Tensor algebras, words, and invariants of polynomials in non-commutative variables

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(Joint work with A. Bergeron-Brlek and C. Hohlweg)

November 24, 2009

Commutative starting point

Let V be a k-vector space with basis $\{x_1, x_2, \ldots, x_n\}$. Then the symmetric algebra $S(V)$ on V over k

$$\begin{aligned} S(V) &= k \oplus V \oplus S^2(V) \oplus S^3(V) \oplus \cdots \\ &\simeq k[x_1, x_2, \ldots, x_n]. \end{aligned}$$

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Let G be a finite subgroup of $GL(V)$. The invariant algebra of G is

$$\begin{aligned} S(V)^G &= k \oplus V^G \oplus S^2(V)^G \oplus S^3(V)^G \oplus \cdots \\ &\simeq k[x_1, x_2, \ldots, x_n]^G \end{aligned}$$

and its Hilbert-Poincaré series is

$$P(S(V)^G) = \sum_{d \geq 0} \dim(S^d(V)^G) q^d.$$

Commutative starting point

Theorem (Molien, Noether, Sheppard-Todd-Chevalley)

V finite dimensional k-vector space
G finite subroup of $GL(V)$

i) *If char $k = 0$, then* $P(S(V)^G) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I - gq)}$.

ii) *As a k-algebra, $S(V)^G$ is finitely generated.*

iii) *If char $k = 0$, then $S(V)^G$ is a free commutative k-algebra (with a homogeneous free generating set) if and only if G is generated by pseudo-reflections.*

Direction: Non-Commutative world

The tensor algebra $T(V)$ on V over k

$$T(V) = k \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$$
$$\simeq k\langle x_1, x_2, \ldots, x_n \rangle ,$$

where

$$T^d(V) = V^{\otimes d} = V \otimes V \otimes \cdots \otimes V \simeq k\langle x_1, x_2, \ldots, x_n \rangle_d .$$

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The invariant algebra of G is

$$T(V)^G = k \oplus V^G \oplus (V^{\otimes 2})^G \oplus (V^{\otimes 3})^G \oplus \cdots$$
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and its Hilbert-Poincaré series is

$$P(T(V)^G) = \sum_{d \geq 0} \dim((V^{\otimes d})^G) q^d .$$

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Theorem (Dick-Formaneck, Kharchenko, Lane)

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i) *If char $k = 0$, then* $P(T(V)^G) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{1 - Tr(g)\, q}$.

ii) *As a k-algebra, $T(V)^G$ finitely generated $\iff$ G is scalar.*

iii) *$T(V)^G$ is a free associative k-algebra (with a homogeneous free generating set).*

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iii) *$T(V)^G$ is a free associative k-algebra (with a homogeneous free generating set).*

- Is there a finite description of the dimensions and generators of $T(V)^G$ when G is not scalar?

Some Examples

Let $V = \mathcal{L}\{x_1, x_2, \ldots, x_n\} \simeq V^{(n-1,1)} \oplus V^{(n)}$ with the permutation action on the variables.
Then $T(V) \simeq k\langle x_1, x_2, \ldots, x_n \rangle$
and $T(V)^{S_n} =$ symmetric functions in non-commutative variables.

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Let $V = \mathcal{L}\{y_1, y_2, \ldots, y_{n-1}\} \simeq V^{(n-1,1)}$ where $y_k = x_k - x_{k+1}$
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Poincarè series counted by oscillating tableaux (see talk by Goupil)

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Poincarè series counted by oscillating tableaux (see talk by Goupil)

Let $V = \mathcal{L}\{x_1, x_2\}$ acting on the Dihedral group generated by elements $r^n = s^2 = 1 = rsrs$ acts on the two elements x_1, x_2.
$T(V) = k\langle x_1, x_2 \rangle$
Poincarè series counted by ???

$T(V)$ is the repeated internal tensor product of of the representations corresponding to V.

The spaces of invariants $T(V)^G$ are the trivial representations inside of these repeated internal tensor products.

Calculating the dimensions of $T^d(V)^G$ is the same as determining the multiplicity of the trivial character in d^{th} Kronecker power of the character corresponding to V.

Cayley graph of G

Let G be a finite group with generating set S and e the identity in G.

A Cayley graph $\Gamma = \Gamma(G, S)$ is a colored directed graph where

- vertices are identified with G
- to each generator $s \in S$ is assigned a color
- for any $g, h \in G$ and $s \in S$,

$$g \bullet\!\!\longrightarrow\!\!\bullet\ h \qquad \text{if } h = gs.$$

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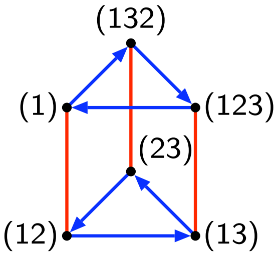
A word which reduces to g is a path along the edges of Γ from e to g.
A word does not cross e if it has no proper prefix which reduces to e.

Cayley graph of S_3

$\Gamma(S_3, \{(12), (132)\})$:



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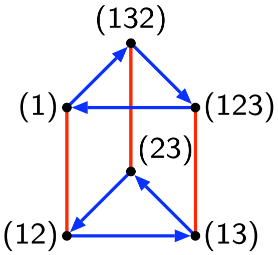

Path: $e = (1) \longrightarrow (132) \longrightarrow (123) \longrightarrow (1) \longrightarrow (12) \longrightarrow (1) \longrightarrow (132) \longrightarrow (23)$

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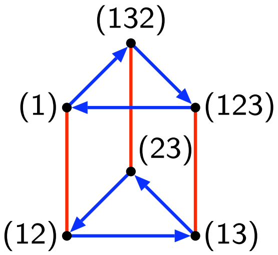

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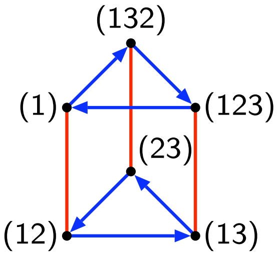

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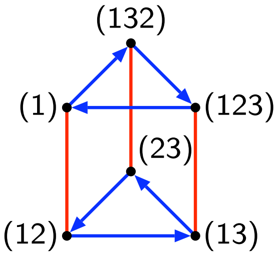

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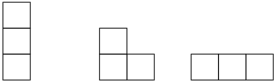
Word: *bbbaaba*   does cross the identity

Path: $e = (1) \longrightarrow (132) \longrightarrow (23) \longrightarrow (12) \longrightarrow (13) \longrightarrow (123)$
Word: *babba*   does not cross the identity

Partitions and Tableaux

A partition λ of a positive integer n is $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$ such that $n = |\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_\ell$. The partitions of 3 are



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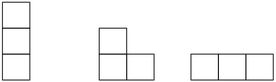

$(1,1,1)$ $(2,1)$ (3)

A tableau of shape $\lambda \vdash n$, is a filling of λ with values in $\{1, 2, \ldots, n\}$. A tableau is standard if its entries form an increasing sequence along each line and each column. $STab_n$ is the set of standard tableau with n boxes.

$$STab_3 = \left\{ \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}\, , \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\, , \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\, , \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array} \right\}$$

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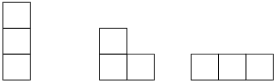

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Robinson-Schensted correspondence $\sigma \longleftrightarrow (P(\sigma), Q(\sigma))$
$P(\sigma)$ insertion tableau and $Q(\sigma)$ recording tableau.

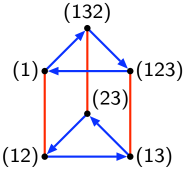
General Theorem for S_n

$\{V^\lambda\}_{\lambda \vdash n}$ forms a complete system of irreducible S_n-modules.

Theorem

The dimension of $T^d(V^{(n-1,1)})^{S_n}$ is equal to the number of words of length d which reduce to e in the Cayley graph $\Gamma(S_n, \{(12), (132), \ldots, (1\, n \cdots 432)\})$.

$\dim(T^4(V^{(2,1)})^{S_3})$ = number of words of length 4 which reduce to e in



$= |\{aaaa, abab, baba\}|$

Free generators for $T(V^{(n-1,1)})^{S_n}$

Proposition

The number of free generators of $T(V^{(n-1,1)})^{S_n}$ as an algebra are counted by the words in $\Gamma(S_n, \{(12), (132), \ldots, (1\, n \cdots 432)\})$ which reduce to the identity without crossing it .

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The free generators of $T(V^{(2,1)})^{S_3}$ are counted by

aa	bbb	$abab$	$abbba$	$abaaab$	$\cdots$
		$baba$	$baabb$	$abbabb$	
			$bbaab$	$baaaba$	
				$babbab$	
				$bbabba$	

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				$babbab$	
				$bbabba$	
1	1	2	3	5	$\cdots$

the Fibonacci numbers.

Proof uses Solomon's descent algebra of S_n

Let $I = \{1, 2, \ldots, n-1\}$ and $Des(\sigma) = \{i \in I \,|\sigma(i) > \sigma(i+1)\}$ be the descent set of $\sigma \in S_n$. For $K \subseteq I$, set

$$d_K = \sum_{\substack{\sigma \in S_n \\ Des(\sigma)=K}} \sigma.$$

For example,

$$d_{\{1\}} = (12) + (132) + (1432) + \cdots + (1\, n \cdots 432).$$

The Solomon's descent algebra $\Sigma(S_n)$ of S_n is a subalgebra of $\mathbb{Z}S_n$ with basis $\{d_K | K \subseteq I\}$. (Solomon) There is an algebra morphism

$$\theta : \Sigma(S_n) \to \mathbb{Z}\mathrm{Irr}(S_n).$$

Proof uses Solomon's descent algebra of S_n

For standard tableau t of shape $\lambda \vdash n$ define

$$z_t = \sum_{\substack{\sigma \in S_n \\ Q(\sigma)=t}} \sigma,$$

where $Q(\sigma)$ is the recording tableau of σ in the Robinson-Schensted corr. For example,

$$z_{\begin{array}{|c|c|c|c|c|}\hline 2 \\ \cline{1-1} 1 & 3 & 4 & \cdots & n \\ \hline\end{array}} = (12) + (132) + (1432) + \cdots + (1\,n\,\cdots 432) = d_{\{1\}}.$$

(Poirier-Reutenaueur) Let $\mathcal{Q}_n = \mathcal{L}\{z_t |\ t \in STab_n\}$. There is a linear map

$$\begin{array}{rcl} \tilde\theta : \mathcal{Q}_n & \longrightarrow & \mathbb{Z}\mathrm{Irr}(S_n) \\ z_t & \mapsto & \chi^{\mathrm{shape}(t)} \end{array}$$

and $\tilde\theta\,|_{\Sigma(S_n)} = \theta$. In particular, $\tilde\theta(z_{\begin{array}{|c|c|c|c|c|}\hline 2 \\ \cline{1-1} 1 & 3 & 4 & \cdots & n \\ \hline\end{array}}) = \theta(d_{\{1\}}) = \chi^{(n-1,1)}$.

Idea of the Proof

Proposition

The dimension of $T^d(V^{(n-1,1)})^{S_n}$ *is equal to the coefficient of* e *in* ${d_{\{1\}}}^d$.

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The dimension of $T^d(V^{(n-1,1)})^{S_n}$ is equal to the coefficient of e in ${d_{\{1\}}}^d$.

Note that $\dim(T^4(V^{(3-1,1)})^{S_3})$=Multiplicity of the trivial in $T^4(V^{(3-1,1)})$.

$$
\begin{aligned}
{d_{\{1\}}}^4 &= 3\,e + 3\,d_{\{2\}} + 2\,d_{\{1\}} + 3\,d_{\{1,2\}} \\
&= 3\,z_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}} + 3\,z_{\begin{array}{cc} 3 & \\ 1 & 2 \end{array}} + 2\,z_{\begin{array}{cc} 2 & \\ 1 & 3 \end{array}} + 3\,z_{\begin{array}{c} 3 \\ 2 \\ 1 \end{array}}
\end{aligned}
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$$\begin{aligned}
{d_{\{1\}}}^4 &= 3\,e + 3\,d_{\{2\}} + 2\,d_{\{1\}} + 3\,d_{\{1,2\}} \\
&= 3\,z_{\young(123)} + 3\,z_{\young(3,12)} + 2\,z_{\young(2,13)} + 3\,z_{\young(3,2,1)}
\end{aligned}$$

$$\Big\downarrow\ \tilde{\theta}$$

$$(\chi^{(2,1)})^4 = 3\,\chi^{(3)} + 3\,\chi^{(2,1)} + 2\,\chi^{(2,1)} + 3\,\chi^{(1,1,1)}$$

Key Lemma

Lemma

The coefficient of e in $d_{\{1\}}{}^d = ((12) + (132) + \ldots + (1\,n \cdots 432))^d$ *is equal to the number of words of length* d *which reduce to e in* $\Gamma(S_n, \{(12), (132), \ldots, (1\,n \cdots 432)\})$.

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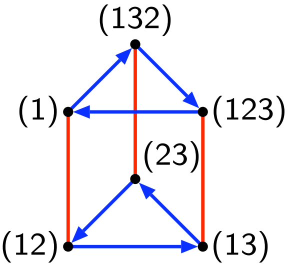
Consider $\Gamma(S_3, \{(12), (132)\})$ and let $a = (12)$ and $b = (132)$. Then

$$ {d_{\{1\}}}^4 = (a+b)^4 = 3\,e + 2\,(12) + 3\,(23) + 3\,(123) + 2\,(132) + 3\,(13) $$



More generally: Multiplicity of V^λ in $(V^{(n-1,1)})^{\otimes d}$

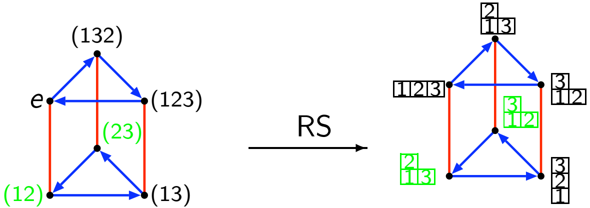
Theorem

Let $\lambda \vdash n$. The multiplicity of V^λ in $(V^{(n-1,1)})^{\otimes d}$ is

$$\sum_{\substack{t \in STab_n \\ sh(t)=\lambda}} |w(\sigma_t, d; \Gamma)|,$$

where $w(\sigma_t, d; \Gamma)$ is the set of words of length d which reduce to σ_t in $\Gamma = \Gamma(S_n, \{(12), (132), \ldots, (1\, n \cdots 432)\})$ and $\sigma_t \in S_n$ is such that $Q(\sigma_t) = t$. In particular, the multiplicity of the trivial is $|w(e, d; \Gamma)|$.

Consider $\Gamma = \Gamma(S_3, \{(12), (132)\})$ and set $a = (12)$ and $b = (132)$.



Decomposition of the S_3-module $(V^{(2,1)})^{\otimes 4}$

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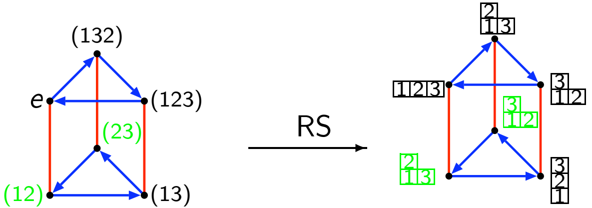

$V^{(3)}$: $\quad |w(\sigma_{\boxed{1|2|3}}, 4; \Gamma)| \quad = |\{aaaa, abab, baba\}| = 3$

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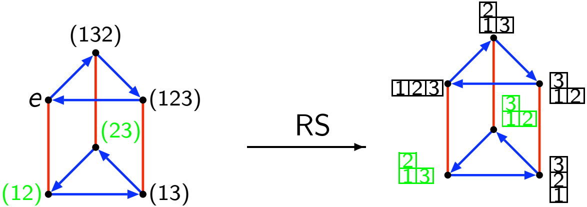

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$$V^{(2,1)} : \quad |w(\sigma_{\begin{array}{l}3\\12\end{array}}, 4; \Gamma)| + |w(\sigma_{\begin{array}{l}2\\13\end{array}}, 4; \Gamma)| = |\{aaba, baaa, bbab\}| + |\{abbb, bbba\}| = 5$$

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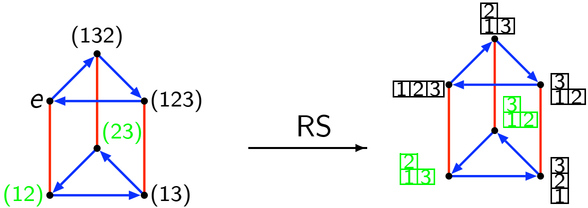

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$$V^{(1,1,1)}: \quad |w(\sigma_{\begin{smallmatrix}3\\2\\1\end{smallmatrix}}, 4; \Gamma)| = |\{aaab, abaa, babb\}| = 3$$

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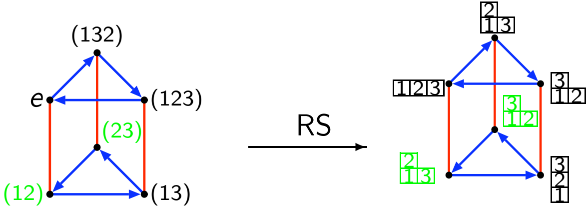

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\end{array}
$$

$$(V^{(2,1)})^{\otimes 4} = 3\, V^{(3)} \oplus 5\, V^{(2,1)} \oplus 3\, V^{(1,1,1)}$$

Kronecker coefficients in general

This idea can be used much more generally than I am considering here. All we need is an embedding of the algebra of representations inside of a group algebra with non-negative integer coefficients and we get for free a combinatorial interpretation of Kronecker coefficients.

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Example: Combinatorial interpretation for Kronecker products with hook shapes.

$$s_{(n-k,1^k)} * s_{(n-\ell,1^\ell)} = \sum_{(\sigma,\tau)} s_{\lambda(Q(\sigma\tau))}$$

where the sum is over (σ, τ) such that $Des(\sigma) = \{1, 2, \ldots, k\}$ and $Des(\tau) = \{1, 2, \ldots, \ell\}$ and $\sigma\tau$ is an involution.

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The combinatorial interpretation is perhaps not useful it is in terms of paths in a graph or alternatively in terms of compositions of group elements.

Applications

Let $[n] = \{1, 2, \ldots n\}$. A set partition of $[n]$, denoted by $A \vdash [n]$, is a family $A_1, A_2, \ldots, A_k \subseteq [n]$ such that $A_1 \cup A_2 \cup \ldots \cup A_k = [n]$.

A set partition A is splitable if $A = B \circ C$, where B and C are non empty and

$$B \circ C = \begin{cases} \{B_1 \cup (C_1 + n), \ldots, B_k \cup (C_k + n), (C_{k+1} + n), \ldots, (C_\ell + n)\} & \text{if } k \leq \ell \\ \{B_1 \cup (C_1 + n), \ldots, B_\ell \cup (C_\ell + n), B_{\ell+1}, \ldots, B_k\} & \text{if } k > \ell. \end{cases}$$

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The set partitions of [3] are

$$\begin{array}{ll} \{\{1\},\{2\},\{3\}\} & \text{nonsplitable} \\ \{\{1\},\{2,3\}\} & \text{nonsplitable} \\ \{\{1,2\},\{3\}\} = \{\{1\}\} \circ \{\{1\},\{2\}\} & \\ \{\{1,3\},\{2\}\} = \{\{1\},\{2\}\} \circ \{\{1\}\} & \\ \{\{1,2,3\}\} = \{\{1\}\} \circ \{\{1\}\} \circ \{\{1\}\} & \end{array}$$

Invariant algebra $T(V^{(n)} \oplus V^{(n-1,1)})^{S_n} \simeq \mathbb{Q}\langle X_n \rangle^{S_n}$

$T(V^{(n)} \oplus V^{(n-1,1)})^{S_n} \simeq \mathbb{Q}\langle X_n \rangle^{S_n}$ is the algebra of Symmetric polynomials in non-commutative variables (Wolf, Rosas and Sagan)

- $\mathbb{Q}\langle X_n \rangle^{S_n} = \mathcal{L}\{\mathbf{m}_A(X_n) \mid A$ set partition with at most n parts$\}$
- $\mathbb{Q}\langle X_n \rangle^{S_n}$ freely generated by $\{\mathbf{m}_A(X_n) \mid A$ non-splitable set partition with at most n parts$\}$ (Wolf)

Invariant algebra $T(V^{(n)} \oplus V^{(n-1,1)})^{S_n} \simeq \mathbb{Q}\langle X_n\rangle^{S_n}$

Corollary

The dimension of $((V^{(n)} \oplus V^{(n-1,1)})^{\otimes d})^{S_n} \simeq \mathbb{Q}\langle X_n\rangle_d^{S_n}$ *is equal to the number of words of length* d *which reduce to the identity in* $\Gamma(S_n, \{e, (12), (132), \ldots, (1\, n \cdots 432)\})$.

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Basis for $\mathbb{Q}\langle x_1, x_2, x_3 \rangle_3^{S_3}$:

- $\mathbf{m}_{\{\{1\},\{2\},\{3\}\}} = x_1x_2x_3 + x_2x_1x_3 + x_1x_3x_2 + x_2x_3x_1 + x_3x_1x_2 + x_3x_2x_1,$
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Words of length 3 which reduce to the identity in $\Gamma(S_3, \{e, (12), (132)\})$:

$$\{bbb, aae, aea, eaa, eee\}.$$

Free generators for $T(V^{(n)} \oplus V^{(n-1,1)})^{S_n}$

Proposition

The number of free generators of $T(V^{(n)} \oplus V^{(n-1,1)})^{S_n}$ as an algebra are counted by the words which reduce to the identity without crossing it in $\Gamma(S_n, \{e, (12), (132), \ldots, (1\, n \cdots 432)\})$.

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The free generators of $T(V^{(3)} \oplus V^{(2,1)})^{S_3}$ are counted by

e	aa	bbb	$abab$	$abbba$	$beaba$	$\ldots$
		aea	$baba$	$baabb$	$baeba$	
			$bebb$	$bbaab$	$babea$	
			$bbeb$	$aebab$	$beebb$	
			$aeea$	$abeab$	$bebeb$	
				$abaeb$	$bbeeb$	
					$aeeea$	

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			$bbeb$	$aebab$	$beebb$	
			$aeea$	$abeab$	$bebeb$	
				$abaeb$	$bbeeb$	
					$aeeea$	
1	1	2	5	13		$\cdots$

the odd indexed Fibonacci numbers.

Link between set partitions and words in Cayley graph

Corollary

The number of set partitions of $[d]$ *into at most* n *parts equals the number of words of length* d *which reduce to the identity in* $\Gamma(S_n, \{e, (12), (132), \ldots, (1\, n \cdots 432)\})$.

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Corollary

The number of nonsplitable set partitions of $[d]$ *into at most* n *parts equals the number of words of length* d *which reduce to the identity without crossing it in* $\Gamma(S_n, \{e, (12), (132), \ldots, (1\, n \cdots 432)\})$.

Invariant algebra $(V^{1^{\otimes d}})^{D_m} \simeq \mathbb{R}\langle x_1, x_2\rangle_d^{D_m}$

Corollary

Let V^1 be the geometric irreducible D_m-module. The dimension of $T^d(V^1)^{D_m}$ is equal to the number of words of length d which reduce to the identity in $\Gamma(D_m, \{r, s\})$.

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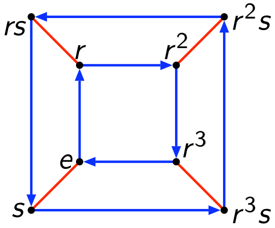
Consider $\Gamma(D_4, \{s, r\})$:



Words of length 4 which reduce to the identity

$$\{rrrr, srsr, rsrs, ssss\}$$

Invariant algebra $(V^{1^{\otimes d}})^{D_m} \simeq \mathbb{R}\langle x_1, x_2\rangle_d^{D_m}$

Consider the dihedral group D_4 acting on $\mathbb{R}\langle x_1, x_2\rangle$ as

$$\begin{aligned} s \cdot x_1 &= -x_1 & r \cdot x_1 &= x_1 + \sqrt{2}\, x_2 \\ s \cdot x_2 &= \sqrt{2}\, x_1 + x_2 & r \cdot x_2 &= -\sqrt{2}\, x_1 - x_2 \end{aligned}$$

Basis for $\mathbb{R}\langle x_1, x_2\rangle_4^{D_4}$:

- $x_1x_2^2x_1 + \frac{\sqrt{2}}{2}x_1x_2^3 + x_2x_1^2x_1 + \frac{\sqrt{2}}{2}x_2x_1x_2^2 + \frac{\sqrt{2}}{2}x_2^2x_1x_2 + \frac{\sqrt{2}}{2}x_2^3x_1 + x_2^4,$
- $x_1^4 + \frac{\sqrt{2}}{2}x_1^3x_2 + \frac{\sqrt{2}}{2}x_1^2x_2x_1 + \frac{\sqrt{2}}{2}x_1x_2x_1^2 - \frac{\sqrt{2}}{2}x_1x_2^3 + \frac{\sqrt{2}}{2}x_2x_1^3 - \frac{\sqrt{2}}{2}x_2x_1x_2^2 - \frac{\sqrt{2}}{2}x_2^2x_1x_2 - \frac{\sqrt{2}}{2}x_2^3x_1 - x_2^4,$
- $x_1^2x_2^2 + \frac{\sqrt{2}}{2}x_1x_2^3 + \frac{\sqrt{2}}{2}x_2x_1x_2^2 + x_2^2x_1^2 + \frac{\sqrt{2}}{2}x_2^2x_1x_2 + \frac{\sqrt{2}}{2}x_2^3x_1 + x_2^4,$
- $x_1x_2x_1x_2 + \frac{\sqrt{2}}{2}x_1x_2^3 + x_2x_1x_2x_1 + \frac{\sqrt{2}}{2}x_2x_1x_2^2 + \frac{\sqrt{2}}{2}x_2^2x_1x_2 + \frac{\sqrt{2}}{2}x_2^3x_1 + x_2^4.$

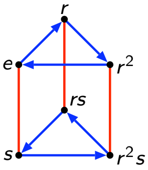
Free generators for $T(V^1)^{D_m}$

Proposition

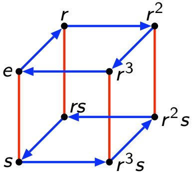
The number of free generators of $T(V^1)^{D_m}$ as an algebra are counted by the words in the Cayley graph $\Gamma(D_m, \{r, s\})$ which reduce to the identity without crossing the identity.

Hilbert-Poincaré series of $T(V^1)^{D_m}$

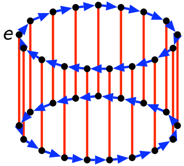
$\Gamma(D_3, \{r, s\})$



$\Gamma(D_4, \{r, s\})$



$\Gamma(D_m, \{r, s\})$



Proposition

$$P(T(V^1)^{D_m}) = 1 + \frac{1}{2}\left(\frac{(2q)^m + \sum_{j=0}^{\lfloor m/2 \rfloor}\left(\binom{m+1}{2j+1} - 2\binom{m}{2j}\right)(1-4q^2)^j}{\sum_{j=0}^{\lfloor m/2 \rfloor}\binom{m}{2j}(1-4q^2)^j - (2q)^m}\right).$$

Irreducible characters of the dihedral group D_m

$D_m = \langle s, r \mid s^2 = r^m = srsr = e \rangle$.

For $m = 2\,k$ even, the irreducible D_m-modules are

$$\{V^{id}, V^{\gamma}, V^{\epsilon}, V^{\gamma\epsilon}, V^1, V^2, \ldots, V^{k-1}\}$$

with irreducible characters $\{id, \gamma, \epsilon, \gamma\epsilon, \chi_1, \chi_2, \ldots, \chi_{k-1}\}$.

For $m = 2\,k + 1$ odd, the irreducible D_m-modules are

$$\{V^{id}, V^{\epsilon}, V^1, V^2, \ldots, V^k\}$$

with irreducible characters $\{id, \epsilon, \chi_1, \chi_2, \ldots, \chi_k\}$.

Surjective algebra morphism from $\mathcal{Q} \subset \mathbb{Z}D_m$ to $\mathbb{Z}\mathrm{Irr}(D_m)$

$D_m = \{e, r, r^2, \ldots, r^{m-1}, s, rs, r^2s, \ldots, r^{m-1}s\}$.
For $m = 2k$ even,

$$\mathcal{Q} = \mathcal{L}\{e, r^k, rs, r^{k+1}s, r^{1-i}s + r^i, r^{-i} + r^{i+1}s\}_{1 \le i \le k-1}.$$

Proposition

$\mathcal{Q}$ is a subalgebra of $\mathbb{Z}D_m$ and there is a surjective algebra morphism

$$\theta : \mathcal{Q} \longrightarrow \mathbb{Z}\mathrm{Irr}(D_m)$$

$$\theta(e) = id$$
$$\theta(rs) = \epsilon$$
$$\theta(r^k) = \gamma$$
$$\theta(r^{k+1}s) = \gamma\epsilon$$

$$\theta(r^{1-i}s + r^i) = \theta(r^{-i} + r^{i+1}s) = \chi_i$$