

## Well-posedness and asymptotic behaviour for the Boussinesq system in $\mathbb{R}^n$ .

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### Resumen

We analyze the well-posedness of the initial value problem for a Convection Problem. Mild solutions are obtained in the weak- $L^p(\mathbb{R}^n)$  spaces and the existence of self-similar solutions is showed, while the only small self-similar solution in the Lebesgue space  $L^p(\mathbb{R}^n)$  is the null solution. The asymptotic stability of solutions is analyzed and, as a consequence, a criterium of self-similarity persistence at large times is obtained.

## 1. Introduction

We consider a viscous incompressible fluid filling the whole space  $\mathbb{R}^n$ ,  $n \geq 2$ . Due to the Boussinesq approximation (Chandrasekhar [3]), density variations are neglected except in the gravitational term (buoyancy term) and they are assumed to be proportional to temperature variations. The relationship among the velocity field  $u(x, t) \in \mathbb{R}^n$ , the pressure  $p(x, t) \in \mathbb{R}$  and the temperature  $\theta(x, t) \in \mathbb{R}$ , can be described by the following initial value problem

$$\frac{\partial u}{\partial t} + u \nabla u - \nu \Delta u + \frac{1}{\rho} \nabla p = \beta \theta f + f_1, \quad x \in \mathbb{R}^n, \quad t > 0, \quad (1)$$

$$\nabla \cdot u = 0, \quad x \in \mathbb{R}^n, \quad t > 0, \quad (2)$$

$$\frac{\partial \theta}{\partial t} + u \nabla \theta - \chi \Delta \theta = h, \quad x \in \mathbb{R}^n, \quad t > 0, \quad (3)$$

$$\theta(x, 0) = \theta_0(x), \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n, \quad (4)$$

where  $f$  represents a gravitational vector field at  $x$ ,  $h$  the reference temperature and  $f_1$  an external force.  $\rho, \nu, \beta, \chi$  are positive physical constants which represent, respectively,

the density, the kinematic viscosity, the coefficient of volume expansion and the thermal conductance. Without loss of generality, we will assume the constants  $\rho, \nu, \beta, \chi$  to be one and the reference temperature  $h$  and the external force  $f_1$  to be zero. The initial data  $u_0$  satisfies the condition  $\nabla \cdot u_0 = 0$  in the distributional sense.

New aspects to studies on the Convection Problem (1)-(4) are considered in this work, in fact, we will study the system (1)-(4) in the whole space  $\mathbb{R}^n$  in the framework of weak- $L^p$  spaces. Firstly we present results of well-posedness in these spaces and make some considerations around the well-posedness in the Lebesgue Spaces  $L^p$  (see [2]). On the other hand, we show some results about the existence, uniqueness, the asymptotic stability and the self-similarity persistence of solutions for the Problem (1)-(4) in weak- $L^p$  spaces. Moreover, as a consequence of results of asymptotic stability, we will show that the only self-similar solutions in Lebesgue spaces  $L^p$  is the null solution, reinforcing the need of more singular initial data to allow the existence of self-similar solutions. These self-similar solutions correspond, for instance, to homogeneous initial functions of degree  $-1$ . Finally, from a physical standpoint, our analysis can be applied for several classes of external forces  $f$ . In fact, we can take  $f$  as the gravitational field

$$f = f(x) = -G\nabla_x \phi = G \frac{x}{|x|^3} \in L^{(\frac{n}{2}, \infty)}(\mathbb{R}^n),$$

where  $G$  is the gravitational constant, in order to show the existence of global solutions  $(u, \theta)$  which are constructed in different functional spaces (see Theorem 3.5, Theorem 3.3 and Remark 4.5). This case can be regarded as an interesting physical case of the Bénard Problem. More details about the physical and mathematical analysis of system (1)-(4) see [2] and the references therein.

## 2. Function Spaces and Definitions

In this section, we introduce the functional spaces relevant to our study of solutions regarding the Cauchy problem for system (1)-(4). For each Lebesgue measurable function  $f$  defined on  $\mathbb{R}^n$ , the rearrangement  $f^*$  is defined by

$$f^*(t) = \inf\{s > 0 : m(\{x \in \mathbb{R}^n : |f(x)| > s\}) \leq t\}, t > 0.$$

The Lorentz space  $L^{(p,q)} \equiv L^{(p,q)}(\mathbb{R}^n)$  is the set of all  $f$  such that

$$\|f\|_{(p,q)} = \begin{cases} \left( \frac{p}{q} \int_0^\infty [t^{\frac{1}{p}} f^{**}(t)]^q dt/t \right)^{\frac{1}{q}}, & \text{if } 1 < p < \infty, 1 \leq q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t) & \text{if } 1 < p \leq \infty, q = \infty. \end{cases}$$

is finite, where  $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds$ , for  $t > 0$ . We observe that  $L^p = L^{(p,p)}$ .  $L^{(p,\infty)}$  are called the Marcinkiewicz spaces or weak- $L^p$  spaces. Moreover,  $L^{(p,q_1)} \subset L^p \subset L^{(p,q_2)} \subset L^{(p,\infty)}$  for  $0 < q_1 \leq p \leq q_2 \leq \infty$ . See [4].

**Proposition 2.1** [4] (*Generalized Holder's inequality*). *Let  $1 < p_1, p_2, r < \infty$ . Let  $f \in L^{(p_1,q_1)}$  and  $g \in L^{(p_2,q_2)}$  where  $\frac{1}{p_1} + \frac{1}{p_2} < 1$ , then the product  $h = fg$  belongs to  $L^{(r,s)}$  where  $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$ , and  $s \geq 1$  is any number such that  $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}$ . Moreover,*

$$\|h\|_{(r,s)} \leq r' \|f\|_{(p_1,q_1)} \|g\|_{(p_2,q_2)},$$

being  $r'$  the conjugate index of  $r$ .

Let us recall the Helmholtz decomposition  $L^r(\Omega) = L_\sigma^r(\Omega) \oplus G^r(\Omega)$ ,  $1 < r < \infty$ , where  $G^r(\Omega) = \{\nabla p \in L^r(\Omega) : p \in L_{loc}^r(\overline{\Omega})\}$ .  $P_r$  (or simply  $P$ ) denotes the projection operator from  $L^r$  onto  $L_\sigma^r$ . The Stokes operator is denoted by  $A_r$  (or simply  $A$  or  $-P\Delta$ ) and the Laplace operator is denoted by  $B_r$  (or simply  $B$  or  $-\Delta$ ). We know that  $-A_r, -B_r$  generate uniformly bounded holomorphic semigroups  $\{e^{-tA_r}\}_{t \geq 0}, \{e^{-tB_r}\}_{t \geq 0}$  of class  $C_0$  in  $L_\sigma^r$  and  $L^r$ , respectively. Borchers and Miyakawa [1] established the following Helmholtz decomposition of the Lorentz spaces. We can extend  $P_r$  to a bounded operator on  $L^{(r,d)}(\Omega)$ , which we denote by  $P_{r,d}$ . Set  $L_\sigma^{(r,d)}(\Omega) = \text{Range}(P_{r,d})$  and  $G^{(r,d)}(\Omega) = \text{Kernel}(P_{r,d})$ . Then,  $L^{(r,d)}(\Omega) = L_\sigma^{(r,d)}(\Omega) \oplus G^{(r,d)}(\Omega)$ . Based on [1],  $-A, -B$  generate uniformly bounded analytic semigroups on  $L_\sigma^{(r,d)}(\Omega)$  and  $L^{(r,d)}(\Omega)$ , respectively. However, notice that these semigroups are not strongly continuous at  $t = 0$  if  $d = \infty$ , since in this case  $D_{r,\infty}(A)$  and  $D_{r,\infty}(B)$  are not dense in  $L_\sigma^{(r,\infty)}$  and  $L^{(r,\infty)}$ , respectively. We recall that in our case  $\Omega = \mathbb{R}^n$ ,  $\{e^{-tB}\}_{t \geq 0}$  is the heat semigroup given as the convolution with the Gauss-Weierstrass kernel:  $G(x, t) = (4\pi t)^{-n/2} \exp(-|x|^2/(4t))$ .

Finally, let  $1 < p < \infty$ ,  $1 < q < \infty$  and  $1 \leq d \leq \infty$ . The following notation is adopted for the norm of product in Lorentz spaces  $L_\sigma^{(p,d)}(\mathbb{R}^n) \times L^{(q,d)}(\mathbb{R}^n)$ :

$$\left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{(p,q),d} = \|u\|_{(p,d)} + \|\theta\|_{(q,d)}.$$

If  $p = q$  and  $d = \infty$ , we simply denote this norm as

$$\left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{(p,\infty)} = \|u\|_{(p,\infty)} + \|\theta\|_{(p,\infty)}.$$

### 3. Results of well-posedness

We define the operator  $M : L_\sigma^{(p,\infty)} \times L^{(q,\infty)} \rightarrow L_\sigma^{(p,\infty)} \times L^{(q,\infty)}$  by

$$M \begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} -P\Delta u \\ -\Delta \theta \end{bmatrix}.$$

With the use of the semigroup  $\{e^{-tM}\}_{t \geq 0}$ , the Cauchy problem (1)-(4) is converted to the integral equation

$$\begin{bmatrix} u(t) \\ \theta(t) \end{bmatrix} = e^{-tM} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} - \int_0^t e^{-(t-s)M} \left( \begin{bmatrix} (u \cdot \nabla u) \\ (u \cdot \nabla \theta) \end{bmatrix} - \begin{bmatrix} (\theta f) \\ 0 \end{bmatrix} \right) ds, \quad t > 0, \quad (5)$$

in  $L_\sigma^{(p,\infty)}(\mathbb{R}^n) \times L^{(q,\infty)}(\mathbb{R}^n)$ . The term in (5)

$$- \int_0^t e^{-(t-s)M} \begin{bmatrix} (u \cdot \nabla u)(s) \\ (u \cdot \nabla \theta)(s) \end{bmatrix} ds$$

will be called the bilinear vector, and the term in (5)

$$\int_0^t e^{-(t-s)P\Delta} (\theta f) ds,$$

we will called of coupling term.

Let us remember the following  $L^{(p,d)} - L^{(r,d)}$  estimates of the semigroup  $\{e^{-tM}\}_{t \geq 0}$  and give the proof by completeness.

**Lemma 3.1** [2] *Let  $1 \leq d \leq \infty$ . For all  $(\varphi, \phi) \in L_\sigma^{(p,d)}(\mathbb{R}^n) \times L^{(q,d)}(\mathbb{R}^n)$ , and all  $t > 0$ , there exists a constant  $C(p, r, s, q)$  such that*

$$\left\| \nabla^j e^{-tM} \begin{bmatrix} \varphi \\ \phi \end{bmatrix} \right\|_{(r,s),d} \leq C t^{-\frac{n}{2}(\gamma + \frac{j}{n})} \left\| \begin{bmatrix} \varphi \\ \phi \end{bmatrix} \right\|_{(p,q),d},$$

where  $\gamma = 1/p - 1/r = 1/q - 1/s$ , with  $1 < p \leq r < \infty$  and  $1 < q \leq s < \infty$ .

Next, let us introduce suitable time dependent functional spaces in which we will need to study the initial value problem (1)-(4).

**Definition 3.2** *Let  $n < q < \infty$  and  $\alpha = 1 - n/q$ . We define the spaces*

$$\begin{aligned} E &\equiv \{(u, \theta) : (u, \theta) \in BC((0, \infty), L_\sigma^{(n,\infty)} \times L^{(n,\infty)})\}, \\ E_q &\equiv \{(u, \theta) \in E : t^{\alpha/2}(u, \theta) \in BC((0, \infty), L_\sigma^{(q,\infty)} \times L^{(q,\infty)})\}, \\ F_q &\equiv \{(u, \theta) : u \in BC((0, \infty); L_\sigma^{(n,\infty)}), t^{\alpha/2}\theta \in BC((0, \infty); L^{(q,\infty)})\}, \end{aligned}$$

which are Banach spaces with the norms in  $E, E_q, F_q$  defined, respectively, as

$$\begin{aligned} \left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_E &= \sup_{t>0} \left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{(n,\infty)}, \quad \left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{E_q} = \left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_E + \sup_{t>0} t^{\alpha/2} \left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{(q,\infty)}, \\ \left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{F_q} &= \sup_{t>0} \|u\|_{(n,\infty)} + \sup_{t>0} t^{\alpha/2} \|\theta\|_{(q,\infty)}. \end{aligned}$$

**Theorem 3.3** (i) *Let  $n > 2$  a positive integer number,  $(u_0, \theta_0)$  be any pair in  $L_\sigma^{(n,\infty)} \times L^{(n,\infty)}$  and  $f$  small enough with respect the following norm*

$$\|f\|_b = \sup_{t>0} t^{\frac{\beta}{2}} \|f(t)\|_{(b,\infty)} < \infty, \quad \beta = 2 - \frac{n}{b}, \quad b > \frac{n}{2}.$$

Then, there are constants  $0 < \tau = \tau(f) < 1$ ,  $\delta > 0$  and  $\varepsilon = \varepsilon(\delta) > 0$  ( $\varepsilon \rightarrow 0$  when  $\delta \rightarrow 0$ ) such that if  $\left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{(n,\infty)} < \delta$ , the initial value problem (1)-(4) has a global solution  $(u(t, x), \theta(t, x)) \in E$  satisfying (5), with

$$\lim_{t \rightarrow 0} (u(t), \phi) = (u_0, \phi), \quad \lim_{t \rightarrow 0} (\theta(t), \varphi) = (\theta_0, \varphi),$$

for all  $\phi \in L_\sigma^{(n',1)}(\mathbb{R}^n)$ ,  $\varphi \in L_\sigma^{(n',1)}(\mathbb{R}^n)$ . Moreover, if  $\left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_E < \frac{2\varepsilon}{1-\tau}$ , then the solution is unique.

(ii) *If we assume that  $(u_0, \theta_0) \in (L_\sigma^{(n,\infty)} \times L_\sigma^{(p,\infty)}) \cap (L^{(n,\infty)} \times L^{(p,\infty)})$  with  $1 < p' < n$ , there are  $0 < \delta_p \leq \delta$  and  $0 < \tau_p = \tau_p(f) \leq \tau$  such that if  $\left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{(p,\infty)} < \delta_p$ , then previous solution  $(u, \theta)$  verifies that  $(u, \theta) \in BC((0, \infty), L_\sigma^{(p,\infty)} \times L^{(p,\infty)})$ .*

**Theorem 3.4** (Regularization) *Under the assumptions of Theorem 3.3, let  $n < q < \infty$ , such that  $\frac{1}{b} + \frac{1}{q} > \frac{1}{n}$ . If  $\|f(t)\|_b = \sup_{t>0} t^{\frac{b}{2}} \|f(t)\|_{(b,\infty)}$  is small enough, there are constants  $0 < \tau_q(f) < 1$  and  $0 < \delta_q \leq \delta$  such that if  $\left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{(n,\infty)} < \delta_q$ , then the solution  $(u, \theta)$  of Theorem 3.3 belongs to  $E_q$ .*

In the case  $n > 2$ , the assumption  $\|f\|_b < \infty$  can be changed by the following one:  $\sup_{t>0} \|f(t)\|_{(\frac{n}{2},\infty)} < \infty$ . Indeed we will prove the following theorem:

**Theorem 3.5** *Let  $(u_0, \theta_0) \in L_\sigma^{(n,\infty)} \times L^{(n,\infty)}$  where  $n > 2$  and assume that  $f$  belongs to  $BC((0, \infty), L^{(\frac{n}{2},\infty)})$ . If  $n < q < \infty$  and  $\sup_{t>0} \|f\|_{(\frac{n}{2},\infty)}$  is sufficiently small, then there are constants  $0 < \tau = \tau(f) < 1$ ,  $\delta > 0$  and  $\varepsilon = \varepsilon(\delta) > 0$  ( $\varepsilon \rightarrow 0$  when  $\delta \rightarrow 0$ ) such that if  $\left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{(n,\infty)} < \delta$ , then the initial value problem for (1)-(4) has a global solution  $(u(t, x), \theta(x, t)) \in F_q$  satisfying (5) together with*

$$\lim_{t \rightarrow 0} (u(t), \phi) = (u_0, \phi), \quad \lim_{t \rightarrow 0} (\theta(t), \varphi) = (\theta_0, \varphi),$$

for all  $\phi \in L_\sigma^{(n',1)}(\mathbb{R}^n), \varphi \in L_\sigma^{(n',1)}(\mathbb{R}^n)$ . Moreover, if  $\left\| \begin{bmatrix} u \\ \theta \end{bmatrix} \right\|_{F_q} \leq \frac{2\varepsilon}{1-\tau}$ , then the solution is unique in the space  $F_q$ .

Furthermore, if we assume that  $(u_0, \theta_0) \in (L_\sigma^{(n,\infty)} \cap L^{(p,\infty)}) \times (L_\sigma^{(n,\infty)} \cap L^{(p,\infty)})$ , with  $q' < p' < \frac{n}{2}$ , there are  $0 < \delta_p \leq \delta$  and  $0 < \tau_p = \tau_p(f) \leq \tau$  such that if  $\left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{(n,\infty)} < \delta_p$ , then previous solution  $(u, \theta)$  satisfies  $(u, \theta) \in BC((0, \infty), L_\sigma^{(p,\infty)} \times L^{(p,\infty)})$ .

### 3.1. Sketch of Proofs of the well-posedness Theorems

The proofs of the well-posedness Theorems follows basically from the next lemma in a generic Banach space and lemmas 3.7, 3.8, 3.9 ( see [2]).

**Lemma 3.6** *Let  $X$  be a Banach space with norm  $\|\cdot\|_X$ ,  $T : X \rightarrow X$  a linear continuous map with norm  $\tau < 1$  and  $B : X \times X \rightarrow X$  a continuous bilinear map, that is, there exists a constant  $K > 0$  such that for all  $x_1$  and  $x_2$  in  $X$   $\|B(x_1, x_2)\|_X \leq K \|x_1\|_X \|x_2\|_X$ . Then, for  $0 < \varepsilon < \frac{(1-\tau)^2}{4K}$  and for any vector  $y \in X$ ,  $y \neq 0$ , such that  $\|y\|_X < \varepsilon$ , there exists a solution  $x \in X$  for the equation  $x = y + B(x, x) + T(x)$  such that  $\|x\|_X \leq \frac{2\varepsilon}{1-\tau}$ . The solution  $x$  is unique in the closed ball  $B_{\frac{2\varepsilon}{1-\tau}} := \overline{B}(0, \frac{2\varepsilon}{1-\tau}) \subset X$ . Moreover, the solution depends continuously on  $y$  in the following sense: If  $\|\tilde{y}\|_X \leq \varepsilon$ ,  $\tilde{x} = \tilde{y} + B(\tilde{x}, \tilde{x}) + T(\tilde{x})$ , and  $\|\tilde{x}\|_X \leq \frac{2\varepsilon}{1-\tau}$ , then  $\|x - \tilde{x}\|_X \leq \frac{1-\tau}{(1-\tau)^2 - 4K\varepsilon} \|y - \tilde{y}\|_X$ .*

**Lemma 3.7** *If  $(u_0, \theta_0) \in L_\sigma^{(n,\infty)} \times L^{(n,\infty)}$ . Then  $e^{-tM} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \in E$ , with  $\left\| e^{-tM} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_E \leq C \left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{(n,\infty)}$  and  $e^{-tM} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \rightarrow \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix}$  when  $t \rightarrow 0^+$ , where the limit is taken in the*

weak-star topology of the  $L_\sigma^{(n,\infty)} \times L^{(n,\infty)}$ . Moreover  $\left\| e^{-tM} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{E_q} \leq C \left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{(n,\infty)}$ , and if  $(u_0, \theta_0) \in (L_\sigma^{(p,\infty)} \times L^{(p,\infty)})$  then  $\left\| e^{-tM} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{(p,\infty)} \leq C \left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{(p,\infty)}$ .

**Lemma 3.8** *Let  $n, b$  be as in the Theorem 3.3 and  $T(\theta) = \int_0^t e^{(t-s)P\Delta}(\theta f)(s) ds$ . Then*

$$\|T(\theta)\|_{(n,\infty)} \leq C \|f\|_b \sup_{t>0} \|\theta\|_{(n,\infty)}, \quad \|T(\theta)\|_{(p,\infty)} \leq C \|f\|_b \sup_{t>0} \|\theta\|_{(p,\infty)}.$$

Moreover, if  $n, b$  satisfy the assumptions of Theorem 3.4, then

$$\|T(\theta)\|_{E_q} \leq C \|f\|_b \sup_{t>0} t^{\frac{\alpha}{2}} \|\theta\|_{(q,\infty)}.$$

**Lemma 3.9** *If  $1 < p < q < \infty$  then for all  $\phi \in L^{(p,1)}(\mathbb{R}^n)$  hold:*

$$s^{\frac{1}{2}(\frac{n}{p}-\frac{n}{q}+1)} \|\nabla e^{-sM} \phi\|_{(q,1)} \leq C \|\phi\|_{(p,1)}, \quad s^{\frac{1}{2}(\frac{n}{p}-\frac{n}{q})} \|e^{-sM} \phi\|_{(q,1)} \leq C \|\phi\|_{(p,1)},$$

$$\int_0^\infty s^{\frac{1}{2}(\frac{n}{p}-\frac{n}{q})-\frac{1}{2}} \|\nabla e^{-sM} \phi\|_{(q,1)} ds \leq C \|\phi\|_{(p,1)}, \quad \int_0^\infty s^{\frac{1}{2}(\frac{n}{p}-\frac{n}{q})-1} \|e^{-sM} \phi\|_{(q,1)} ds \leq C \|\phi\|_{(p,1)}.$$

## 4. Self-Similarity

Assuming that  $f(t, x) = \lambda^2 f(\lambda^2 t, \lambda x)$  is smooth and that  $(u(t, x), \theta(t, x))$  is a smooth solution of the convection problem (1)-(4), it is straightforward to check that  $(u, \theta)_\lambda(t, x) = \lambda(u(\lambda^2 t, \lambda x), \theta(\lambda^2 t, \lambda x))$  is also a solution of the System (1)-(4). In fact, we can look for particular solutions of the System (1)-(4) satisfying

$$(u(t, x), \theta(t, x)) = (u(t, x), \theta(t, x))_\lambda(t, x), \quad (6)$$

for any  $t > 0$ ,  $x \in \mathbb{R}^n$  and  $\lambda > 0$ . These solutions are called self-similar solutions of the system and it is clear that taking  $t \rightarrow 0^+$ , formally in (6),  $(u(0, x), \theta(0, x))$  should be a homogeneous function of degree  $-1$ . This remark gives the hint that a suitable space to find self-similar solutions should be one containing homogeneous functions with that exponent. The space  $L^{(n,\infty)}$  is the only weak- $L^p$  space such that  $|x|^{-1} \in L^{(p,\infty)}$ . Moreover, in case that such a self-similar solution exists, its norm is invariant by the scaling transformation,

$$(u(t, x), \theta(t, x)) \rightarrow (u(t, x), \theta(t, x))_\lambda = \lambda(u(\lambda^2 t, \lambda x), \theta(\lambda^2 t, \lambda x)).$$

Moreover, homogenous functions of any order do not belong to any strong  $L^p$  space. All of these facts reinforce the idea that weak- $L^p$  spaces with the right homogeneity are the most relevant spaces for finding global non-trivial self-similar solutions to the Convection Problem.

### 4.1. Decay Estimates in weak- $L^p$ and $L^p$

**Theorem 4.1** *Let  $(u_0, \theta_0)$  as in the Theorem 3.4 and  $r \geq p$  is finite and satisfies  $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} < \frac{1}{n}$  and  $\frac{1}{b} + \frac{1}{p} - \frac{1}{r} < \frac{2}{n}$ . Then the solution of the Theorem 3.4 satisfies*

$$t^{(\frac{n}{2p}-\frac{n}{2r})} u \in BC((0, \infty); L_\sigma^{(r,\infty)})^n, \quad t^{(\frac{n}{2p}-\frac{n}{2r})} \theta \in BC((0, \infty); L^{(r,\infty)}).$$

Moreover, this theorem is true by relaxing the assumptions to  $n \geq 2$ , and even substituting weak- $L^p$  spaces by their stronger counterparts.

#### 4.1.1. Proof of Theorem 4.1.

Let  $(u_0, \theta_0) \in L_\sigma^{(p,\infty)} \times L^{(p,\infty)}$ . As a direct consequence of Lemma 3.1 we have that  $\sup_{t>0} t^{\frac{r-p}{2\gamma}} \left\| e^{-tM} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{(r,\infty)} \leq C \left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{(p,\infty)}$ . Now, by the second part of Theorem 3.3, we already know that if the initial data  $(u_0, \theta_0) \in (L_\sigma^{(p,\infty)} \cap L_\sigma^{(n,\infty)}) \times (L^{(p,\infty)} \cap L^{(n,\infty)})$ , then the solution  $(u(t), \theta(t))$  satisfies  $\sup_{t>0} (\|u(t)\|_{(p,\infty)} + \|\theta(t)\|_{(p,\infty)}) < \infty$ . Thus, in order to conclude the proof of the Theorem 4.1, we need a lemma where we estimate the norm  $\sup_{t>0} t^{\frac{r-p}{2\gamma}} \|\cdot\|_{r,\infty}$  of bilinear vector term and the linear operator term  $T(\theta)$ , using the norm  $\|\cdot\|_{E_q} + \sup_{t>0} \|\cdot\|_{p,\infty}$  of the solution. For this, we prove the following lemmas.

**Lemma 4.2** *Let  $p$  and  $b$  as in the Theorem 4.1 and  $r \geq p$ , then*

$$\sup_{t>0} t^{(\frac{n}{2p} - \frac{n}{2r})} \|T(\theta)(t)\|_{(r,\infty)} \leq C \|f\|_b \sup_{t>0} \|\theta\|_{(p,\infty)}.$$

**Lemma 4.3** *Let  $p$  as in the Theorem 4.1 and  $r \geq p$ , then*

$$\sup_{t>0} t^\rho \left\| \int_0^t e^{-(t-s)M} \begin{bmatrix} u_1 \cdot \nabla u_2 \\ u_2 \cdot \nabla \theta_1 \end{bmatrix} \right\|_{(r,\infty)} \leq C \sup_{t>0} \left\| \begin{bmatrix} u_1 \\ \theta_1 \end{bmatrix} \right\|_{(p,\infty)} \sup_{t>0} t^{\frac{\alpha}{2}} \|u_2\|_{(q,\infty)},$$

where  $\rho = (\frac{n}{2p} - \frac{n}{2r})$ .

## 4.2. Self-Similar Solution in the spaces $L^{(n,\infty)}$ .

The aim of this subsection is to describe the principal results relative to the existence and the uniqueness of self-similarity solutions in the  $L^{(n,\infty)}$ -spaces.

**Theorem 4.4** *Let  $(u_0, \theta_0) \in L_\sigma^{(n,\infty)} \times L^{(n,\infty)}$ . Assume that  $u_0, \theta_0$  are homogeneous functions of degree  $-1$ , that is,  $u_0(\lambda x) = \lambda^{-1}u_0(x), \theta_0(\lambda x) = \lambda^{-1}\theta_0(x)$  for all  $x \in \mathbb{R}^n, x \neq 0$  and all  $\lambda > 0$  and  $f$  as in Theorem 3.3 and Theorem 3.5, satisfies the scale relation  $f(t, x) = \lambda^2 f(\lambda^2 t, \lambda x)$ . Then, if  $\left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{(n,\infty)} < \varepsilon$  the solution  $(u, \theta)$  given by Theorem 3.3 and Theorem 3.5 is self-similar, i.e.,  $u(t, x) = \lambda u(\lambda^2 t, \lambda x), \theta(t, x) = \lambda \theta(\lambda^2 t, \lambda x)$ ,*

*for all  $x \in \mathbb{R}^n, x \neq 0$  and all  $\lambda > 0$ . Moreover, in case of Theorem 3.3, if the initial data is smaller  $\left\| \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} \right\|_{(n,\infty)} < \varepsilon_q$ , the previous unique self-similar solution becomes regularized.*

**Remark 4.5** [Bénard Problem] *Note that, in the case of Theorem 3.5, we can take  $f$  as the gravitational field  $f = f(x) = -G \nabla_x (\frac{1}{|x|}) = G \frac{x}{|x|^3} \in L^{(\frac{n}{2}, \infty)}(\mathbb{R}^n)$ , where  $G$  is the gravitational constant. This case can be regarded as the Bénard problem (see [3]) which corresponds to the interesting physical case. This consideration is also true for the modified Theorem 3.3, where we assume  $f \in BC([0, \infty); L^{(\frac{n}{2}, \infty)}(\mathbb{R}^n))$ , instead of  $\sup_{t>0} t^{\frac{\beta}{2}} \|f(t)\|_{(b,\infty)} < \infty$ , and we search solution in the space  $E_q$ .*

## 5. Stability in $L^{(n,\infty)}$ .

We analyze the large time behavior of solutions of Section 3. In short, we will show that perturbations of the initial data are negligible for large times.

**Theorem 5.1** *Assume that  $(u, \theta)$  and  $(v, \phi)$  are solutions of (1)-(4) as in the Theorem 3.3 corresponding to the initial conditions  $(u_0, \theta_0)$  and  $(v_0, \phi_0) \in L_\sigma^{(n,\infty)} \times L^{(n,\infty)}$ , respectively. Suppose that*

$$\lim_{t \rightarrow \infty} \|e^{t\Delta}(\theta_0 - \phi_0)\|_{(n,\infty)} = \lim_{t \rightarrow \infty} \|e^{tP\Delta}(u_0 - v_0)\|_{(n,\infty)} = 0,$$

then

$$\lim_{t \rightarrow \infty} \|u(t) - v(t)\|_{(n,\infty)} = 0, \quad \lim_{t \rightarrow \infty} \|\theta(t) - \phi(t)\|_{(n,\infty)} = 0.$$

Moreover, assume  $(u, \theta)$  and  $(v, \phi)$  are solutions of (1)-(4) given by Theorem 3.4 corresponding to initial conditions  $(u_0, \theta_0)$  and  $(v_0, \phi_0) \in L_\sigma^{(n,\infty)} \times L^{(n,\infty)}$  satisfying that

$$\lim_{t \rightarrow \infty} t^{\frac{\alpha}{2}} \|e^{t\Delta}(\theta_0 - \phi_0)\|_{(q,\infty)} = \lim_{t \rightarrow \infty} \|e^{tP\Delta}(u_0 - v_0)\|_{(q,\infty)} = 0,$$

then

$$\lim_{t \rightarrow \infty} t^{\frac{\alpha}{2}} \|u(t) - v(t)\|_{(q,\infty)} = 0, \quad \lim_{t \rightarrow \infty} t^{\frac{\alpha}{2}} \|\theta(t) - \phi(t)\|_{(q,\infty)} = 0.$$

**Theorem 5.2** *Assume that  $(u, \theta)$  and  $(v, \phi)$  are solutions of (1)-(4) as in the Theorem 3.5 corresponding to the initial conditions  $(u_0, \theta_0)$  and  $(v_0, \phi_0) \in L_\sigma^{(n,\infty)} \times L^{(n,\infty)}$ , respectively. Suppose that  $\lim_{t \rightarrow \infty} t^{\frac{\alpha}{2}} \|e^{t\Delta}(\theta_0 - \phi_0)\|_{(q,\infty)} = 0$  and that  $\lim_{t \rightarrow \infty} \|e^{tP\Delta}(u_0 - v_0)\|_{(n,\infty)} = 0$ , then*

$$\lim_{t \rightarrow \infty} \|u(t) - v(t)\|_{(n,\infty)} = 0, \quad \lim_{t \rightarrow \infty} t^{\frac{\alpha}{2}} \|\theta(t) - \phi(t)\|_{(q,\infty)} = 0.$$

**Corollary 5.3** *Let  $(u_0, \theta_0) \in L_\sigma^n \times L^n$  (Lebesgue space) be as in the Theorem 3.4. Then the corresponding solution satisfies  $\lim_{t \rightarrow \infty} \|u(t)\|_{L^n} = 0$ ,  $\lim_{t \rightarrow \infty} \|\theta(t)\|_{L^n} = 0$ . As a consequence, the unique self-similar solution in  $L^n(\mathbb{R}^n) \times L^n(\mathbb{R}^n)$  is the null solution.*

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