systems, in particular, of stochastic differential systems. The main purpose of the paper is the analysis of the stability of stochastic differential equations by using Lyapunov functions when the origin is not necessarily an equilibrium point. The global uniform boundedness and the global practical uniform exponential stability of solutions of stochastic differential equations based on Lyapunov techniques are investigated. Furthermore, an example is given to illustrate the applicability of the main result.

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1   Introduction

The main aim of this paper is to establish some criteria for the global practical uniform exponential stability of a class of nonlinear stochastic differential equations of the form

$$dx(t) = f(x(t), t)dt + g(x(t), t)dW(t). \tag{1.1}$$

As it is well known, stochastic differential equations are useful for modeling physical, technical, biological and economical dynamical systems in which significant uncertainly is present.

To investigate stability for stochastic differential equations, we usually study the stability and attractivity of solutions in a neighborhood of the origin considered as an equilibrium point. The equations are worth being investigated since the solutions with different initial values have similar large-time properties. When $f(0, t) = g(0, t) = 0$, the stability and the exponential stability of equation (1.1) have received very much attention in the literature. We would like to mention here the references [5]-[14], among others.

Several interesting and important variants to Lyapunov's original concepts of practical stability were proposed in [1]-[4]. When the origin is not necessarily an equilibrium point, we can study the asymptotic stability of solutions with respect to a small neighborhood of the origin. The goal is to analyze systems whose desired behavior is asymptotic stability about a small ball of the state space or a close approximation to this. Almost all the state trajectories are bounded and approach a sufficiently small neighborhood of the origin. One also wishes that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner, for instance, exponentially. This property is referred to as practical stability.

Therefore, in this paper we state some sufficient conditions to ensure the almost sure practical global uniform asymptotic stability and the exponential stability of the solution of equation (1.1). The content of the paper is as follows. In Section 2, we introduce some auxiliary facts and results. In Section 3, we prove the main theorem about the global practical uniform exponential stability of stochastic differential equation and exhibit an example as an application of this study.

2   Auxiliary Facts and Results

Consider the following d-dimensional stochastic differential equation (SDE):

$$dx(t) = f(x(t), t)dt + g(x(t), t)dW_t, \quad \forall t \geq 0, \tag{2.1}$$

where $f : \mathbb{R}^d \times \mathbb{R}_+ \longrightarrow \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{R}_+ \longrightarrow \mathbb{R}^{d \times m}$, $x(t) = (x_1(t), ..., x_d(t))^T \in \mathbb{R}^d$ and $W_t = (W_1(t), ..., W_m(t))^T$ is an m-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The Euclidian norm of a vector $x \in \mathbb{R}^d$ is denoted by $|x|$.

We assume that both f and g satisfy the following conditions:

$$|f(x, t)| \leq C(1 + |x|); \quad |g(x, t)| \leq C(1 + |x|), \text{ for all } t \geq 0, x \in \mathbb{R}^d, \tag{2.2}$$

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$$|f(x,t) - f(y,t)| \leq C|x - y|; \quad |g(x,t) - g(y,t)| \leq C|x - y|, \text{ for all } t \geq 0, x, y \in \mathbb{R}^d. \quad (2.3)$$

Hence, for any given initial value $x_0 \in \mathbb{R}^d$, the SDE has a unique global solution denoted by $x(t, t_0, x_0)$, such that $x(t_0) = x_0$, and defined for all $t \geq t_0 \geq 0$. In what follows we use $x(t, t_0, x_0)$, or simply $x(t)$ if no confusion is possible, to denote the solution of (2.1) with initial value x_0 at the initial time t_0.

We suppose that there exists t such that $f(0, t) \neq 0$ and $g(0, t) \neq 0$. In this case, the SDE does not possess the trivial solution $x(t, t_0, 0) = 0$. Therefore, we will study the asymptotic stability of the SDE when 0 in not an equilibrium point, but in a small neighborhood of the origin in terms of convergence of solution in probability to a small ball $B_r := \{x \in \mathbb{R}^d : |x| \leq r\}$, $r > 0$.

Definition 2.1. The solution of (2.1) is said to be globally uniformly bounded in probability if, for each $\alpha > 0$, there exists $c = c(\alpha) > 0$ (independent of t_0) such that,

$$\text{for every } t_0 \geq 0, \text{ and all } x_0 \in \mathbb{R}^d \text{ with } \quad |x_0| \leq \alpha, \quad \sup_{t \geq t_0} |x(t, t_0, x_0)| \leq c(\alpha), \quad \text{a.s.} \quad (2.4)$$

Definition 2.2. (i) The ball B_r is said to be uniformly stable in probability if, for each $\epsilon \in]0, 1[$ and $k > r$, there exists $\delta = \delta(\epsilon, k) > 0$ such that

$$\mathbb{P}\big(|x(t, t_0, x_0)| < k, \forall t \geq t_0 \geq 0\big) \geq 1 - \epsilon \quad \text{for all} \quad |x_0| < \delta. \quad (2.5)$$

(ii) The ball B_r is said to be globally uniformly stable in probability if it is uniformly stable in probability and the solution of (2.1) is globally uniformly bounded in probability.

Definition 2.3. The ball B_r is said to be globally uniformly attractive in probability if for each $\epsilon \in]0, 1[$, $k > r$ and $c > 0$ (independent of t_0), there exists $T = T(\epsilon, c) > 0$ such that, for all $t_0 \geq 0$, it holds

$$\mathbb{P}\big(|x(t, t_0, x_0)| < k, \forall t \geq t_0 + T\big) \geq 1 - \epsilon \quad \text{for all } x_0 \in \mathbb{R}^d \text{ such that } |x_0| < c. \quad (2.6)$$

Let us now denote by $C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+, \mathbb{R}_+)$ the family of all nonnegative functions $V(x, t)$ defined on $\mathbb{R}^d \times \mathbb{R}_+$ which are twice continuously differentiable in x and once in t. Thanks to the Itô formula it follows

$$dV(x(t), t) = LV(x(t), t)dt + V_x(x(t), t)g(x(t), t)dW_t,$$

where

$$LV(x, t) = V_t(x, t) + V_x(x, t)f(x, t) + \frac{1}{2}trace[g(x, t)^T V_{xx}(x, t)g(x, t)],$$

$$V_t(x, t) = \frac{\partial V}{\partial t}(x, t) \; ; \; V_x(x, t) = \big(\frac{\partial V}{\partial x_1}(x, t), ..., \frac{\partial V}{\partial x_d}(x, t)\big); \; V_{xx}(x, t) = \Big(\frac{\partial^2 V}{\partial x_i \partial x_j}(x, t)\Big)_{d \times d}.$$

Let us also recall that by $\mathcal{K}$ it is usual to denote the family of all continuous nondecreasing functions $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(0) = 0$, and by $\mathcal{K}_\infty$ the class of all functions $\mu \in \mathcal{K}$ such that $\mu(r) \to +\infty$, $r \to +\infty$.

Now we can prove our first result.

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Theorem 2.1. *Assume that there exist* $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+, \mathbb{R}_+)$, $\mu_1, \mu_2 \in \mathcal{K}_\infty$, *with* $\mu_1 : \mathbb{R}_+^* \to \mathbb{R}_+^*$ *and* $\mu_3 \in \mathcal{K}$, $M > 0$, *such that for all* $t \geq 0$, *and all* $x \in \mathbb{R}^d$,

$$\mu_1(|x|) \leq V(x, t) \leq \mu_2(|x|),$$

$$LV(x, t) \leq \varrho(t) - \mu_3(|x|),$$

where $\varrho(t)$ *is a continuous nonnegative function with*

$$\lim_{t \to +\infty} \varrho(t) = 0,$$

and

$$\int_0^{+\infty} \varrho(t) dt \leq M < +\infty.$$

Then, system (2.1) is globally uniformly practically stable in probability.

Proof. Let $\epsilon \in]0, 1[$, and take a small positive real number r, and $k > r$. Assume that there exists $\delta = \delta(\epsilon, k) \in]0, k[$ such that, for $x_0 \in S_\delta$, $S_\delta = \{x \in \mathbb{R}^d / |x| < \delta\}$ and $\mu_1(k) > \dfrac{M}{\epsilon}$, we deduce

$$\frac{V(x_0, 0)}{\epsilon} \leq \mu_1(k) - \frac{M}{\epsilon}.$$

Let us write $x(t, t_0, x_0) = x(t)$ for simplicity. Define $\tau = \inf\{t \geq 0; |x(t)| \geq k\}$ (Throughout this paper we set $\inf \emptyset = \infty$). By the Itô formula, for any $t \geq 0$, we have

$$V(x(\tau \wedge t), \tau \wedge t) = V(x_0, 0) + \int_0^{\tau \wedge t} LV(x(s), s) ds + \int_0^{\tau \wedge t} V_x(x(s), s) g(x(s), s) dW_s.$$

Then, $0 \leq \mathbb{E}\big(V(x(\tau \wedge t), \tau \wedge t)\big) = V(x_0, 0) + \mathbb{E}\big(\int_0^{\tau \wedge t} LV(x(s), s) ds\big)$ and

$$\mathbb{E}\big(V(x(\tau \wedge t), \tau \wedge t)\big) \leq V(x_0, 0) + \mathbb{E}\big(\int_0^{\tau \wedge t} \varrho(s) - \mu_3(|x(s)|) ds\big)$$

$$\leq V(x_0, 0) + \mathbb{E}\big(\int_0^{\tau \wedge t} \varrho(s) ds\big)$$

$$\leq V(x_0, 0) + \mathbb{E}\big(\int_0^{+\infty} \varrho(s) ds\big)$$

$$\leq V(x_0, 0) + M.$$

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Notice that $|x(\tau \wedge t)| = |x(\tau)| = k$ if $\tau \leq t$.
Therefore, we can obtain

$$
\begin{aligned}
\mathbb{E}\big(V(x(\tau \wedge t), \tau \wedge t)\big) &\geq \mathbb{E}\big(1_{\{\tau \leq t\}} V(x(\tau), \tau)\big) \\
&\geq \mathbb{E}\big(1_{\{\tau \leq t\}} \mu_1(|x(\tau)|)\big) \\
&\geq \mathbb{E}\big(1_{\{\tau \leq t\}} \mu_1(k)\big) \\
&= \mu_1(k)\mathbb{E}\big(1_{\{\tau \leq t\}}\big) \\
&= \mu_1(k)\mathbb{P}\big(\tau \leq t\big).
\end{aligned}
$$

Then,

$$
\mu_1(k)\mathbb{P}\big(\tau \leq t\big) \leq V(x_0, 0) + M \leq \epsilon \mu_1(k).
$$

Thus,

$$
\mathbb{P}\big(\tau \leq t\big) \leq \epsilon.
$$

Letting $t \to +\infty$, we arrive at

$$
\mathbb{P}\big(\tau < +\infty\big) \leq \epsilon.
$$

Thus, for all $\epsilon \in]0,1[$, $r \geq 0$ and $k > r$, there exists $\delta = \delta(\epsilon, k) \in]0, k[$, such that

$$
\mathbb{P}\big(|x(t)| < k, \forall t \geq t_0 \geq 0\big) \geq 1 - \epsilon, \quad \text{for all } |x_0| < \delta. \tag{2.7}
$$

In particular, for a small $r \in]0, \delta[$ and $|x_0| < r$, we can take $k = k(r) > 0$, such that for all $t \geq t_0 \geq 0$, we have $|x(t)| < k$, a.s., for all $|x_0| < r$, and thus,

$$
\sup_{t \geq t_0} |x(t)| < k \quad \text{for all } |x_0| < r \quad a.s.
$$

Therefore, B_r is globally uniformly stable in probability.

Let us now prove the uniform attractivity of B_r. To this end, it is sufficient to prove that, for each $\epsilon \in]0,1[$, $h > r$ and $c > 0$, there exists $T = T(\epsilon, c) > 0$ (independent of t_0), such that for all $t \geq t_0 + T$, we have

$$
\mathbb{P}\big(|x(t)| < h\big) \geq 1 - \frac{\epsilon}{2}, \text{ for all } |x_0| < c. \tag{2.8}
$$

Pick $\epsilon \in]0,1[$, $h > r$ large enough and $c > 0$ (independent of t_0), such that

$$
|x_0| < c, \ \mu_1(h) > \frac{2M}{\epsilon} \ \text{ and } \ \mu_1(h) - \frac{2M}{\epsilon} \geq \frac{2V(x_0, 0)}{\epsilon}.
$$

Suppose now that there exists $T = T(\epsilon, c) > 0$ such that, for all $t_0 \geq 0$,

$$
\tau_h = \inf\{t \geq t_0 + T; |x(t)| \geq h\}.
$$

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Noting that, $|x(\tau_h \wedge t)| = |x(\tau_h)| = h$ when $\tau_h \leq t$, and applying the Itô formula, we obtain, for all $t \geq t_0 + T$,

$$\mathbb{E}\big(V(x(\tau_h \wedge t), \tau_h \wedge t)\big) \leq V(x_0, 0) + \mathbb{E}\Big(\int_0^{\tau_h \wedge t} (\varrho(s) - \mu_3(|x(s)|))ds\Big),$$
$$\leq V(x_0, 0) + M,$$
$$\leq \frac{\epsilon}{2}\mu_1(h) - M + M,$$
$$\leq \frac{\epsilon}{2}\mu_1(h).$$

We also have

$$\mathbb{E}\big(V(x(\tau_h \wedge t), \tau_h \wedge t)\big) \geq \mathbb{E}\big(1_{\{\tau_h \leq t\}}V(x(\tau_h), \tau_h)\big)$$
$$\geq \mathbb{E}\big(1_{\{\tau_h \leq t\}}\mu_1(|x(\tau_h)|)\big)$$
$$\geq \mathbb{E}\big(1_{\{\tau_h \leq t\}}\mu_1(h)\big)$$
$$= \mu_1(h)\mathbb{E}\big(1_{\{\tau_h \leq t\}}\big)$$
$$= \mu_1(h)\mathbb{P}\big(\tau_h \leq t\big).$$

Therefore,

$$\mu_1(h)\mathbb{P}\big(\tau_h \leq t\big) \leq \frac{\epsilon}{2}\mu_1(h),$$

which implies that

$$\mathbb{P}\big(\tau_h \leq t\big) \leq \frac{\epsilon}{2}.$$

Then, for any $\epsilon \in]0, 1[$, $h > r$ and $c > 0$ (independent of t_0), for a fixed $T = T(\epsilon, c) > 0$, we have, for all $t \geq t_0 + T$, $\mathbb{P}\big(\tau_h \leq t\big) \leq \frac{\epsilon}{2}$, for all $|x_0| < c$. Letting $t \to +\infty$, we arrive at

$$\mathbb{P}\big(\tau_h < +\infty\big) \leq \frac{\epsilon}{2}.$$

Thus,

$$\mathbb{P}\big(|x(t)| < h; \forall t \geq t_0 + T\big) \geq 1 - \frac{\epsilon}{2}, \quad \text{for all } |x_0| < c.$$

Then, B_r is globally uniformly attractive in probability, as desired. □

Remark 2.2. Note that if $r \to 0$ or $\varrho(t) \to 0$ when $t \to +\infty$, we then recover the classical stability results when 0 is an equilibrium point.

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3 The global practical uniform exponential stability of stochastic differential equation

Let us start by recalling a technical lemma which will be useful in our analysis.

Lemma 3.1. *For all $x_0 \in \mathbb{R}^d$ such that $x_0 \neq 0$ it holds*

$$\mathbb{P}\big(x(t, t_0, x_0) \neq 0, \forall t \geq 0\big) = 1. \tag{3.1}$$

Proof. Arguing by contradiction, if the lemma were false, there would exist some $x_0 \neq 0$ such that $\mathbb{P}\{\tau < \infty\} > 0$, where

$$\tau = \inf\{t \geq 0; x(t) = 0\},$$

where we write $x(t, t_0, x_0) = x(t)$ for simplicity. Thus, we can find a pair of constants $T > 0$ and $\theta > 1$, sufficiently large, such that $\mathbb{P}(B) > 0$, where

$$B = \{\tau \leq T \quad \text{and} \quad |x(t)| \leq \theta - 1, \quad \text{for all} \quad 0 \leq t \leq \tau\}.$$

But, by the standing hypotheses, there exist positive constants K_θ, r, and a small positive constant c such that

$$|f(x, t)| \vee |g(x, t)| \leq K_\theta |x| + r \quad \text{for all } 0 < c \leq |x| \leq \theta, \quad 0 \leq t \leq T.$$

Let $V(x, t) = |x|^{-1}$. Then, for $0 < c \leq |x| \leq \theta$ and $0 \leq t \leq T$,

$$\begin{aligned}
LV(x, t) &= -|x|^{-3} x^T f(x, t), \\
&\quad + \frac{1}{2}\Big(-|x|^{-3}|g(x, t)|^2 + 3|x|^{-5}|x^T g(x, t)|^2 \Big) \\
&\leq |x|^{-2}|f(x, t)| + |x|^{-3}|g(x, t)|^2 \\
&\leq |x|^{-2}\big(K_\theta |x| + r\big) + |x|^{-3}\big(K_\theta |x| + r\big)^2 \\
&\leq K_\theta(1 + K_\theta)V(x, t) + r|x|^{-1}\big(|x|^{-1} + 2K_\theta|x|^{-1} + r|x|^{-2}\big) \\
&\leq K_\theta(1 + K_\theta)V(x, t) + rV(x, t)\Big(\frac{1}{c} + \frac{2K_\theta}{c} + \frac{r}{c^2}\Big) \\
&\leq \big[K_\theta(1 + K_\theta) + \beta\big]V(x, t),
\end{aligned}$$

where $\beta = r\Big(\dfrac{1}{c} + \dfrac{2K_\theta}{c} + \dfrac{r}{c^2}\Big)$.

Now, for any $\epsilon \in]0, |x_0|[$, define the stopping time

$$\tau_\epsilon = \inf\{t \geq 0; |x(t)| \in] -\infty, \epsilon] \cup [\theta, +\infty[\}.$$

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Again by the Itô formula,

$$\mathbb{E}\Big[\exp\Big\{-\big[K_\theta(1+K_\theta)+\beta\big](\tau_\epsilon \wedge T)\Big\}V\big(x(\tau_\epsilon \wedge T),\tau_\epsilon \wedge T\big)\Big] - V(x_0,0),$$

$$= \mathbb{E}\Big[\int_0^{\tau_\epsilon \wedge T}\exp\Big\{-\big[K_\theta(1+K_\theta)+\beta\big]s\Big\}\Big[-\big(K_\theta(1+K_\theta)+\beta\big)V(x(s),s)+LV(x(s),s)\Big]ds,$$

$$\leq 0.$$

Note that for $\omega \in B$, it follows that $\tau_\epsilon \leq T$ and $|x(\tau_\epsilon)| = \epsilon$. The above inequality therefore implies that

$$\mathbb{E}\Big[\exp\Big\{-\big[K_\theta(1+K_\theta)+\beta\big]T\Big\}\epsilon^{-1}1_B\Big] \leq |x_0|^{-1}.$$

Hence $\mathbb{P}(B) \leq \epsilon|x_0|^{-1}\exp\Big\{\big[K_\theta(1+K_\theta)+\beta\big]T\Big\}$. Letting $\epsilon \longrightarrow 0$ yields that $\mathbb{P}(B) = 0$, but this contradicts the definition of B. The proof is complete. $\qquad\square$

The next lemma, whose proof can be found in [15], will be crucial in the proof of our main result.

Lemma 3.2. *Let $g = (g_1, ..., g_m) \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, and let T, α, β be any positive numbers. Then*

$$\mathbb{P}\Big(\sup_{0 \leq t \leq T}\Big[\int_0^t g(s)dW_s - \frac{\alpha}{2}\int_0^t |g(s)|^2 ds\Big] > \beta\Big) \leq \exp(-\alpha\beta).$$

Definition 3.1. The ball B_r is said to be almost surely globally practically uniformly exponentially stable if, for any x_0 such that $0 < |x(t,t_0,x_0)| - r$, for all $t \geq t_0 \geq 0$, it holds that

$$\limsup_{t\to\infty}\frac{1}{t}\ln(|x(t,t_0,x_0)| - r) < 0, \text{ a.s.} \tag{3.2}$$

System (2.1) is said to be almost surely globally practically uniformly exponentially stable if there exists $r > 0$ such that B_r is almost surely globally practically uniformly exponentially stable.

Now, we can establish and prove our main result in this paper.

Theorem 3.3. *Assume that there exist a function $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+, \mathbb{R}^*_+)$ and constants $p \in \mathbb{N}^*$, $c_1 \geq 1$, $\varrho \geq c_1$, $\gamma \geq 0$ and $c_2 \in \mathbb{R}$, $c_3 \geq 0$ such that for all $t \geq t_0 \geq 0$, and $x \in \mathbb{R}^d$, the following conditions hold:*

$$c_1|x|^p \leq V(x,t),$$

$$LV(x,t) \leq c_2 V(x,t) + \varrho,$$

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$$|V_x(x,t)g(x,t)|^2 \geq c_3 V^2(x,t) + \gamma.$$

Then

$$\limsup_{t \to +\infty} \frac{1}{t} \ln \left(|x(t,t_0,x_0)| - (\frac{\varrho}{c_1})^{\frac{1}{p}} \right) \leq -\frac{[c_3 - 2(c_2+1)]}{2}, \quad a.s., \ for \ all \ x_0 \in \mathbb{R}^d$$

In particular, if $c_3 > 2(c_2+1)$, then the solution of (2.1) is almost surely globally practically uniformly exponentially stable with $r = (\frac{\varrho}{c_1})^{\frac{1}{p}}$.

Proof. Fix $x_0 \neq 0$ in $\mathbb{R}^d$ and write $x(t,t_0,x_0) = x(t)$. By Lemma 3.1, $x(t) \neq 0$, for all $t \geq 0$ almost surely.

Observe that, for any $x \in \mathbb{R}^d$, we have

$$
\begin{aligned}
c_1|x|^p - \varrho &= c_1\left(|x|^p - \frac{\varrho}{c_1}\right), \\
&= c_1\left(|x|^p - \left((\frac{\varrho}{c_1})^{\frac{1}{p}}\right)^p\right), \\
&= c_1\left(|x| - (\frac{\varrho}{c_1})^{\frac{1}{p}}\right)\left(|x|^{p-1} + |x|^{p-2}(\frac{\varrho}{c_1})^{\frac{1}{p}} + \ldots + (\frac{\varrho}{c_1})^{\frac{p-1}{p}}\right), \\
&\geq c_1\left(|x| - (\frac{\varrho}{c_1})^{\frac{1}{p}}\right).
\end{aligned}
$$

Since $c_1 \geq 1$, we obtain

$$V(x,t) \geq c_1|x|^p \geq c_1|x|^p - \varrho \geq \left(|x| - (\frac{\varrho}{c_1})^{\frac{1}{p}}\right).$$

Therefore, $\left(|x| - (\frac{\varrho}{c_1})^{\frac{1}{p}}\right) \leq V(x,t)$ and $\ln\left(|x| - (\frac{\varrho}{c_1})^{\frac{1}{p}}\right) \leq \ln\left(V(x,t)\right)$, for all $t \geq 0, x \in \mathbb{R}^d$. Applying the Itô formula once more and taking into account the assumptions, we obtain that, for all $t \geq 0$

$$
\begin{aligned}
d\left(\ln(V(x(t),t))\right) &= \frac{LV(x(t),t)}{V(x(t),t)}dt + \frac{V_x(x(t),t)g(x(t),t)}{V(x(t),t)}dW_t, \\
&\quad - \frac{1}{2}\frac{|V_x(x(t),t)g(x(t),t)|^2}{V^2(x(t),t)}dt,
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t d\left(\ln(V(x(s),s))\right)ds &= \int_0^t \frac{LV(x(s),s)}{V(x(s),s)}ds + \int_0^t \frac{V_x(x(s),s)g(x(s),s)}{V(x(s),s)}dW_s, \\
&\quad - \frac{1}{2}\int_0^t \frac{|V_x(x(s),s)g(x(s),s)|^2}{V^2(x(s),s)}ds,
\end{aligned}
$$

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$$\ln(V(x(t),t)) = \ln(V(x(0),0)) + \int_0^t \frac{LV(x(s),s)}{V(x(s),s)} ds + M(t),$$
$$-\frac{1}{2} \int_0^t \frac{|V_x(x(s),s)g(x(s),s)|^2}{V^2(x(s),s)} ds,$$
$$\leq \ln(V(x(0),0)) + \int_0^t \frac{c_2 V(x(s),s) + \varrho}{V(x(s),s)} ds + M(t),$$
$$-\frac{1}{2} \int_0^t \frac{|V_x(x(s),s)g(x(s),s)|^2}{V^2(x(s),s)} ds,$$
$$\leq \ln(V(x(0),0)) + c_2 t + \int_0^t \frac{\varrho}{V(x(s),s)} ds + M(t),$$
$$-\frac{1}{2} \int_0^t \frac{|V_x(x(s),s)g(x(s),s)|^2}{V^2(x(s),s)} ds,$$
$$\leq \ln(V(x(0),0)) + c_2 t + \int_0^t \frac{\varrho}{c_1 |x(s)|^p} ds + M(t),$$
$$-\frac{1}{2} \int_0^t \frac{|V_x(x(s),s)g(x(s),s)|^2}{V^2(x(s),s)} ds,$$
$$\leq \ln(V(x(0),0)) + c_2 t + t + M(t) - \frac{1}{2} \int_0^t \frac{|V_x(x(s),s)g(x(s),s)|^2}{V^2(x(s),s)} ds,$$

$$\ln(V(x(t),t)) \leq \ln(V(x(0),0)) + (c_2+1)t + M(t) - \frac{1}{2} \int_0^t \frac{|V_x(x(s),s)g(x(s),s)|^2}{V^2(x(s),s)} ds, \quad (3.3)$$

where $M(t) = \int_0^t \frac{V_x(x(s),s)g(x(s),s)}{V(x(s),s)} dW_s$ is a continuous martingale with initial value $M(0) = 0$. Assign $\epsilon \in]0,1[$ arbitrarily and let $n = 1, 2, ...$ By Lemma 3.2,

$$\mathbb{P}\left\{ \sup_{0 \leq t \leq n} \left[M(t) - \frac{\epsilon}{2} \int_0^t \frac{|V_x(x(s),s)g(x(s),s)|^2}{V^2(x(s),s)} ds \right] > \frac{2}{\epsilon} \ln(n) \right\} \leq \frac{1}{n^2}.$$

Applying the Borel-Cantelli lemma we see that, for almost all $\omega \in \Omega$, there exists an integer $n_0 = n_0(\omega)$, such that if $n \geq n_0$,

$$M(t) \leq \frac{2}{\epsilon} \ln(n) + \frac{\epsilon}{2} \int_0^t \frac{|V_x(x(s),s)g(x(s),s)|^2}{V^2(x(s),s)} ds, \text{ for all } 0 \leq t \leq n.$$

Then, since $\frac{\epsilon-1}{2} \int_0^t \frac{\gamma}{V^2(x(s),s)} ds \leq 0$, inequality (3.3) becomes, for all $0 \leq t \leq n$, $n \geq n_0$

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almost surely, as

$$\ln(V(x(t),t)) \leq \ln(V(x(0),0)) + (c_2+1)t + \frac{2}{\epsilon}\ln(n) + \frac{\epsilon-1}{2}\int_0^t \frac{|V_x(x(s),s)g(x(s),s)|^2}{V^2(x(s),s)}ds,$$

$$\leq \ln(V(x(0),0)) + (c_2+1)t + \frac{2}{\epsilon}\ln(n) + \frac{\epsilon-1}{2}c_3 t + \frac{\epsilon-1}{2}\int_0^t \frac{\gamma}{V^2(x(s),s)}ds,$$

$$\leq \ln(V(x(0),0)) + \frac{2}{\epsilon}\ln(n) - \frac{1}{2}\big[(1-\epsilon)c_3 - 2(c_2+1)\big]t.$$

Consequently, for almost all $\omega \in \Omega$, if $n-1 \leq t \leq n$ and $n \geq n_0$, we deduce

$$\frac{1}{t}\ln(V(x(t),t)) \leq -\frac{1}{2}\big[(1-\epsilon)c_3 - 2(c_2+1)\big] + \frac{\ln(V(x(0),0)) + \frac{2}{\epsilon}\ln(n)}{n-1}.$$

This implies that,

$$\lim_{t\to+\infty}\sup\frac{1}{t}\ln(V(x(t),t)) \leq -\frac{1}{2}\big[(1-\epsilon)c_3 - 2(c_2+1)\big] \quad \text{a.s.}$$

Hence,

$$\lim_{t\to+\infty}\sup\frac{1}{t}\ln(|x(t)| - (\frac{\varrho}{c_1})^{\frac{1}{p}}) \leq -\frac{1}{2}\big[(1-\epsilon)c_3 - 2(c_2+1)\big] \quad \text{a.s.}$$

and the required assertion follows since $\epsilon > 0$ is arbitrary. $\qquad\square$

Remark 3.4. Note that, as the origin $x = 0$ may not be an equilibrium point of the system (2.1), then we can no longer study the stability of the origin as an equilibrium point nor should we expect the solution of the system to approach the origin almost surely as $t \to +\infty$. The inequality (3.2) implies that $x(t)$ will be ultimately bounded by a small bound $r > 0$, that is, $|x(t)|$ will be small for sufficiently large t. This can be viewed as a robustness property of convergence almost surely to the origin provided that f and g satisfies $f(0,t) = 0$ and $g(0,t) = 0$, $\forall t \geq 0$. In this case the origin becomes an equilibrium point.

Example 3.5. *Although our theory may be applied to a more general situation, we will exhibit now a very simple situation in order to illustrate how our main result works. Indeed, let us consider the Langevin equation (see [5], [16]):*

$$dx(t) = f(x(t))dt + g(x(t))dW_t = \alpha x(t)dt + \beta dW_t, \tag{3.4}$$

where W_t is a one dimensional standard Brownian motion and $f(x,t) = \alpha x$, $g(x,t) = \beta$, $V(x,t) = x^2$, $\beta \in \mathbb{R}$, $|x| > \sqrt{\beta^2+1}$ and $\alpha < -\frac{1}{2}$. Then, it is easy to check that

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}g(x,t)^2 V_{xx}(x,t) = 2\alpha x^2 + \beta^2.$$

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$$|V_x(x,t)g(x,t)|^2 = |2\beta x|^2 = 4\beta^2 x^2.$$

Therefore,

$$|x|^2 \le V(x,t),$$
$$LV(x,t) \le 2\alpha V(x,t) + \beta^2 + 1,$$
$$|V_x(x,t)g(x,t)|^2 = 4\beta^2 x^2 \ge 0.$$

Thus, the constants in Theorem 3.3 become $c_1 = 1$, $c_2 = 2\alpha$, $c_3 = 0$, $p = 2$, $\varrho = \beta^2 + 1$, $\gamma = 0$.

Consequently,

$$c_3 = 0 > 2(c_2 + 1) = 4\alpha + 2,$$

and the assumptions of Theorem 3.3 are fulfilled. Then, (3.4) is almost sure globally practically uniformly exponentially stable with $r = \sqrt{\beta^2 + 1}$.

Remark 3.6. 1. As we mentioned in the Introduction, when the origin is not a solution of the stochastic equation, the practical stability means that almost all the paths of each solution starting in deterministic initial values, remain inside a neighborhood of certain ball centered at the origin. This definition is therefore weaker than the usual one of almost sure stability, in which the initial values are allowed to be non-deterministic.

2. On the other hand, and in some sense, one could interpret the global practical stability established in Theorem 3.3 as a result ensuring the existence of an absorbing set for the solutions starting in deterministic initial values. Therefore, it would be very interesting to analyze whether this kind of results are related to the theory of random attractors (see e.g. [17], [11], [12]). However, as the forward convergence stated in Theorem 3.3 is not necessarily uniform it is not clear that we might ensure the existence of a forward random attractor.

3. Nevertheless, in the previous example it is known the existence of a pullback random attractor given by the so-called Ornstein-Uhlenbeck process, which is a special solution of the equation (3.4), corresponding to a non-deterministic initial value, and which attracts any other solution of the equation in the pullback sense (see [11] for more details).

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