KRONECKER POWERS
AND
CHARACTER POLYNOMIALS

A. Goupil,
with
C. Chauve and A. Garsia,

Menu

- **Introduction : Kronecker products**
- **Tensor powers**
- **Character Polynomials**
- **Perspective : Duality with product of conjugacy classes**

• **Tensor Product of representations of S_n**

$$A : S_n \to Aut(V) \quad B : S_n \to Aut(W)$$
$$\sigma \mapsto A(\sigma) \qquad \sigma \mapsto B(\sigma)$$

$$A \otimes B : S_n \to Aut(V \otimes W)$$
$$\sigma \mapsto A(\sigma) \otimes B(\sigma)$$

• **Definition**. $A \otimes B$ is called the Kronecker product of the *representations* A and B.

When A^λ and A^μ are irreducible representations, then $A^\lambda \otimes A^\mu$ is, in general, not irreducible and

$$A^\lambda \otimes A^\mu = \sum_\alpha t^\alpha_{\lambda,\mu} A^\alpha$$

The question of finding an easy computation and a combinatorial interpretation of the coefficients $t^\alpha_{\lambda,\mu}$ goes back to the beginning of representation theory.

To compute the coefficients $t^\alpha_{\lambda,\mu}$, we need the characters of the irreducible representations :

$$t^\alpha_{\lambda,\mu} = \chi^\lambda \otimes \chi^\mu \big|_{\chi^\alpha} = \sum_\gamma \frac{|C_\gamma|}{n!} \chi^\lambda(\gamma)\chi^\mu(\gamma)\chi^\alpha(\gamma)$$

but we are looking for a combinatorial computation …

1- Tensor powers

Expansion of the Kronecker powers of $\chi^{(n-1,1)}$:

If P is the permutation representation then

$$\chi^{P} = \chi^{(n-1,1)} \oplus \chi^{(n)}$$

Notation : $\chi^{(n-1,1)^{\otimes k}} = \chi^{(n-1,1)} \otimes \chi^{(n-1,1)} \otimes \cdots \otimes \chi^{(n-1,1)}$

let $\chi^{(n-1,1)^{\otimes k}} \big|_{\chi^{\lambda}} = t^{\lambda}_{(n-1,1)^k}$

then we have the exponential generating function

$$\sum_{k \geq |\bar{\lambda}|} t^{\lambda}_{(n-1,1)^k} \frac{x^k}{k!} = \frac{f^{\bar{\lambda}}}{|\bar{\lambda}|!} e^{e^x - x - 1} (e^x - 1)^{|\bar{\lambda}|} \quad \text{for all } n \geq kn + \lambda_2$$

where $|\bar{\lambda}| = \lambda_2 + \lambda_3 + \cdots$

Observe : $t^{\lambda}_{(n-1,1)^k}$ depends only on $|\bar{\lambda}|$ and $f^{\bar{\lambda}}$.

We carry the Kronecker product in the ring of symmetric functions and we use the Schur functions basis.

Ingredients needed :

s_{λ} : Schur functions

$s_{\lambda}^{\perp}$: operators adjoint to multiplication by s_{λ} :

Recall : $s_{\gamma}^{\perp}(s_{\lambda}) = s_{\lambda/\gamma} = \sum_{\alpha} LR_{\gamma,\alpha}^{\lambda} s_{\alpha}$ if $\gamma \subseteq \lambda$

Combinatorial operator.

$$\chi^{(n-1,1)} \otimes \chi^{\mu} = (s_{(1)}s^{\perp}_{(1)} - 1)s_{\mu}{}^{*}$$

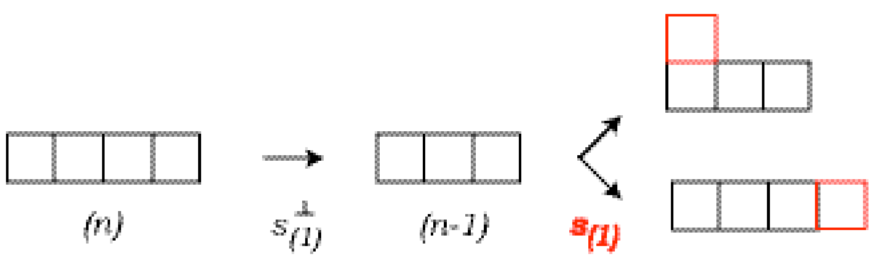
$$\Rightarrow \chi^{(n-1,1)^{\otimes k}} = (s_{(1)}s^{\perp}_{(1)} - 1)^{k} s_{(n)}$$

* There exists such an operator on Schur functions for each $\chi^{\lambda} \otimes \chi^{\mu}$

$s^{\perp}_{(1)}s_{\mu}$: remove one cell from the border of the diagram μ so that the remaining cells is a Ferrers diagram.

$s_{(1)}s_{\mu}$: add one cell to the border of the diagram μ so that the new set of cells is a Ferrers diagram.

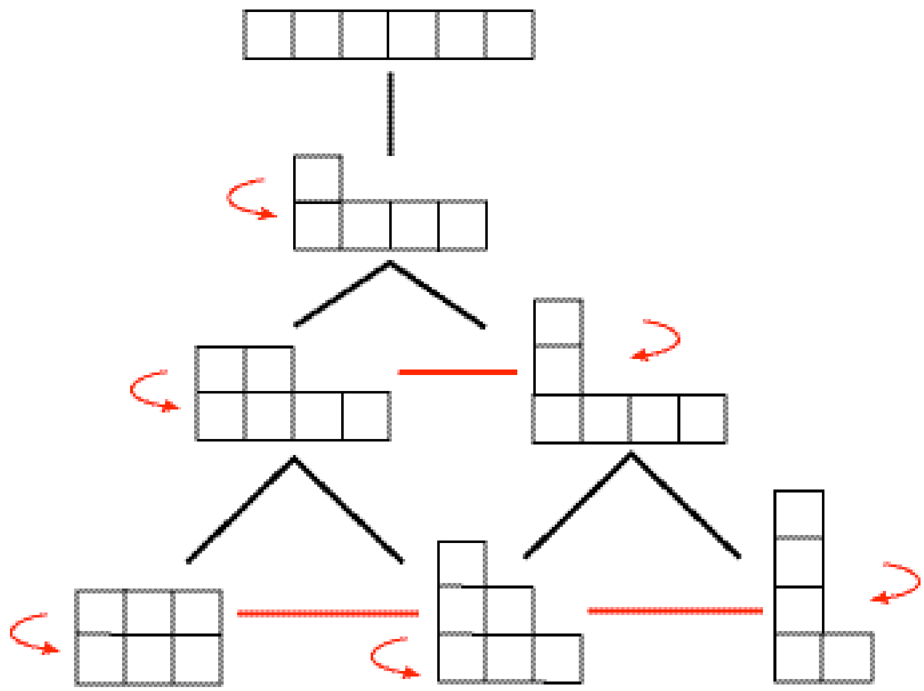
Example : $(s_{(1)}s^{\perp}_{(1)} - 1)s_{(n)}$:



$$\Rightarrow (s_{(1)}s^{\perp}_{(1)} - 1)s_{(n)} = (s_{(n-1,1)} + s_{(n)}) - s_{(n)}$$
$$= s_{(n-1,1)}$$

Apply this combinatorial operator *k* times and count …

We obtain a walk in Young's Lattice (***black*** edges) augmented with the ***red*** edges from (n) to λ :



$$\chi^{(n-1,1)^{\otimes k}}\Big|_{\chi^{\lambda}} = \text{number of walks of length } k \text{ from } (n) \text{ to } \lambda.$$

$k > \bar{\lambda}$: **oscillating** and **stationary** tableaux of shape λ (S. Sundaram 1986, L. Favreau & al. 1988)

● **black** edges :

f_k^{λ}= Number of **Oscillating T**ableaux of shape λ and length k.

$$= 1 \cdot 3 \cdot 5 \ldots (k-|\lambda|-1) \binom{k}{|\lambda|} f^{\lambda}$$

$$= card \;\; OT_k^{\lambda}$$

Idea of proof :

▮ : Oscillating tableaux:

$$OT_k^\lambda \xrightarrow{\text{bijection}} \left\{ \left(\begin{pmatrix} j_1 < \cdots < j_r \\ \vee \qquad\qquad \vee \\ i_1, \quad \ldots \ , \quad i_r \end{pmatrix}, Q_\lambda \right) : |2r| + |\lambda| = k \right\}$$

Right side: pairs

(involution with no fixed point, standard tableau)

on disjoint complementary subsets of [k].

▮ + ▮ : Oscillating and stationary tableaux:

$$OST_k^\lambda \xrightarrow{\text{bijection}} \left\{ \left(\pi, Q_\lambda \right) \right\}$$

- 1^{st} component : π = set partition with parts of size ≥ 2.
- 2^{nd} component : Q_λ = Standard tableau on a subset of [k].

  each entry in Q_λ is the largest entry in a part of π.
- Entries in π and Q_λ do not form disjoint sets.
- Each number in [k] appears once or twice in a pair (π, Q_λ)

$$|OST_k^\lambda| = f^\lambda \sum_{m_1=0}^{|\lambda|} \binom{|\lambda|}{m_1} \sum_{m_2=|\bar{\lambda}|-m_1}^{\lfloor (k-m_1)/2 \rfloor} \binom{m_2}{|\bar{\lambda}|-m_1} p_2(k-m_1, m_2)$$

$$= \left.\left(\chi^{(n-1,1)}\right)^{\otimes k}\right|_{\chi^\lambda}$$

where $p_2(a,b)$ = number of partitions of a set of size a in b parts of size ≥ 2.

$$\Rightarrow \sum_{k \geq |\lambda|} \left.\left(\chi^{(n-1,1)}\right)^{\otimes k}\right|_{\chi^\lambda} \frac{y^k}{k!} = \frac{f^{\bar{\lambda}}}{|\bar{\lambda}|} e^{e^y - y - 1} (e^y - 1)^{|\bar{\lambda}|}$$

Question: *Can we extend these methods to other Kronecker powers ?*

2- Character Polynomials

-Defined by Specht in 1960,
-Little known (Kerber gave a table)

Definition. For each partition $\lambda = \lambda_1 \geq \lambda_2 \geq, \ldots, \geq \lambda_k$ *of n* and $\bar{\lambda} = \lambda_2, \ldots, \lambda_k$, there is a (unique) polynomial $q_{\bar{\lambda}}(x_1,\ldots,x_n)$, the **Character Polynomial of λ**, such that for all permutations $\sigma \in S_n$ with cyclic type $\mu = 1^{m_1} \cdots n^{m_n}$, we have

$$q_\lambda(m_1,\ldots,m_n) = \chi^{(n-|\lambda|,\lambda)}_{1^{m_1}\cdots n^{m_n}}$$

- Examples.

1-Well known: $\chi_\mu^{(n-1,1)}$ = number of fixed points of $C_\mu - 1$

$= m_1 - 1$ if $\mu = 1^{m_1} 2^{m_2} \cdots n^{m_n}$

$$\Rightarrow \quad q_{(1)}(\boldsymbol{x}) = x_1 - 1$$

2- $\chi_\mu^{(n-2,2)} = m_2 + \binom{m_1}{2} - m_1$

$$\Rightarrow \quad q_{(2)}(\boldsymbol{x}) = x_2 + \binom{x_1}{2} - x_1$$

3- $\chi_\mu^{(n-2,1^2)} = -m_2 + \binom{m_1 - 1}{2}$

$$\Rightarrow \quad q_{(1,1)}(\boldsymbol{x}) = -x_2 + (x_1-1)(x_1-2)/2$$

Consequence of the definition :

Products of character polynomials decompose precisely as Kronecker products :

$$q_\lambda q_\mu = \sum_\alpha t^{\alpha}_{\lambda,\mu} q_\alpha \text{ for all } \alpha, \lambda, \mu \vdash n$$

Example 1 (suite).

$$q_1(\mathbf{x}) \bullet q_1(\mathbf{x}) \longleftrightarrow \chi^{(n-1,1)} \otimes \chi^{(n-1,1)}$$

$$= \chi^{(n-2,2)} + \chi^{(n-2,1^2)} + \chi^{(n-1,1)} + \chi^{(n)}$$

$$(x_1-1)^2 = \left[x_2 + \binom{x_1}{2} - x_1\right] + \left[-x_2 + \binom{x_1-1}{2}\right] + \left[x_1 - 1\right] + 1$$

How to compute characters polynomials $q_\lambda(x)$?

Recipy 1. Using the umbral operators of Rota

a) Write s_λ in the power sums basis $\{p_\mu\}_{\mu \vdash n}$

b) Replace p_i by $(ix_i - 1)$ in each $p_\mu = (p_1)^{m_1}(p_2)^{m_2}\cdots$

c) Expand $\prod_{i\geq 1}(ix_i - 1)^{m_i}$ as a sum $\sum_\theta c_\theta \prod_i x_i^{\theta_i}$

d) Replace each $x_i^{\theta_i}$ par $\downarrow x_i^{\theta_i} = (x_i)_{\theta_i}$
(umbral operator)

•Example. $q_{(3)}(\boldsymbol{x})$

a) $$s_{(3)} = \frac{1}{6}(p_{1^3} + 3p_{21} + 2p_3)$$

b) $$\frac{1}{6}((x_1 - 1)^3 + 3(2x_2 - 1)(x_1 - 1) + 2(3x_3 - 1))$$

c) $$\frac{1}{6}(x_1^3 - 3x_1^2 + 6x_1x_2 - 6x_2 + 6x_3)$$

d) $$q_{(3)}(\boldsymbol{x}) = \frac{1}{6}\left[(x_1)_3 - 3(x_1)_2 + 6x_1x_2 - 6x_2 + 6x_3\right]$$

Recipy 2. Recursive Calculus à la Murnaghan-Nakayama

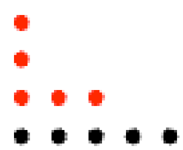
a) Compute $q_{\bar{\lambda}}(x_1,0,\ldots,0) = f^{(x_1-|\bar{\lambda}|,\bar{\lambda})}$ as a polynomial in x_1.

b) The term containing $\binom{x_i}{j}$ and no variable x_k, with $k > i$ is

$$\binom{x_i}{j} \sum_{S=(\bar{\lambda}=\lambda^0,\lambda^1,\ldots,\lambda^j),} (-1)^{ht(S)} q_{\lambda^j}(x_1,\ldots,x_{i-1},0,\ldots)$$

where the sum is over all (j+1)-tuples of partitions obtained by succesively removing from $\bar{\lambda}$ j border strips of length i.

Example. $q_{(3,1,1)}(x_1, x_2,\ldots)$:



$$f^{(x_1-5,3,1,1)} + \binom{x_5}{1} - 2\binom{x_2}{2} q_{(1)}(\mathbf{x}) + \binom{x_2}{1} q_{(1,1,1)}(\mathbf{x}) - \binom{x_2}{1} q_{(3)}(\mathbf{x}) =$$

$$\frac{x_1!}{(x_1-2)(x_1-5)(x_1-6)(x_1-8)!\,20} + x_5 - 2\binom{x_2}{2}(x_1-1)$$

$$+ x_2\left[\binom{x_1-1}{3} - \left(\binom{x_1}{3} - \binom{x_1}{2}\right)\right]$$

Proof. Symmetric functions, plethystic substitution …

The Algebra of Character Polynomials

- The set $\{q_\lambda(x_1,x_2,\ldots)\}_\lambda$ is a basis of $Q[x_1,x_2,\ldots] = Q[\boldsymbol{x}]$

- Scalar product in $Q[\boldsymbol{x}]$:

for $f(x_1,x_2,\ldots,x_k), g(x_1,x_2,\ldots,x_k)$, with $\sum_i i\,deg(x_i) \le k$

if $$\langle f,g\rangle_{Q[\boldsymbol{x}]} = \sum_{\alpha=1^{a_1}2^{a_2}\cdots n^{a_n}\vdash n} \frac{f(a_1,\ldots a_k)g(a_1,\ldots a_k)}{1^{a_1}\cdots n^{a_n}a_1!\,a_2!\cdots a_n!},\quad n\ge 2k$$

then $\{q_\lambda(\boldsymbol{x})\}_\lambda$ becomes an orthonormal basis.

$\Rightarrow$ We can use the scalar product $\langle\ ,\ \rangle_{Q[\boldsymbol{x}]}$ to compute the expansion of any polynomial $f \in Q[\boldsymbol{x}]$

$$f = \sum_\lambda c_\lambda q_\lambda(\boldsymbol{x}) \;\Rightarrow\; c_\lambda = \langle f, q_\lambda(\boldsymbol{x})\rangle_{Q[\boldsymbol{x}]}$$

- Using truncated partitions, we can define the projective limit $Z = \langle \chi^\lambda \rangle$ of the centers $Z_n = \langle \chi^{n-|\lambda|,\lambda}\rangle$ of the group algebras of the symmetric groups S_n.

- In summary, the application

$$q : (Z, +, \otimes) \to (Q[\boldsymbol{x}], +, \bullet),\; q(\chi^\lambda) = q_\lambda(\boldsymbol{x})$$

is an algebra isomorphism and an isometry.

Remark. We have also defined and computed character polynomials for Hecke Algebras of S_n

Example.

$$\chi_\mu^{(n-1,1)} = x_1 - 1$$

$$\chi_\mu^{(n-1,1)}(q) = q^{n-\ell(\mu)-1}\left[x_1 + (\ell(\mu)-1)(q-1)-1\right]$$

APPLICATIONS

1- Coverings of {1,2,…, n}

Bell numbers

$$B_k = \left\langle (h_1 h_{n-1})^{\otimes k}, h_n \right\rangle_\Lambda = \sum_{\mu = 1^{m_1} 2^{m_2} \dots k^{m_k} \vdash k} \frac{m_1{}^k}{z_\mu}$$

= number of walks of length *k* obtained from the action of

$$s_{(1)} s_{(1)}^{\perp}$$

on Ferrers diagrams starting and ending at Identity.

*h*₂ : Multigraphs (G. Labelle)

w_k^*= number of multigraphs with *k* labelled edges and no loop

$$= \left\langle (h_2 h_{n-2})^{\otimes k}, h_n \right\rangle_\Lambda = \sum_{\mu = 1^{m_1} 2^{m_2} \dots k^{m_k} \vdash 2k} \frac{\left[m_2 + \binom{m_1}{2} \right]^k}{z_\mu}$$

= number of walks of length *k* obtained from the action of

$$\boldsymbol{h_2} = s_{(2)} s_{(2)}^{\perp} + s_{(1,1)} s_{(1,1)}^{\perp}$$

on Ferrers diagrams starting and ending at Identity.

h_3 : Russian dolls (D. Zeilberger,)

number of coverings with *k* labelled triangles

$$=\left\langle (h_3 h_{n-3})^{\otimes k}, h_n \right\rangle_\Lambda = \sum_{\mu=1^{m_1}2^{m_2}\cdots k^{m_k} \vdash 3k} \frac{1}{z_\mu} \left(\sum_{\nu=1^{n_1}2^{n_2}\cdots r^{n_r} \vdash 3} \left[\prod_{i=1}^{3} \binom{m_i}{n_i} \right]^k \right)$$

= number of walks of length *k* obtained from the action of h_3 on Ferrers diagrams.

2- Permutations

The character polynomial

$$q_{(e_1^k)}(n) = \sum_{r=0}^{k} \binom{k}{r} (-1)^r n(n-1)\cdots(n-k+1)$$

= number of permutations $\sigma \in S_n$ with longest increasing subsequence of size *n-k* present at the beginning of σ :

$\sigma(1) < \sigma(2) < \ldots < \sigma(n-k) = n.$

Class Polynomials

Kronecker product is « dual » to product of conjugacy classes :

Analogies.

1- Stability.

2- There exists a family $\{\omega_\mu\}_\mu$ of symmetric polynomials with ordinary product analogous to product of conjugacy classes

Example.

$$C(1^{n-2},2)*C(1^{n-3},3) = 4C(1^{n-4},4)+C(1^{n-5},3,2)+2(n-2)C(1^{n-2},2)$$

First examples known to Frobenius, 1901 and Ingram, 1950.

$$\omega_{(1^{n-2},2)} = p_1(\boldsymbol{x})$$

$$\omega_{(1^{n-3},3)} = p_2(\boldsymbol{x}) - \binom{n}{2}$$

$$\omega_{(1^{n-4},4)} = p_3(\boldsymbol{x}) - (2n-3)p_1(\boldsymbol{x})$$

$$\omega_{(1^{n-5},3,2)} = p_{(2,1)}(\boldsymbol{x}) - 4p_3(\boldsymbol{x}) - (\binom{n}{2} - 6n+8)p_1(\boldsymbol{x})$$

$$\omega_{(1^{n-2},2)}\, \omega_{(1^{n-3},3)} = 4\, \omega_{(1^{n-4},4)} + \omega_{(1^{n-5},3,2)} + 2(n-2)\, \omega_{(1^{n-2},2)}$$

Fundamental Property of the $\omega_\mu(x)$:

The value of $\omega_\mu(x)$ on the *contents* $C(\lambda)=\{(j-i)\}_{(i,j)\in\lambda}$ of a Ferrers diagram λ, is

$$\omega_\mu(C(\lambda)) = \frac{|C_\mu(n)|}{f^\lambda}\chi_\mu^\lambda$$

Class polynomials $w_\mu(\boldsymbol{x})$

vs

Character polynomials $p_\mu(\boldsymbol{x})$

Can we establish a combinatorial link between Kronecker coefficients and Class coefficients ?

References

A. Goupil, C. Chauve, *Combinatorial operators for Kronecker powers of representations of Sn*, Séminaire lotharingien de combinatoire, no 54, 2006.

A. Garsia, A.Goupil, *Character Polyominals, their q-analogs and the Kronecker product*, The Electronic Journal of Combinatorics, no 16 (2), 2009.