Kronecker coefficients, convexity, and generating functions

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November 27, 2009

Outline

Definitions of Kronecker coefficients

partitions $\lambda = [\lambda_1, \lambda_2, \ldots]$ where $\lambda_i \in \mathbb{N} \cup \{0\}$ and $\lambda_i \geq \lambda_{i+1}$

length $\ell(\lambda) = \max\{\, i \mid \lambda_i \neq 0\}$ and degree $|\lambda| = \sum_i \lambda_i$

irreducible representations

- general linear group $\mathrm{GL}(m, \mathbb{C})$:
  irred. polynomial repr. U_λ indexed by partitions λ of length $\ell(\lambda) \leq m$
- symmetric group S_k:
  irred. repr. V_λ indexed by partitions λ of degree $|\lambda| = k$

Kronecker coefficient $k^{\lambda}_{\mu,\nu}$ for partitions λ, μ, and ν

- definition w.r.t. $\mathrm{GL}(m \cdot n, \mathbb{C}) \supset \mathrm{GL}(m, \mathbb{C}) \otimes \mathrm{GL}(n, \mathbb{C})$:
  $U_\lambda = \bigoplus_{\mu,\nu} k^{\lambda}_{\mu,\nu} U_\mu \otimes U_\nu$ (outer tensor product)
- definition w.r.t. S_k: $V_\mu \otimes V_\nu = \bigoplus_\lambda k^{\lambda}_{\mu,\nu} V_\lambda$ (inner tensor product)

Kronecker coefficients: symmetric group

SCHUR (`http://schur.sourceforge.net/`)

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$\{111\} + \{101^2\} + \{93\} + \{921\} + \{831\} + \{821^2\} + \{75\} + \{741\} + \{732\} + \{72^21\} + \{651\} + \{641^2\} + \{63^2\} + \{6321\} + \{5^22\} + \{543\} + \{5421\} + \{532^2\} + \{4^231\} + \{43^22\}$

irred. representations V_λ of the symmetric group S_{12}

$V_{[7,5]} \otimes V_{[6,6]} =$ (inner tensor product)

$V_{[11,1]} + V_{[10,1,1]} + V_{[9,3]} + V_{[9,2,1]} + V_{[8,3,1]} + V_{[8,2,1,1]} + V_{[7,5]} + V_{[7,4,1]} + V_{[7,3,2]} + V_{[7,2,2,1]} + V_{[6,5,1]} + V_{[6,4,1,1]} + V_{[6,3,3]} + V_{[6,3,2,1]} + V_{[5,5,2]} + V_{[5,4,3]} + V_{[5,4,2,1]} + V_{[5,3,2,2]} + V_{[4,4,3,1]} + V_{[4,3,2]}$

Kronecker coefficients: special unitary group (1/2)

general linear group $\mathrm{GL}(m, \mathbb{C})$

irred. polynomial repr. U_λ indexed by partitions λ of length $\ell(\lambda) \leq m$

special unitary group $\mathrm{SU}(m)$ [and special linear group $\mathrm{SL}(m, \mathbb{C})$]

- irred. polynomial repr. U_μ indexed by partitions $\mu = [\lambda_1 - \lambda_m, \ldots, \lambda_{m-1} - \lambda_m]$ of length $\ell(\mu) \leq m - 1$
- irred. polynomial repr. U_t indexed by highest weights $t = (t_1, \ldots, t_{m-1}) = [\lambda_1 - \lambda_2, \ldots, \lambda_{m-1} - \lambda_m] = [\mu_1 - \mu_2, \ldots, \mu_{m-2} - \mu_{m-1}, \mu_{m-1}]$

example

$\lambda = [6 + x, 4 + x, 2 + x, x] \Leftrightarrow \mu = [6, 4, 2] \Leftrightarrow t = (2, 2, 2)$

Kronecker coefficients: special unitary group (2/2)

LiE (http://wwwmathlabo.univ-poitiers.fr/~maavl/LiE/)

$\mathtt{m} = [[1,1],[2,0],[1,1]]$
$\mathtt{v} = [2,2,2]$
$\mathtt{branch}(\mathtt{v}, \mathtt{A1A1}, \mathtt{m}, \mathtt{A3})$
$1\mathtt{X}[0,0] + 2\mathtt{X}[0,4] + 1\mathtt{X}[0,6] + 1\mathtt{X}[0,8] + 3\mathtt{X}[2,2] + 3\mathtt{X}[2,4] + 3\mathtt{X}[2,6] + 1\mathtt{X}[2,8] + 2\mathtt{X}[4,0] + 3\mathtt{X}[4,2] + 4\mathtt{X}[4,4] + 2\mathtt{X}[4,6] + 1\mathtt{X}[4,8] + 1\mathtt{X}[6,0] + 3\mathtt{X}[6,2] + 2\mathtt{X}[6,4] + 1\mathtt{X}[6,6] + 1\mathtt{X}[8,0] + 1\mathtt{X}[8,2] + 1\mathtt{X}[8,4]$

$\mathrm{SU}(m \cdot n, \mathbb{C}) \supset \mathrm{SU}(m, \mathbb{C}) \otimes \mathrm{SU}(n, \mathbb{C})$

$$U_{(2,2,2)} = U_{(0)} \otimes U_{(0)} + 2U_{(0)} \otimes U_{(4)} + U_{(0)} \otimes U_{(6)} + U_{(0)} \otimes U_{(8)} + 3U_{(2)} \otimes U_{(2)} + 3U_{(2)} \otimes U_{(4)} + 3U_{(2)} \otimes U_{(6)} + U_{(2)} \otimes U_{(8)} + 2U_{(4)} \otimes U_{(0)} + 3U_{(4)} \otimes U_{(2)} + 4U_{(4)} \otimes U_{(4)} + 2U_{(4)} \otimes U_{(6)} + U_{(4)} \otimes U_{(8)} + U_{(6)} \otimes U_{(0)} + 3U_{(6)} \otimes U_{(2)} + 2U_{(6)} \otimes U_{(4)} + U_{(6)} \otimes U_{(6)} + U_{(8)} \otimes U_{(0)} + U_{(8)} \otimes U_{(2)} + U_{(8)} \otimes U_{(4)}$$

Outline

Algebraic complexity theory (1/2)

the permanent and the determinant

- permanent $\mathrm{perm}(A) = \sum_{\pi \in \mathrm{S}_n} \prod_{i=1}^{n} A_{i,\pi(i)}$ of a $n \times n$-dimensional matrix A
- determinant $\det(B) = \sum_{\pi \in \mathrm{S}_m} \mathrm{sgn}(\pi) \prod_{i=1}^{m} B_{i,\pi(i)}$ of a $m \times m$-dimensional matrix B

simple substitution p ($\mathbb{F}$ = field)

- $A_{i,j} = x_{i,j}$, where $x_{i,j}$ are variables from $\mathbb{F}[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$
- $B_{i,j} = y_{i,j}$, where $y_{i,j}$ are variables from $\mathbb{F}[y_{1,1}, y_{1,2}, \ldots, y_{m,m}]$
- $p : \{y_{1,1}, y_{1,2}, \ldots, y_{m,m}\} \mapsto \{x_{1,1}, x_{1,2}, \ldots, x_{n,n}\} \cup \mathbb{F}$

algebraic version of the P vs. NP question (Valiant 1979)

For each n, is there a $m = \mathrm{poly}(n)$ and a p s.t. $\mathrm{perm}(A) = p[\det(B)]$?

Algebraic complexity theory (2/2)

quasi–polynomials $\tilde{k}^{\lambda}_{\mu,\nu}(N)$ for Kronecker coefficients $k^{\lambda}_{\mu,\nu}$

- $\tilde{k}^{\lambda}_{\mu,\nu} : N \in \mathbb{N} \setminus \{0\} \mapsto k^{N\lambda}_{N\mu,N\nu}$, where $N\lambda = [N\lambda_1, N\lambda_2, \ldots]$
- $\tilde{k}^{\lambda}_{\mu,\nu}(N)$ is a quasi–polynomial, i.e., $\tilde{k}^{\lambda}_{\mu,\nu}(N) = f_i(N)$ if $N \equiv i \mod M$ for polynomials $f_1, \ldots, f_M$ and a period $M > 0$

Mulmuley's saturation conjecture (Mulmuley 2009)

- $\tilde{k}^{\lambda}_{\mu,\nu}(N)$ is strictly saturated if $f_i(M) = 0$ (for some M) $\Rightarrow f_i \equiv 0$
- saturation index $s(\tilde{k}^{\lambda}_{\mu,\nu})$ = smallest $M \in \mathbb{N} \cup \{0\}$ s.t. $\tilde{k}^{\lambda}_{\mu,\nu}(M+N)$ is strictly saturated
- conjecture: $s(\tilde{k}^{\lambda}_{\mu,\nu}) = \mathrm{poly}(\max\{|\lambda|, |\mu|, |\nu|\})$ (bit lengths)
- comments:
  - more conjectures; connections to permanent–determinant problem
  - Briand/Orellana/Rosas (2009): sometimes $s(\tilde{k}^{\lambda}_{\mu,\nu}) \neq 0$

Reduced density matrices (1/3)

mixed quantum systems: the density matrix ρ

- $\rho =$ complex matrix which is hermitian ($\rho = \rho^\dagger$), positive–semidefinite ($x^\dagger \rho x \geq 0$ for all vectors x), and of trace one
- ρ can be written as $\frac{1}{m}\mathrm{Id}_m + H$ where $-iH \in \mathfrak{su}(m)$ $\qquad [\mathrm{Tr}(H) = 0]$

partial trace and reduced density matrices

- partial trace $\mathrm{Tr}_V : \mathrm{L}(V \otimes W) \to \mathrm{L}(W)$, $C \mapsto \mathrm{Tr}_V(C) = \mathrm{Tr}_V(\sum_i A_i \otimes B_i) = \sum_i \mathrm{Tr}(A_i) B_i$
- reduced density matrix $\rho^W = \mathrm{Tr}_V(\rho)$ $\qquad [\rho \in \mathrm{L}(V \otimes W)]$

compatibility relations for ρ, ρ^V, and ρ^W

What combinations of $\mathrm{spec}\,(\rho)$, $\mathrm{spec}\,(\rho^V)$, and $\mathrm{spec}\,(\rho^W)$ are possible?

Reduced density matrices (2/3)

two qubits [Bravyi (2004)]

- partitions $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$, $[\mu_1, \mu_2]$, and $[\nu_1, \nu_2]$
- $\mathrm{spec}\,(\rho) = [\lambda_1, \lambda_2, \lambda_3, \lambda_4]/(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$
- $\mathrm{spec}\,(\rho^V) = [\mu_1, \mu_2]/(\mu_1 + \mu_2)$ and $\mathrm{spec}\,(\rho^W) = [\nu_1, \nu_2]/(\nu_1 + \nu_2)$

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 1 & -1 \\ 1 & -1 \\ 1 & -1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ 1 & -1 \\ -1 & 1 \\ -1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \leq \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 2 & 0 & 0 & -2 \\ 0 & 2 & 0 & -2 \\ 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ 2 & 0 & -2 & 0 \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}$$

Reduced density matrices (3/3)

variant for highest weights

- $t = (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 - \lambda_4)$
- $u = (\mu_1 - \mu_2)$ and $v = (\nu_1 - \nu_2)$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \cdot u + \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \cdot v \leq \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

Control algorithms (1/2)

efficient control algorithm for $U \in \mathrm{SU}(2^n)$ with evolution time t

- **instantaneous** operations $K_\ell \in \mathfrak{K} = \mathrm{SU}(2)^{\otimes n} = \mathrm{SU}(2) \otimes \cdots \otimes \mathrm{SU}(2)$
- time-evolution w.r.t. a coupling Hamilton operator H $(-iH \in \mathfrak{su}(2^n))$
- $U = \left[\prod_{\ell=1}^{m} \left(K_\ell \exp(-iHt_\ell) K_\ell^{-1}\right)\right] K_0$ and $t = \sum_{\ell=1}^{m} t_\ell$ $\qquad (t_\ell \geq 0)$

two qubits: $\mathfrak{su}(4) = \text{local} \oplus \text{nonlocal} = \mathfrak{k} \oplus \mathfrak{p}$ $\qquad (n = 2)$

- $\mathfrak{k} = [\mathfrak{su}(2) \otimes \mathrm{Id}] \oplus [\mathrm{Id} \otimes \mathfrak{su}(2)]$ and $\mathfrak{p} = \mathfrak{su}(2) \otimes \mathfrak{su}(2)$

Kostant's convexity theorem (1973)

- condition: $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ $\qquad$ (true for two qubits)
- $\mathfrak{a}$ = max. commutative subalgebra in $\mathfrak{p} = \bigcup_{K \in \mathfrak{K}} \mathrm{Ad}(K)(\mathfrak{a})$
- Kostant: What is with $\{KHK^{-1} \colon K \in \mathfrak{K}\}$ for $H \in \mathfrak{p}$?
  orthogonal projection to $\mathfrak{a}$ = convex closure of the intersection with $\mathfrak{a}$

→ time-optimal controls for two qubits [Khaneja et al. (2001)]

Control algorithms (2/2)

three qubits: $\mathfrak{su}(8) = \mathfrak{k} \oplus \mathfrak{m} = \mathfrak{k} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4$

- $\mathfrak{k} = [\mathfrak{su}(2) \otimes \mathrm{Id} \otimes \mathrm{Id}] \oplus [\mathrm{Id} \otimes \mathfrak{su}(2) \otimes \mathrm{Id}] \oplus [\mathrm{Id} \otimes \mathrm{Id} \otimes \mathfrak{su}(2)]$
- $\mathfrak{m}_1 = \mathfrak{su}(2) \otimes \mathfrak{su}(2) \otimes \mathrm{Id}$, $\mathfrak{m}_2 = \mathfrak{su}(2) \otimes \mathrm{Id} \otimes \mathfrak{su}(2)$, $\mathfrak{m}_3 = \mathrm{Id} \otimes \mathfrak{su}(2) \otimes \mathfrak{su}(2)$, and $\mathfrak{m}_4 = \mathfrak{su}(2) \otimes \mathfrak{su}(2) \otimes \mathfrak{su}(2)$
- $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, but NOT $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$

Kronecker coefficients vs. control algorithms

- Kronecker coefficients: restrict repr. from $\mathrm{SU}(2^n)$ to $\mathrm{SU}(2)^{\otimes n}$
- control algorithms: seek a generalization of Kostant's convexity theorem which describes the adjoint action of $\mathrm{SU}(2)^{\otimes n}$ on $\mathfrak{m} = \mathfrak{k}^{\perp}$ ($\rightarrow$ difficult sub-riemannian geodesics)
- entanglement: want to characterize $\mathrm{SU}(2^n)/\mathrm{SU}(2)^{\otimes n}$ (is a symmetric space for $n = 2$)

Outline

Convexity theorems (1/2)

Heckman (1980/1982): restrict the group $\mathfrak{G}$ to the subgroup $\mathfrak{K}$

- projection $p\colon \mathfrak{g} \to \mathfrak{k}$
- max. commutative subalgebras $\mathfrak{t}_g$, $\mathfrak{t}_\mathfrak{k}$ $[p(\mathfrak{t}_\mathfrak{g}) \subset \mathfrak{t}_\mathfrak{k}]$
- notation:$t \in \mathfrak{t} \Rightarrow Nt = (Nt_1, \dots, Nt_{m-1})$
- following problems are equivalent:
  1. find all restricted adjoint orbits: find all $(t, t') \in (\mathfrak{t}_\mathfrak{g}, \mathfrak{t}_\mathfrak{k})$ s.t. $t' \subset p(t)$
  2. find all asymptotic decompositions of representations: find all rational $(t, t') \in (\mathfrak{t}_\mathfrak{g}, \mathfrak{t}_\mathfrak{k})$ s.t.
     - $\exists N \in \mathbb{N}$ s.t. Nt' and Nt are integral (i.e. weights)
     - $Nt' \subset p(Nt)$

example: $\mathfrak{G} = \mathrm{SU}(4)$ and $\mathfrak{K} = \mathrm{SU}(2) \otimes \mathrm{SU}(2)$

- $t = (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 - \lambda_4)$ and $t' = (u, v) = (\mu_1 - \mu_2, \nu_1 - \nu_2)$

$\to$ asymptotic Kronecker coefficients $k_{Nu,Nv}^{Nt}$

Convexity theorems (2/2)

Guillemin/Sternberg (1982/1984), Kirwan (1984), ...

❷ find all asymptotic decompositions of representations:
find all rational $(t, t') \in (\mathfrak{t}_{\mathfrak{g}}, \mathfrak{t}_{\mathfrak{k}})$ s.t.

- $\exists N \in \mathbb{N}$ s.t. Nt' and Nt are integral (i.e. weights)
- $Nt' \subset p(Nt)$

$\Rightarrow$ solution for $t \geq 0$, $t' \geq 0$ is a rational convex polytope

Berenstein/Sjamaar (2000) [here $\mathrm{SU}(m) \supset \mathrm{SU}(m_1) \otimes \mathrm{SU}(m_2)$]

❸ $t' \in \tilde{\omega} p(\omega^{-1} t - v C_{\mathfrak{su}(m)})$ for all
$(\tilde{\omega}, \omega, v) \in (\mathrm{S}_{m_1} \times \mathrm{S}_{m_2}, \mathrm{S}_m, \mathcal{W}_{\mathrm{rel}})$ s.t. $\mathrm{Y}_{\tilde{\omega}} \subset \phi^*(v \mathrm{X}_{\omega v})$

- $C_{\mathfrak{su}(m)}$ = cone spanned by $(2, -1, 0, \ldots, 0), (-1, 2, -1, 0, \ldots, 0), \cdots,$
  $(0, \ldots, 0, -1, 2, -1), (0, \ldots, 0, -1, 2)$ and $\mathcal{W}_{\mathrm{rel}} \subset \mathrm{S}_m$
- Schubert condition $\mathrm{Y}_{\tilde{\omega}} \subset \phi^*(v \mathrm{X}_{\omega v})$, where ϕ^* is a projection on cohomology rings (basis $= \{\mathrm{X}_w : w \in \mathrm{S}_m\}$) induced by p

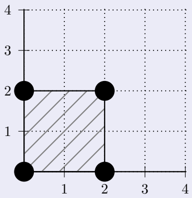
$\rightarrow$ refinements by Ressayre (2009)

Example computations (1/2)

- given $0 \leq t \in \mathfrak{t}_\mathfrak{g}$ all rational $0 \leq t' \in \mathfrak{t}_\mathfrak{k}$ s.t. $k_{Nt'}^{Nt} \neq 0$ for some $N \in \mathbb{N}$
- •'s denote positive Kronecker coefficients $k_{t'}^{t}$

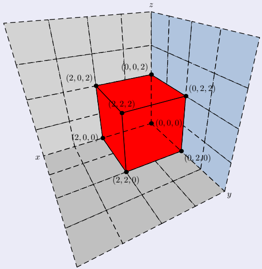
$\mathrm{SU}(4) \supset \mathrm{SU}(2) \otimes \mathrm{SU}(2)$

adjoint repr. $t = (1,0,1) \cong [2,1,1]$



$\mathrm{SU}(8) \supset \mathrm{SU}(2) \otimes \mathrm{SU}(2) \otimes \mathrm{SU}(2)$

adjoint repr. $t = (1,0,0,0,0,0,1) \cong [2,1,1,1,1,1,1]$

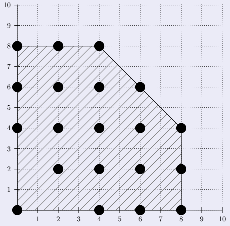


Example computations (2/2)

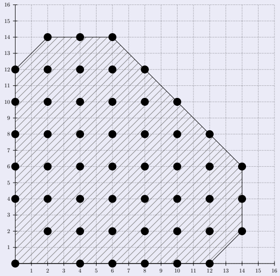
- given $0 \leq t \in \mathfrak{t}_{\mathfrak{g}}$ all rational $0 \leq t' \in \mathfrak{t}_{\mathfrak{k}}$ s.t. $k^{Nt}_{Nt'} \neq 0$ for some $N \in \mathbb{N}$
- $\bullet$'s denote positive Kronecker coefficients $k^{t}_{t'}$

$\mathrm{SU}(4) \supset \mathrm{SU}(2) \otimes \mathrm{SU}(2)$

$t = (2,2,2) \cong [6,4,2]$



$t = (4,4,2) \cong [10,6,2]$



both have "holes" at $(2,0)$ and $(0,2)$ $\rightarrow$ Briand/Orellana/Rosas (2009)

Outline

Kostant's multiplicity formula (1/3)

roots = highest weights of the adjoint representation

- pos. roots $\Delta_{\mathfrak{G}}^{+}$ = simple roots and sums of them; $\Delta_{\mathfrak{G}} = \Delta_{\mathfrak{G}}^{+} \cup \{-\Delta_{\mathfrak{G}}^{+}\}$ [for $\mathrm{SU}(m)$, simple roots $= \{(2,-1,0,\ldots,0),(-1,2,-1,0,\ldots,0)$ $,\cdots,(0,\ldots,0,-1,2,-1),(0,\ldots,0,-1,2)\}$]
- half sum $\rho_{\mathfrak{G}} = \frac{1}{2}\sum_{\alpha\in\Delta_{\mathfrak{G}}^{+}} \alpha$
- example $\mathrm{SU}(3)$: simple roots $\alpha_1 = (2,-1)$ and $\alpha_2 = (-1,2)$; positive roots α_1, α_2, $\alpha_1+\alpha_2$; half sum $\rho = \alpha_1 + \alpha_2$

restriction of Lie groups $\mathfrak{G} \supset \mathfrak{K}$ [e.g. $\mathrm{SU}(4) \supset \mathrm{SU}(2) \otimes \mathrm{SU}(2)$]

- projection $p\colon \mathfrak{g} \to \mathfrak{k}$; $\Delta = \{\alpha \in \Delta_{\mathfrak{G}}\colon\ p(\alpha) = 0\}$; $\Delta^{+} = \Delta \cap \Delta_{\mathfrak{G}}^{+}$
- multiset $A = p(\Delta_{\mathfrak{G}}^{+} \setminus \Delta) \setminus \Delta_{\mathfrak{K}}^{+}$; $m_\alpha =$ multiplicity of α in A
- Kostant's partition function: $P_A(\beta)$ = number of ways of writing β in the lattice $L = \sum_{a\in A} c_a a$, where $c_a \in \mathbb{N}\cup\{0\} \Rightarrow$ generating function $\frac{1}{\prod_{\alpha\in A}(1-z^{\alpha})^{m_\alpha}} = \sum_{\beta\in L} P_A(\beta) z^{\beta}$

Kostant's multiplicity formula (2/3)

- projection $p: \mathfrak{g} \to \mathfrak{k}$; $\Delta = \{\alpha \in \Delta_{\mathfrak{G}} : p(\alpha) = 0\}$; $\Delta^+ = \Delta \cap \Delta_{\mathfrak{G}}^+$
- multiset $A = p(\Delta_{\mathfrak{G}}^+ \setminus \Delta) \setminus \Delta_{\mathfrak{K}}^+$; $m_\alpha =$ multiplicity of α in A
- $\frac{1}{\prod_{\alpha\in A}(1-z^\alpha)^{m_\alpha}} = \sum_{\beta\in L} P_A(\beta) z^\beta$

- Weyl group $\mathcal{W}_{\mathfrak{G}} =$ symmetry group of roots and weights; e.g. $\mathcal{W}_{\mathrm{SU}(m)} = \mathrm{S}_m$, $\mathcal{W}_{\mathrm{SU}(m_1)\otimes\mathrm{SU}(m_2)} = \mathrm{S}_{m_1} \times \mathrm{S}_{m_2}$
- $\mathcal{W}_\Delta \subset \mathcal{W}_{\mathfrak{G}}$ w.r.t. root system Δ; half sum ρ_Δ
- define $\mathcal{W}$ by $\mathcal{W}_{\mathfrak{G}} = \mathcal{W}_\Delta \cdot \mathcal{W}$ s.t. the length of $\omega \in \mathcal{W}$ is minimum [length is defined w.r.t. a certain generating set]
- $D(s) = \prod_{\alpha\in\Delta^+} \frac{(s,\alpha)}{(\rho_\Delta,\alpha)}$, where $(,)$ is a scalar product

multiplicity formula for $k_{t'}^t$

$$\sum_{\omega\in\mathcal{W}} \mathrm{sgn}(\omega) D[\omega(t+\rho_{\mathfrak{G}})] P_A\{p[\omega(t+\rho_{\mathfrak{G}})] - t' - p(\rho_{\mathfrak{G}})\}$$

Kostant's multiplicity formula (3/3)

multiplicity formula for $k_{t'}^{t}$

$\sum_{\omega \in \mathcal{W}} \mathrm{sgn}(\omega) D[\omega(t + \rho_{\mathfrak{G}})] P_A\{p[\omega(t + \rho_{\mathfrak{G}})] - t' - p(\rho_{\mathfrak{G}})\}$

history

- builds on Weyl's character formula
- due to Kostant (early sixties, unpublished)
- for most general form see Vogan (1978) or Heckman (1982)

comments

- the formula is NOT positive, because of the $\mathrm{sgn}(\omega)$
- the sum has $|\mathcal{W}| \leq |\mathcal{W}_{\mathfrak{G}}|$ elements; $|\mathcal{W}_{\mathrm{SU}(m)}| = m!$

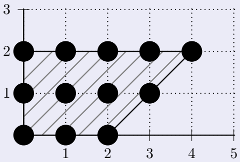
Example $\mathrm{SU}(4) \supset \mathrm{SU}(2) \otimes \mathrm{SU}(2)$ [Patera/Sharp (1980)]

generating function for $k^{t}_{t'}$ where $t = (a_1, a_2, a_3), t' = (b_1, b_2)$

$$\mathrm{magic}(a_1, a_2, a_3, b_1, b_2) = [a_1^3 a_2^2 a_3^3 b_1^4 b_2^4 + a_1^3 a_2 a_3^2 b_1^3 b_2^3 + a_1^2 a_2 a_3^3 b_1^3 b_2^3 + a_1^2 a_2 a_3^2 b_1^4 b_2^2 + a_1^2 a_2 a_3^2 b_1^2 b_2^4 - 2 a_1^2 a_2 a_3^2 b_1^2 b_2^2 - a_1^2 a_2 a_3 b_1^3 b_2 - a_1^2 a_2 a_3 b_1 b_2^3 - a_1^2 a_3^2 b_1^2 b_2^2 - a_1 a_2^2 a_3 b_1^2 b_2^2 - a_1 a_2 a_3^2 b_1^3 b_2 - a_1 a_2 a_3^2 b_1 b_2^3 - 2 a_1 a_2 a_3 b_1^2 b_2^2 + a_1 a_2 a_3 b_1^2 + a_1 a_2 a_3 b_2^2 + a_1 a_2 b_1 b_2 + a_2 a_3 b_1 b_2 + 1]/[(1 - a_1^2)(1 - a_2^2)(1 - a_3^2)(1 - a_2 b_1^2)(1 - a_2 b_2^2)(1 - a_1 b_1 b_2)(1 - a_3 b_1 b_2)(1 - a_1 a_3 b_2^2)(1 - a_1 a_3 b_1^2)]$$

Interlude: rational convex geometry (1/3)

Brion's theorem (1988) [ex. Beck/Haase/Sottile (2008)]



$$\frac{1}{(1-x)(1-y)} + \frac{y^2}{(1-x)(1-y^{-1})} + \frac{x^4y^2}{(1-x^{-1})(1-x^{-1}y^{-1})}$$
$$+\frac{x^2}{(1-xy)(1-x^{-1})} =$$
$$y^2 + xy^2 + x^2y^2 + x^3y^2 + x^4y^2$$
$$+y + xy + x^2y + x^3y$$
$$+1 + x + x^2$$

Lawrence-Varchenko theorem

$$\frac{1}{(1-x)(1-y)} - \frac{y^3}{(1-x)(1-y)} + \frac{x^6y^3}{(1-x)(1-xy)} - \frac{x^3}{(1-xy)(1-x)} =$$
$$y^2 + xy^2 + x^2y^2 + x^3y^2 + x^4y^2$$
$$+y + xy + x^2y + x^3y$$
$$+1 + x + x^2$$

Interlude: rational convex geometry (2/3)

Barvinok (1994), Barvinok/Pommersheim (1999)

polynomial time algorithm (fixed dimension d):

input: polyhedron P (without lines)

output: $f_P(\mathbf{x}) = \sum_{i \in I} \epsilon_i \frac{\mathbf{x}^{v_i}}{\prod_{1 \le j \le d}(1-\mathbf{x}^{u_{ij}})}$, where $f_P(\mathbf{x}) = \sum_{m \in P \cap \mathbb{Z}^d} \mathbf{x}^m$,
$\epsilon_i \in \{-1, 1\}$, $v_i, u_{ij} \in \mathbb{Z}^d$, $u_{ij} \neq 0$.

Barvinok/Woods (2003)

polynomial time algorithm (fixed q):

input: $f_\ell(\mathbf{x}) = \sum_{i \in I} \epsilon_{\ell,i} \frac{\mathbf{x}^{v_{\ell,i}}}{\prod_{1 \le j \le q}(1-\mathbf{x}^{u_{\ell,ij}})} = \sum_{m \in \mathbb{Z}^d} \beta_{\ell,m} \mathbf{x}^m$,
where $\epsilon_{\ell,i} \in \{-1, 1\}$, $v_{\ell,i}, u_{\ell,ij} \in \mathbb{Z}^d$, $u_{\ell,ij} \neq 0$, $\ell \in \{1, 2\}$

output: Hadamard product $g(\mathbf{x}) = \sum_{i \in I} \gamma_i \frac{\mathbf{x}^{v_i}}{\prod_{1 \le j \le r}(1-\mathbf{x}^{u_{ij}})} =$
$f_1(\mathbf{x}) \odot f_2(\mathbf{x}) = \sum_{m \in \mathbb{Z}^d} \beta_{1,m} \beta_{2,m} \mathbf{x}^m$,
where $\gamma_i \in \mathbb{Q}$, $v_i, u_{ij} \in \mathbb{Z}^d$, $u_{ij} \neq 0$, $r \le 2q$

Interlude: rational convex geometry (3/3)

Barvinok (1994), Barvinok/Woods (2003)

polynomial time algorithm (fixed q):

input 1: $f(\mathbf{x}) = \sum_{i \in I} \gamma_i \frac{\mathbf{x}^{v_i}}{\prod_{1 \le j \le q}(1-\mathbf{x}^{u_{ij}})}$,
where $\gamma_i \in \mathbb{Q}$, $v_i, u_{ij} \in \mathbb{Z}^n$, $u_{ij} \neq 0$

input 2: monomial map $\phi : \mathbb{C}^n \to \mathbb{C}^d$, $(z_1, \ldots, z_n) \mapsto (x_1, \ldots, x_d)$
where $x_i = \mathbf{z}^{l_i}$, $l_i \in \mathbb{Z}^n$

assumption: image of ϕ lies not entirely in the set of poles of $f(\mathbf{x})$

output: $g(\mathbf{z}) = \sum_{i \in I} \gamma_i' \frac{\mathbf{z}^{f_i}}{\prod_{1 \le j \le r}(1-\mathbf{z}^{e_{ij}})} = f[\phi(\mathbf{x})]$,
where $\gamma_i' \in \mathbb{Q}$, $f_i, e_{ij} \in \mathbb{Z}^d$, $e_{ij} \neq 0$, $r \le q$

implementations

- barvinok (http://www.kotnet.org/~skimo/barvinok/)
- LattE (http://www.math.ucdavis.edu/~latte/)

Continue example for $\mathrm{SU}(4) \supset \mathrm{SU}(2) \otimes \mathrm{SU}(2)$

generating function for $k^{t}_{t'}$ where $t = (a_1, a_2, a_3), t' = (b_1, b_2)$

$$\mathrm{magic}(a_1, a_2, a_3, b_1, b_2) = [a_1^3 a_2^2 a_3^3 b_1^4 b_2^4 + a_1^3 a_2 a_3^2 b_1^3 b_2^3 + a_1^2 a_2 a_3^3 b_1^3 b_2^3 + a_1^2 a_2 a_3^2 b_1^4 b_2^2 + a_1^2 a_2 a_3^2 b_1^2 b_2^4 - 2a_1^2 a_2 a_3^2 b_1^2 b_2^2 - a_1^2 a_2 a_3 b_1^3 b_2 - a_1^2 a_2 a_3 b_1 b_2^3 - a_1^2 a_3^2 b_1^2 b_2^2 - a_1 a_2^2 a_3 b_1^2 b_2^2 - a_1 a_2 a_3^2 b_1^3 b_2 - a_1 a_2 a_3^2 b_1 b_2^3 - 2a_1 a_2 a_3 b_1^2 b_2^2 + a_1 a_2 a_3 b_1^2 + a_1 a_2 a_3 b_2^2 + a_1 a_2 b_1 b_2 + a_2 a_3 b_1 b_2 + 1]/[(1 - a_1^2)(1 - a_2^2)(1 - a_3^2)(1 - a_2 b_1^2)(1 - a_2 b_2^2)(1 - a_1 b_1 b_2)(1 - a_3 b_1 b_2)(1 - a_1 a_3 b_2^2)(1 - a_1 a_3 b_1^2)]$$

compute $f: N \mapsto k^{Nt}_{Nt'}$ for $t = (2, 2, 2)$ and $t' = (0, 2)$

- compute Hadamard product (using `barvinok`):

$$\mathrm{magic}(a_1, a_2, a_3, b_1, b_2) \odot \frac{a_1^2 a_2^2 a_3^2 b_2^2}{1 - a_1^2 a_2^2 a_3^2 b_2^2} = \frac{-a_1^6 a_2^6 a_3^6 b_2^6 + 2a_1^4 a_2^4 a_3^4 b_2^4}{(1 - a_1^2 a_2^2 a_3^2 b_2^2)(1 - a_1^4 a_2^4 a_3^4 b_2^4)}$$

$\Rightarrow$ generating function for f:

$$\frac{-N^3 + 2N^2}{(1-N)(1-N^2)} = 2N^2 + N^3 + 3N^4 + 2N^5 + 4N^6 + 3N^7 + 5N^8 + \ldots$$

http://www.org.chemie.tu-muenchen.de/people/zeier/

Thank you for your attention!