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Sliding collapse in masonry structures: a numerical model

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ABSTRACT

A probabilistic Non-Standard Limit Analysis based method is proposed to do safety assessments of historical masonry structures when collapse develops under sliding movements. The problem is presented as an Unilateral Contact Problem, formulated as a Complementarity Problem and solved as a Sequential Linear Programming with random starting points. The implemented method is a Monte Carlo type method that consists in: 1) fixing random points of application; 2) calculating the maximum value of the point load by Linear Programming; 3) making sure that the solution is in the onset of collapse; 4) rejecting the solutions that do not meet the requirements; 5) calculating the minimum value of the point load that would promote the collapse while keeping the same values of the limit conditions. The results are compared with a set of thirty three sliding tests carried out on dry masonry walls.

Keywords: Non-standard Limit Analysis, Monte Carlo methods, sliding collapse, masonry structures.

1. INTRODUCTION

The safety assessment of historical masonry structures is still a controversial matter nowadays (Roca et al., 2010). Global failure is often due to instability when yield conditions appear at a certain number of points, turning the structure into a mechanism, even when compressive stresses could still be under their limit value. In these cases and when no sliding occurs, many authors [1,2,3] have proposed the application of the Limit Analysis theorems since they constitute an excellent simplified tool.

The equivalence of both Limit Analysis theorems and two dual Linear Programs were proved [4,5,6] at the beginning of plasticity theory, and Limit Analysis by Linear Programming [7,8] allowed to obtain the actual load of the onset of collapse.

However, when the collapse includes sliding [9,10,11], the standard Limit Analysis theorems are no longer valid and the load factor of the onset of collapse is not necessarily unique. Hence many researchers have been looking for the solution of the onset of collapse which load factor is minimal. This is a very hard problem, in its simplest formulation is presented as a Linear Complementarity Problem [10,12]. Its computational complexity is NP-hard [13,14,15] and there is no method that guarantees to obtain a solution if exists, or to prove there is no solution. On the other hand, some researchers have pointed out that this minimal solution is perhaps too conservative [3,16], and recent

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numeric results obtained by applying Monte Carlo simulation [17] allows to conjecture that when there are multiple solutions the minimum and maximum values are not the most probable solutions.

At the end of 2013 in the Building Structures Department Laboratory, ETSAM, Technical University of Madrid, a wall of dry masonry was subjected to a series of thirty three load tests, the actions were their self-weight and a horizontal load. The objective was to obtain plenty of sliding tests which could enable to form a statistical judgment of the results and compare them with the results of several numerical methods. The obtained results show a wide dispersion (Fig. 1), and all of them not being equally probable. Even when all of them are in the range of Non-Standard Limit Analysis solutions it seems appropriate the implementation of a method that allows obtaining more accurate results to resolve problems such as the one treated.

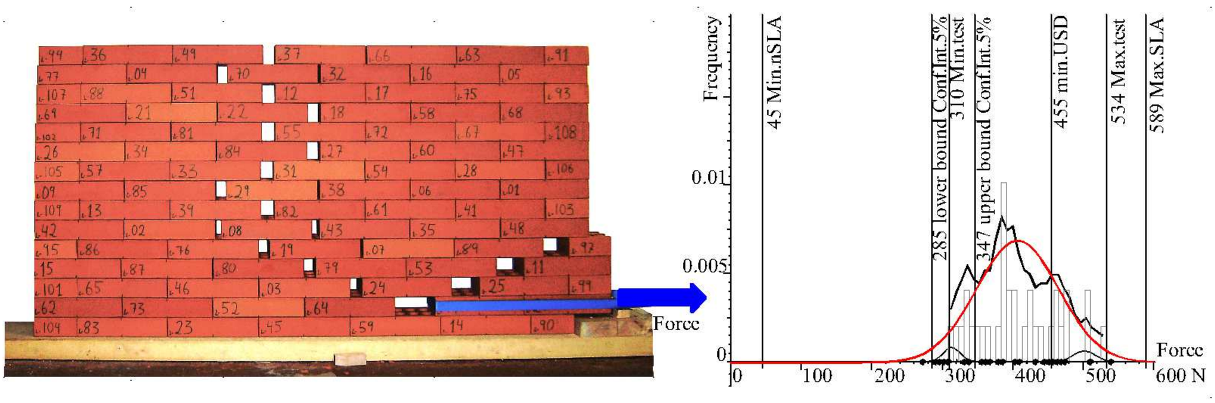

Figure 1. *An example of one of the experimental tests and the comparison between all the results.*

2. METHOD

A Non-Standard Limit Analysis based method is proposed, presented as an Unilateral Contact Problem, formulated as a Complementarity Problem [10] and solved as a Sequential Linear Programming from random starting points. In [17], the calculation of all the extreme solutions of the onset of collapse appear to be equivalent to the calculation of all the possible combinations of the points of application of the resultant contact forces in the contact surfaces, rejecting those that do not meet the requirements. The implemented method is a Monte Carlo type method that consists in: 1) fixing random points of application; 2) calculating the maximum value of the point load by Linear Programming; 3) making sure that the solution is in the onset of collapse; 4) rejecting the solutions that do not meet the requirements; 5) calculating the minimum value of the point load that would promote the collapse while keeping the same values of the limit conditions. The theoretical basis of the method and its implementation is detailed below.

2.1. Theoretical basis

The proposed model is classified as an "Advanced Computer Developments Based on Limit Analysis: Analysis of Blocky Structures", by Roca et al. [18] in his recent review of applicable studies in historical masonry constructions, and specifically, the ones that consider the collapse by sliding and that are formulated using a non-associative friction law [3,10,11,12,16,19,20].

A set of rigid bodies with unilateral contact is used for modelling the structure, with finite friction and following a non-associative friction law.

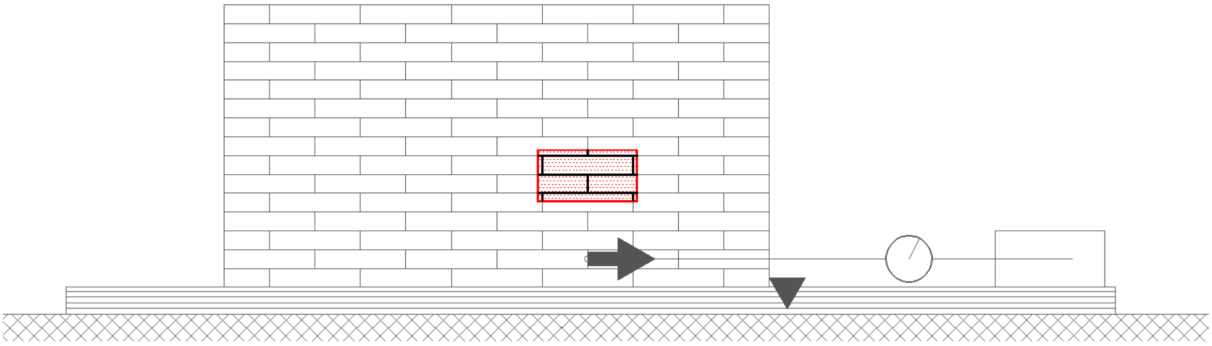

Figure 2. *Layout of experimental test.*

This model intends to represent a material that resists compressions much higher than the subjected ones and little or null tension resistance. Not considering the contribution of the tension resistance of the joint material is allowable in terms of safety. Moreover, it is common in historical constructions the lack of damages records that materials suffer over time, and in some cases, these materials do not even exists currently. In Figure 3 there is the representation of the corresponding numerical model.

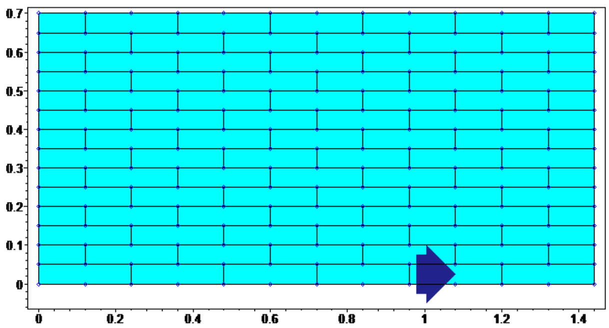

Figure 3. *Numerical model.*

At least from the 80s these kind of Unilateral Contact problems have been formulated as Complementarity problems. Their principal characteristic is to include, at least, a Complementarity constraint, that could be express as a requirement of orthogonality between two positive vectors, in other words, as a null dot product, written in equation (1), being in this case **y** vectors purely static and **z** vectors purely kinematic.

$$\mathbf{y} \cdot \mathbf{z} = 0 \ ; \ \ \mathbf{y} \geq \mathbf{0} \, ; \, \mathbf{z} \geq \mathbf{0} \tag{1}$$

These kinds of problems are hard and they could: have no solution, have one solution or multiple solutions. It is easy to check a previously calculated solution, but there are no deterministic method that guarantees to find a solution, if this one exists, or evidence that it does not exists. In addition, the main challenge from the practical point of view is [15] that many programs are capable of obtaining without any guarantee of success some kind of solution, but they are not capable of giving the quality of the obtained solution, in other words, they do not find out if the solution is near the corresponding global minimum. It seems necessary a method that rates the obtained solutions.

In [17], the method proposed consists in studying all the possible interval of the onset of collapse solutions, to analyze afterwards the probability of each of the load factors associated to the solutions. With this procedure the probability of collapse generation, for a smaller or equal load factor value compared with the one given, can be calculated.

This procedure requires the calculation of the volume of the set of solutions with load factors smaller or equal to the ones given. This calculation can be solve through exact determinists methods [21] or probabilistic methods [22] being both equivalent from the practical point of view. The first method finds out the implicit form for all the solutions of the onset of collapse and the second method consists in a Monte Carlo simulation that tries to meet with a huge amount of extreme solutions of the onset of collapse. As well, it has been shown that when there is a contact that is imperfect, the solutions are extreme, and the calculation of all the extreme solutions of the onset of collapse appear to be equivalent to the calculation of all the possible combinations of the points of application of the resultant contact forces in the contact surfaces, rejecting those that do not meet the requirements.

The mentioned probabilistic method, although it is less expensive than the exact determinist method, it is still too expensive in computational terms, since the extreme solutions search is done with deterministic methods; therefore, the application is limited for small examples.

A new method is propose where instead of calculating all the extreme possible solutions, it calculates a random wide sample that is significant enough, using the found relationship between contact imperfections and extreme solutions, and the relationship between extreme solutions and the points of application of contact forces. It is propose a random sampling of the extreme solutions assigning in a random way points of application of the contact force. From the physical point of view, we can expect that the unavoidable imperfections in the contact surface will establish, in a random mode, the points of application of the contact force.

2.2. Implementation

The claim of this work is to obtain a wide sampling for the possible solutions for the onset of collapse, with the aim of studying the distribution of the destabilizing action values. In a simple way, the onset of collapse solution should have a valid static solution, that we represent as $\mathbf{y} \in \{\mathrm{E_y}\}$, and it must be a kinematic valid solution represented as $\mathbf{z} \in \{\mathrm{K_z}\}$.

Besides, unilateral contact must be a requirement, so that the elemental displacement is not achievable, in a point of contact, unless the states of stresses have reached the limit value. The conditions already mentioned, since the static and the kinematic solutions have been formulated as a function of two positive vectors $\mathbf{y},\mathbf{z}$, they can be written as a null dot product $\mathbf{y}\cdot\mathbf{z}=0$. Therefore a solution for the onset of collapse it is a solution that meets the requirements (2).

$$\mathbf{y}\in\{\mathrm{E_y}\}\ ;\ \mathbf{y}\cdot\mathbf{z}=0\ ;\ \mathbf{z}\in\{\mathrm{K_z}\} \tag{2}$$

Due to the special characteristics of the constraints that govern the problem (1), once one part of the solution is substituted in the original problem, it turns into a linear problem, being reduced into a equation system and linear inequalities. When we choose a solution of the equilibrium limit $\mathbf{y}=\mathbf{y}'$, the problem is reduced to a linear problem with kinematic variables $\mathbf{z}$ (3).

$$\mathbf{y}=\mathbf{y}' \Rightarrow \mathbf{z}\in\{\mathrm{E_z}\}\ ;\ \mathbf{y}'\cdot\mathbf{z}=0 \tag{3}$$

The proposed implementation consists in obtaining multiple values of the point force that would promote the collapse, repeating the following steps:

1) Obtain a limit equilibrium solution $\mathbf{y}=\mathbf{y}'$, fixing randomly the points of application for the contact forces. (Fig. 4)

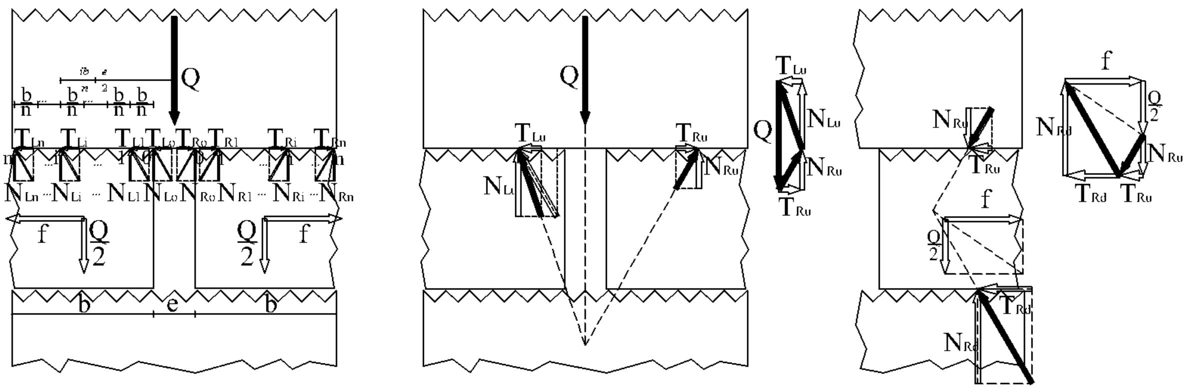

Figure 4. *Equilibrium solution with random points of application of the contact forces.*

2) Find the maximum value of the point load that promotes collapse calculated by Linear Programming (Fig. 5)

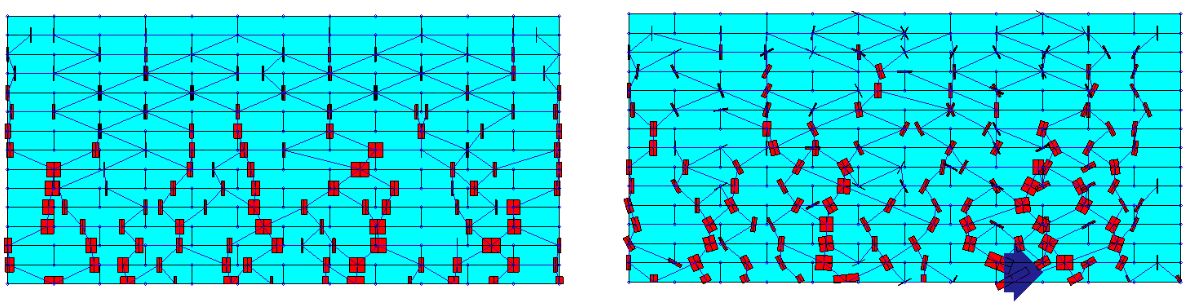

Figure 5. *Two random stress states, the first only with their self-weight and the second one with their self-weight and a point load.*

3) Make sure that the solution is in the onset of collapse.
4) Reject all the solutions that do not meet the requirements.
5) Calculate the minimum value of the point load that would promote the collapse while keeping the same values of the limit conditions. $\mathbf{y}_i = 0$.

The procedure is repeated as many times as it is considered to obtain a large enough sample.

3. RESULTADOS

Although the main objective of this work is to obtain the numerical values of point loads that cause the onset of collapse, we have also obtained the mechanisms, yield lines, corresponding to these loads. In Figure 6 there are represented four of these mechanisms, being the upper left mechanism the one that is produced more frequently compared with the rest.

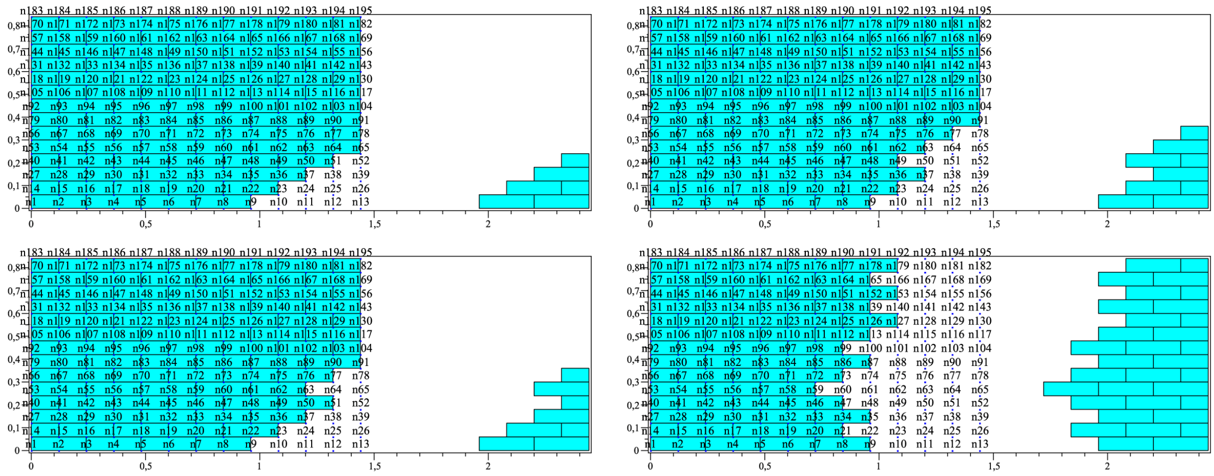

Figure 6. *Different collapse mechanisms with simple yield lines.*

Figure 7 shows four of the onset of collapse mechanisms from the tests.

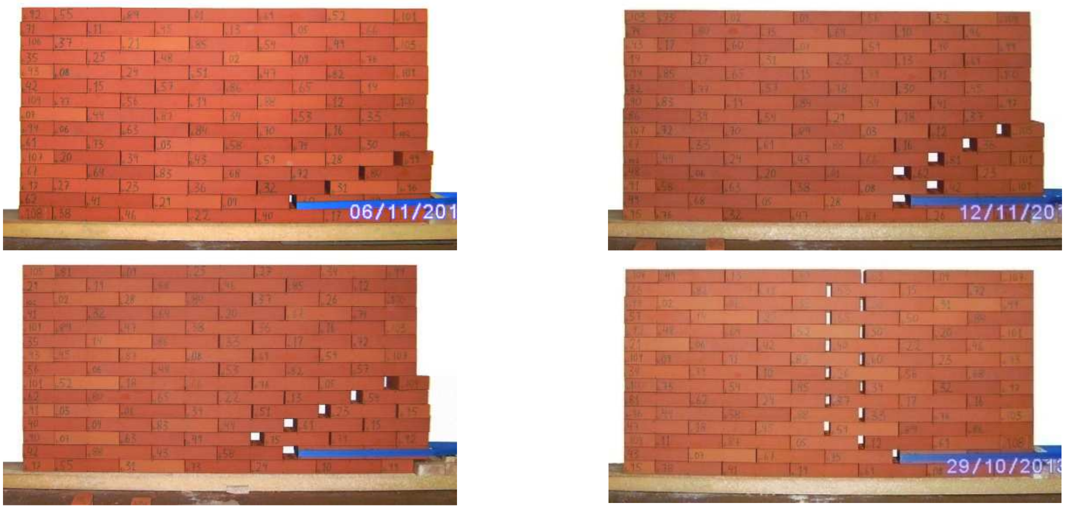

Figure 7. *Different collapse mechanisms with simple yield lines*

Again it is the upper left mechanism the most repeated one and these same yield lines are included in many other ways of more complex collapse.

As to numerical results, from 1000 obtained solutions by the cited method the results have been compared with the 33 test results. In Figure 8, the model results are in red and the test results in black, in the chart of the left side, relative frequencies are compared, in the central chart we compare the best fit Gaussian Distribution for the experimental results and the Gaussian Kernel Density Estimation for the numerical simulation, and finally in the chart in the right side we compare the best fit Gaussian Distributions for the experimental results and for the numerical results grouped in 33 data classes.

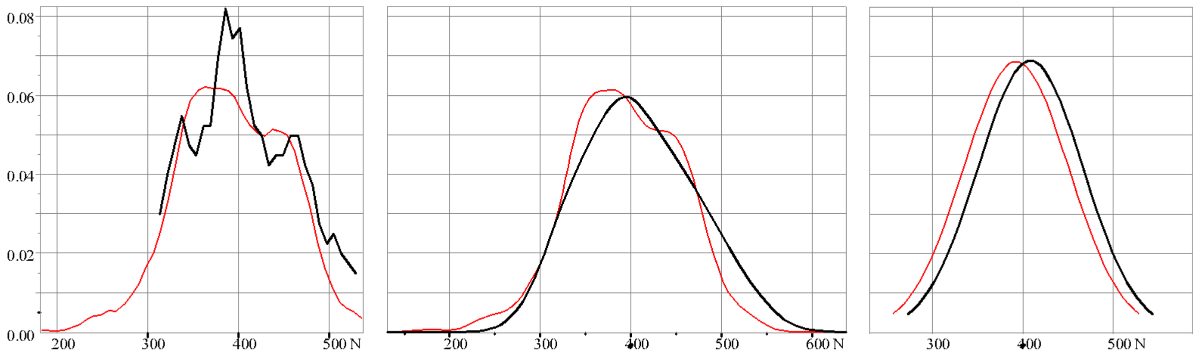

Figure 8. *Comparisons of the results of the experimental tests (black) and the model (red):1) the frequencies, 2) the Gaussian Kernel Density Estimation of model results vs. the best-fit Gaussian (Normal) Distribution of test results and 3) the best-fit Gaussian Distributions of test results vs. model results grouped in 33 data classes.*

In Figure 9, it is represented the Relative Cumulative Frequencies.

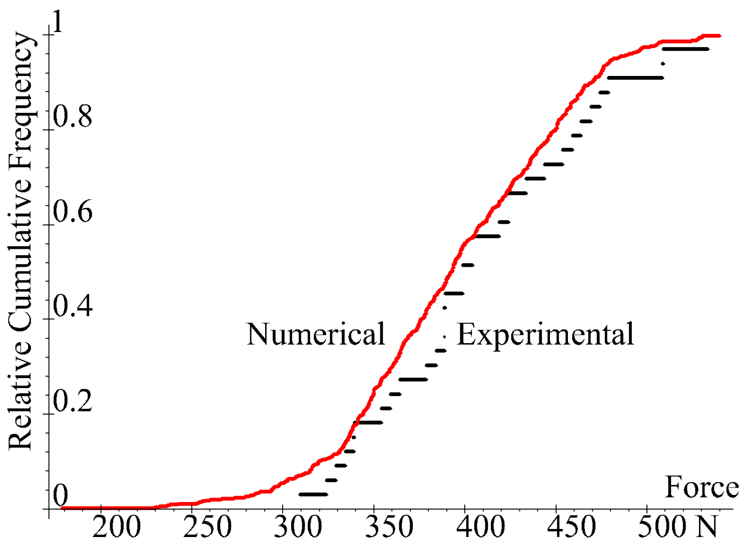

Figure 9. *Comparison between Relative Cumulative Frequencies of numerical and experimental results.*

As it is shown the Relative Cumulative Frequencies of the numerical method is near but always above the corresponding of the experimental results. The two sample Kolmogorov-Smirnov test [23] confirms this situation, giving a p-value of 0.994 for the "Numerical model < Experimental test" hypothesis. This concludes that the use of the model results is reliable in contrast to the experimental results.

The results have also been compared with the ones obtained by several numerical methods of single solution (Fig. 10). In addition, the best fit Gaussian Distribution for the experimental results and the Gaussian Kernel Density Estimation for the numerical simulation are shown in black and red.

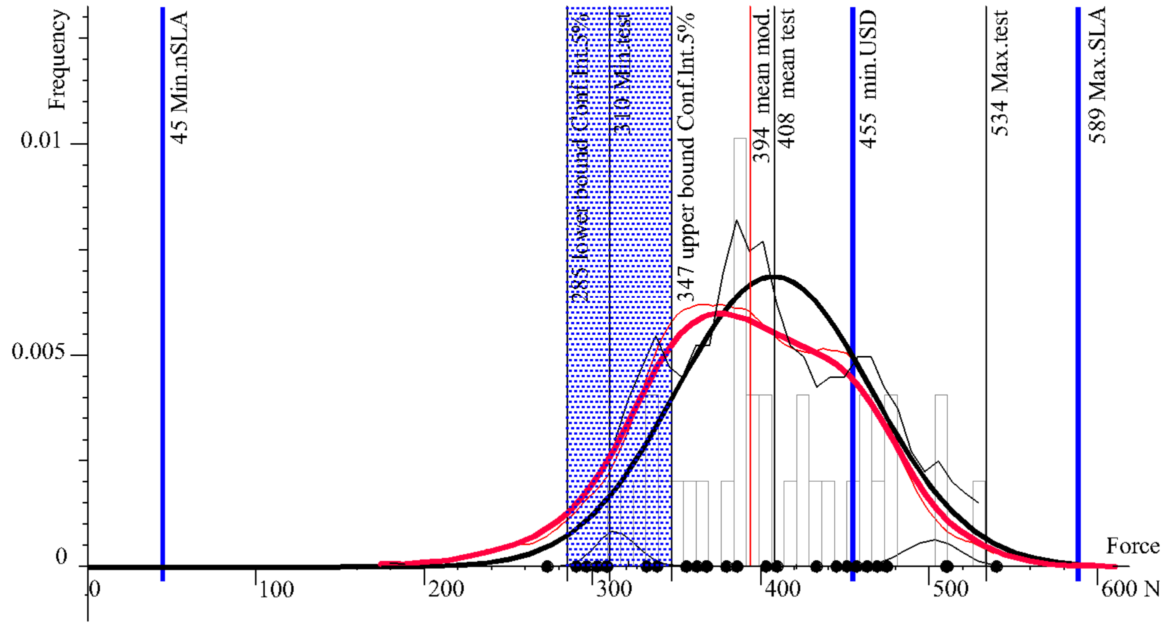

Figure 10. *Comparison between: the frequency of test results (thin black line), the frequency of model results (thin red line),the best-fit Normal (Gaussian) Distribution of test results (thick black line), the Gaussian Kernel Density Estimation of model results (thick red line), the confidence interval of characteristic value of test results and the single value which is obtained by other numerical methods.*

The Relative Frequencies Distribution corresponding to the onset of collapse of the numeric model and of the experimental test are represented too (Fig 10).

In Table 1, it is shown the comparison between the characteristic values, meaning the 5% percentiles, between the model and the test. Once more, the values are quite accurate and in the safe side.

Table 1. Comparison between the characteristic values of numerical methods and experimental results.

Numerical model			**Experimental test**	
5% percentile	**99% confidence interval 5% percentile**		**99% confidence interval 5% percentile**	
298 N	282 N	315 N	285 N	347 N

3. DISCUSSION

The structure where the tests were executed and the load conditions corresponds to a particular case. However the present case has a practical interest, since this mode of failure by sliding can occur in those parts of the historic buildings subjected to strong thrust in the direction of the joint and at low compression along the perpendicular direction. This could happen in the upper parts of buttresses, the upper bands of masonry shear walls under thrust of roofs or transverse walls, or the anchorage zones for tension ties.

All the solutions obtained in the tests are within the range of feasible solutions to the Non-Standard Limit Analysis, nevertheless, in the studied case this range is wide. In this case, the maximum load factor obtained by Standard Limit Analysis may be unsafe. The minimum load factor obtained by non-Standard Limit Analysis, is not always the actual load factor of the onset of collapse, in fact it never happened for the current tests. Therefore, their use may produce overly conservative results, besides being very expensive computationally.

Other methods such as Rankine, Uniform Stress Distribution, that produce intermediate results gives values that are closer to the characteristic value obtained in the tests but may be unsafe.

The proposed method, which ignores the elastic deformation of the material but considers the discontinuity of the material and the irregular surfaces contact, gets tighter results and reflects in a more suitable way the dispersion of the experimental results.

A model that also incorporates elastic deformations would be even tighter, however would not give safer results for each of the studied random cases, because the minimum value of the load that will cause the collapse has been calculated and the incorporation of new constraints will give a new minimum equal or higher than the calculated one.

4. CONCLUSION

As a conclusion, it should be noticed that in the studied case of dry masonry or similar structures when the collapse is caused by sliding and the actions are point loads, the approach that considers randomness in contact conditions as the proposed here and [17], could give best fit results with experimental data rather than with other methods whose solution is unique.

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