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Dynamics for a non-linear and non-autonomous compartmental system

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Abstract

We study the long-time behavior of the amount of material within the compartments of a compartmental system for which the flow of material does not have to be instantaneous and may even take an infinite time to occur. Results on the estructure of minimal sets for monotone skew-product semiflows, previously obtained by the authors, are applied to this description.

1 Some previous results

In this section, we focus on some properties of monotone skew-product semiflows determined by a family of functional differential equations with infinite delay. They were proved by Novo *et al.* [7] and provide infinite delay version of significative results obtained by Jiang and Zhao [4].

Let $(\Omega,\sigma,\mathbb{R})$ be a minimal flow over a compact metric space (Ω,d) and denote $\sigma(t,\omega)=\omega{\cdot}t$ for each $\omega\in\Omega$ and $t\in\mathbb{R}$. In $\mathbb{R}^n$, we take the maximum norm $\|v\|=\max_{j=1,\dots,n}|v_j|$ and the usual partial order relation

$$\begin{aligned} v\le w &\iff v_j\le w_j \quad \text{for } j=1,\dots,n\,,\\ v<w &\iff v\le w \quad\text{and}\quad v_j<w_j \quad \text{for some } j\in\{1,\dots,n\}\,. \end{aligned}$$

We consider the Fréchet space $X=\mathcal{C}((-\infty,0],\mathbb{R}^n)$ endowed with the compact-open topology, i.e. the topology of uniform convergence over compact subsets, which is a metric space for the distance

$$\mathsf{d}(x,y)=\sum_{m=1}^{\infty}\frac{1}{2^m}\frac{\|x-y\|_m}{1+\|x-y\|_m}\,,\qquad x,y\in X\,,$$

where $\|x\|_m = \sup_{s\in[-m,0]} \|x(s)\|$. The subset

$$X_+ = \{x \in X \mid x(s) \geq 0 \ \text{ for each } \ s \in (-\infty, 0]\}$$

is a positive cone in X, because it is a nonempty closed subset $X_+ \subset X$ satisfying $X_+ + X_+ \subset X_+$, $\mathbb{R}^+ X_+ \subset X_+$ and $X_+ \cap (-X_+) = \{0\}$. Besides, it is normal and has an empty interior. As usual, a partial order relation in X is induced, given by

$$\begin{aligned} x \leq y \quad &\Longleftrightarrow \quad x(s) \leq y(s) \quad \text{for each } s \in (-\infty, 0]\,, \\ x < y \quad &\Longleftrightarrow \quad x \leq y \quad \text{and} \quad x \neq y\,. \end{aligned}$$

Let $BU \subset X$ be the Banach space

$$BU = \{x \in X : x \text{ is bounded and uniformly continuous}\}$$

with the supremum norm $\|x\|_\infty = \sup_{s\in(-\infty,0]} \|x(s)\|$. Given $r > 0$ we will denote

$$B_r = \{x \in BU : \|x\|_\infty \leq r\}\,.$$

As usual, given $I = (-\infty, a] \subset \mathbb{R}$, $t \in I$, and a continuous function $z : I \to \mathbb{R}^n$, z_t will denote the element of X defined by $z_t(s) = z(t+s)$ for $s \in (-\infty, 0]$.

We consider the family of non-autonomous infinite delay functional differential equations

$$z'(t) = F(\omega{\cdot}t, z_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,, \qquad (1)_\omega$$

defined by a function $F\colon \Omega \times BU \to \mathbb{R}^n$, $(\omega, x) \mapsto F(\omega, x)$ satisfying the following conditions:

(H1) F is continuous on $\Omega \times BU$ and locally Lipschitzian in x for the norm $\|\cdot\|_\infty$.

(H2) For each $r > 0$, $F(\Omega \times B_r)$ is a bounded subset of $\mathbb{R}^n$.

(H3) For each $r > 0$, $F\colon \Omega \times B_r \to \mathbb{R}^n$ is continuous when we take the restriction of the compact-open topology to B_r, i.e. if $\omega_m \to \omega$ and $x_m \xrightarrow{\mathsf{d}} x$ as $m \to \infty$ with $x \in B_r$, then $\lim_{m\to\infty} F(\omega_m, x_m) = F(\omega, x)$.

(H4) If $x, y \in BU$ with $x \leq y$ and $x_j(0) = y_j(0)$ holds for some $j \in \{1, \ldots, n\}$, then $F_j(\omega, x) \leq F_j(\omega, y)$ for each $\omega \in \Omega$.

(H5) If $x, y \in BU$ with $x \leq y$ and $x_i(0) < y_i(0)$ holds for some $i \in \{1, \ldots, n\}$, then $z_i(t, \omega, x) < z_i(t, \omega, y)$ for each $t \geq 0$ and $\omega \in \Omega$.

(H6) There is an $r > 0$ such that all the trajectories with initial data in B_r are uniformly stable in B_r and relatively compact for the product metric topology.

From Hypothesis (H1), the standard theory of infinite delay functional differential equations (see Hino *et al.* [1]) assures that for each $x \in BU$ and each $\omega \in \Omega$ the system $(1)_\omega$ locally admits a unique solution $z(t, \omega, x)$ with initial value x, i.e. $z(s, \omega, x) = x(s)$ for each $s \in (-\infty, 0]$. Therefore, the family $(1)_\omega$ induces a local skew-product semiflow

$$\begin{aligned} \tau \quad : \mathbb{R}^+ \times \Omega \times BU \quad &\longrightarrow \quad \Omega \times BU \\ (t, \omega, x) \qquad &\mapsto \quad (\omega{\cdot}t, u(t, \omega, x))\,, \end{aligned} \qquad (2)$$

where $u(t,\omega,x) \in BU$ and $u(t,\omega,x)(s) = z(t+s,\omega,x)$ for $s \in (-\infty, 0]$.

Hypothesis (H4) implies that the semiflow is *monotone*, i.e. $u(t,\omega,x) \le u(t,\omega,y)$ provided that $x \le y$ and whenever they are defined.

Let $z(t,\omega_0,x_0)$ be a bounded solution of equation $(1)_{\omega_0}$. From Hypotheses (H1-H3) the omega-limit set of the trajectory of the point (ω_0, x_0)

$$\mathcal{O}(\omega_0,x_0) = \{(\omega,x) \in \Omega \times BU : \exists\, t_m \uparrow \infty \text{ with } \omega_0{\cdot}t_m \to \omega\,,\ u(t_m,\omega_0,x_0) \xrightarrow{\mathsf{d}} x\}$$

is a positively τ-invariant compact subset for the product metric topology, which admits a flow extension.

Under Hypotheses (H1–H6), the following result was proved in [7].

Theorem 1.1. *Assume that* Hypotheses (H1–H6) *hold and let* $(\omega_0,x_0) \in \Omega \times B_r$ *be such that* $\mathcal{O}(\omega_0,x_0) \subset \Omega \times B_r$. *Then* $\mathcal{O}(\omega_0,x_0) = \{(\omega, c(\omega)) : \omega \in \Omega\}$ *is a copy of the base and*

$$\lim_{t\to\infty} \mathsf{d}(u(t,\omega_0,x_0), c(\omega_0{\cdot}t)) = 0\,,$$

where $c : \Omega \to BU$ *is a continuous map.*

2 Application to compartmental systems

Compartmental systems have been widely used as a mathematical model for the study of the dynamical behavior of many processes in biological and physical sciences which depend on local mass balance conditions (see Jacquez and Simon [2, 3] for a review of compartmental systems with or without delay, Györi [6], Györi and Eller [5] and Wu and Freedman [8]).

Firstly, we introduce the model with which we are going to deal as well as some notation. Let us suppose that we have a system formed by n compartments $C_1,\dots,C_n$ among which there is a flow of material through some pipes; we denote by $P_{i,j}$ the pipe taking material from the ith comparment to the jth one for $i,j \in \{1,\dots,n\}$ with $i \neq j$. We call C_0 the environment surrounding the system and, for each $i \in \{1,\dots,n\}$, z_i will denote the amount of material within the compartment C_i. Let $\tilde{g}_{i,j} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$ be the so-called *transport function* determining the volume of material flowing from C_j to C_i given in terms of the time t and the value of z_j in t for $i \in \{0,\dots,n\}, j \in \{1,\dots,n\}$. For each $i \in \{1,\dots,n\}$, we will assume that there exists an incoming flow of material $\tilde{I}_i$ from the environment into the compartment C_i which only depends on time.

Thus, considering a non-instantaneous flow of material among the compartments leads us to a model governed by the following system of infinite delay differential equations:

$$z_i'(t) = -\tilde{g}_{0,i}(t,z_i(t)) - \sum_{\substack{j=1\\ j\neq i}}^{n} \tilde{g}_{j,i}(t,z_i(t)) + \sum_{\substack{j=1\\ j\neq i}}^{n} \int_{-\infty}^{t} \tilde{g}_{i,j}(\tau, z_j(\tau))h_{i,j}(t-\tau)d\tau + \tilde{I}_i(t), \quad (3)$$

$i \in \{1,\dots,n\}$, where $h_{i,j} \in L^1([0,+\infty),\mathbb{R}^+)$ is the transit time density function, which has integral 1 on $[0,+\infty)$ and satisfies $\int_0^\infty s\, h_{i,j}(s)ds < +\infty$.

2.1 The system on the hull

For simplicity, we define $\tilde{g}_{i,0} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(t, v) \mapsto \tilde{I}_i(t)$ for $i \in \{1, \dots, n\}$ and denote $\tilde{g} = (\tilde{g}_{i,j})_{i,j} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m$, where $m = n(n+1)$. For each $t \in \mathbb{R}$, let $\tilde{g}_t : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m$, $(s, v) \mapsto \tilde{g}(t+s, v)$ as usual.

Hereafter, the following property will be assumed:

(C1) For all $i, j \in \{0, \dots, n\}$, $\tilde{g}_{i,j} \in \mathcal{C}(\mathbb{R}^2, \mathbb{R}^+)$ and is uniformly continuous and bounded on all the sets of the form $\mathbb{R} \times \{v_0\}$ for $v_0 \in \mathbb{R}$.

Under Hypothesis (C1), it is easy to check that the family $\{\tilde{g}_t : t \in \mathbb{R}\}$ is equicontinuous and that the sets $\{\tilde{g}_t(s, v) : t \in \mathbb{R}\}$ have compact closure in $\mathbb{R}^m$ for all $s, v \in \mathbb{R}$. Consequently, since $\mathbb{R}^2$ is separable and $\mathbb{R}^m$ is complete, the set $\Omega = \mathrm{cls}\{\tilde{g}_t : t \in \mathbb{R}\}$ turns out to be compact and metrizable when the compact-open topology is considered on $\mathcal{C}(\mathbb{R}^2, \mathbb{R}^m)$ (see [1]). Henceforth, Ω will be considered to be endowed with this topology.

Now, we define a flow on Ω by

$$\begin{array}{rccc} \sigma : & \mathbb{R} \times \Omega & \longrightarrow & \Omega \\ & (t, (\omega_{i,j})_{i,j}) & \mapsto & (\omega_{i,j} \cdot t)_{i,j}, \end{array}$$

where $\omega_{i,j} \cdot t(s, v) = \omega_{i,j}(t+s, v)$ for all i, j and all $t, s, v \in \mathbb{R}$. Again, we will use the notation $\sigma(t, \omega) = \omega \cdot t$, $t \in \mathbb{R}$, $\omega \in \Omega$. It is clear that the flow σ is continuous.

Let us introduce some more notation. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}^m$, $(\omega, v) \mapsto \omega(0, v)$. This map is continuous on Ω. We denote $g = (g_{i,j})_{i,j}$. It is easy to check that, for all $\omega = (\omega_{i,j})_{i,j} \in \Omega$ and all $i \in \{1, \dots, n\}$, $\omega_{i,0}$ is a function dependent only on t; thus, we can define $I_i = \omega_{i,0}$, $i \in \{1, \dots, n\}$. Let $F : \Omega \times BU \to \mathbb{R}^n$ be the map defined by

$$F_i(\omega, x) = -g_{0,i}(\omega, x_i(0)) - \sum_{\substack{j=1 \\ j \neq i}}^{n} g_{j,i}(\omega, x_i(0)) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \int_{-\infty}^{0} g_{i,j}(\omega \cdot \tau, x_j(\tau)) h_{i,j}(-\tau) d\tau + I_i(\omega),$$

for $(\omega, x) \in \Omega \times BU$ and $i \in \{1, \dots, n\}$.

We have a family of infinite delay differential equations corresponding to $(1)_\omega$

$$z'(t) = F(\omega{\cdot}t, z_t), \quad t \geq 0, \ \omega \in \Omega. \qquad (1)_\omega$$

If $\omega = \tilde{g}$ is taken in $(1)_\omega$, we recover equation (3).

2.2 Conditions for the system

More properties of the map $\tilde{g}$ will be needed in order to apply the aforementioned results of [7]. The following hypotheses will be assumed throughout the reminder of this section.

(C2) $\tilde{g}_{i,j}$ is locally Lipschitz in its second variable with Lipschitz constant independent from t.

(C3) $\tilde{g}_{i,j}(t, v) \leq \tilde{g}_{i,j}(t, w)$ for all $t, v, w \in \mathbb{R}$ and all i, j whenever $v \leq w$.

(C4) $\tilde{g}_{i,j}(t, 0) = 0$ for all $t \in \mathbb{R}$ and all i, j.

(C5) $\tilde{g} = (\tilde{g}_{i,j})_{i,j}$ is a recurrent function, i.e. the flow σ is minimal on Ω.

This last property holds for instance if $\tilde{g}$, i.e. all the transport and input functions are almost periodic or almost automorphic in t.

Now, we can define the *total mass* of the system $(1)_\omega$ as

$$\begin{array}{rccl} M: & \Omega \times BU & \longrightarrow & \mathbb{R} \\ & (\omega, x) & \mapsto & \displaystyle\sum_{i=1}^{n} x_i(0) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \int_{-\infty}^{0} \left(\int_{s}^{0} g_{j,i}(\omega \cdot \tau, x_i(\tau)) d\tau \right) h_{i,j}(-s) ds. \end{array}$$

Under assumptions (C1–C5), σ is a minimal flow on Ω and hypotheses (H1–H5) are satisfied. In particular, the skew-product semiflow (2) can be defined.

Proposition 2.1. *The total mass M is a continuous function on all the sets of the form $\Omega \times B_r$ with $r > 0$. Moreover, we have that*

$$\frac{d}{dt} M(\tau_t(\omega, x)) = \sum_{i=1}^{n} (I_i(\omega \cdot t) - g_{0,i}(\omega \cdot t, z_i(t, \omega, x))) \quad \text{for } t \geq 0.$$

Proposition 2.2. *Let $(\omega, x) \in \Omega \times BU$. If there exists $r > 0$ such that $z_t(\omega, x) \in B_r$ for all $t \geq 0$, then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\| z(t + s, \omega, x) - z(t, \omega \cdot s, y)) \| < \varepsilon$$

for all $t \geq 0$ whenever $y \in B_r$ and $\mathsf{d}(z_s(\omega, x), y) < \delta$.

Corollary 2.3. *If there exists a bounded solution of equation $(1)_\omega$, then all solutions of $(1)_\omega$ are bounded as well.*

Thus, Hypothesis (H6) is satisfied and Theorem 1.1 may be applied to our equation. In particular, when a bounded solution exists, and hence all the solutions are bounded, we deduce that they are asymptotically of the same type as the transport functions, i.e. asymptotically almost periodic (resp. almost automorphic) if $\tilde{g}$ is almost periodic (resp. almost automorphic). This result is a generalization of the asymptotic constancy of solutions for the autonomous case.

2.3 Long-time behavior of the solutions

In order to state precisely some subsequent results, two definitions on the way pipes connect compartments will be needed. Let $I = \{1, \ldots, n\}$.

Definition 2.4. Given $i, j \in I$, a pipe $P_{i,j}$ is said to *carry material (from compartment C_j to compartment C_i)* if, for all $t, v \in \mathbb{R}$, $v = 0$ whenever $\tilde{g}_{i,j}(t, v) = 0$.

Definition 2.5. Let $\tau : \mathcal{P}(I) \to \mathcal{P}(I)$, $J \mapsto \cup_{i \in J} \{ j \in I \setminus \{i\} : P_{j,i} \text{ carries material} \}$. A subset J of I is said to be *irreducible* if $\tau(J) \subset J$ and no proper subsets of J have that property.

Note that $\tau(I) \subset I$, so there is always some irreducible subset of I. Our next result gives a useful property of the irreducible sets.

Proposition 2.6. *If a subset J of I is irreducible, then, for all $i, j \in J$ with $i \neq j$, there exist $s \in \mathbb{N}$ and $i_1, \ldots, i_s \in J$ such that $P_{i_1,i}$, $P_{i_2,i_1}, \ldots, P_{i_s,i_{s-1}}$ and P_{j,i_s} carry material.*

Let $J_1, \ldots, J_k$ be all the irreducible subsets of I and let $J_0 = I \setminus \cup_{l=1}^{k} J_l$. These sets reflect the geometry of the compartmental system in a good enough way as to describe the long-time behavior of the solutions, as we will see below.

Let K be a minimal subset of $\Omega \times BU$ for the semiflow (2). In virtue of Theorem 1.1, K will be of the form $K = \{(\omega, x(\omega)) : \omega \in \Omega\}$ where x is a continuous map from Ω into BU. It is clear that, in such circumstances, x gives the solution of $(1)_\omega$ in that $z(t, \omega, x(\omega)) = x(\omega{\cdot}t)(0)$ for all $\omega \in \Omega$ and all $t \in \mathbb{R}$.

Let us check that, on any minimal subset of $\Omega \times BU$ for the semiflow (2), each compartment is either empty all the time or it is never empty.

Proposition 2.7. *If there exist $i \in J$, $\omega \in \Omega$ and $t \in \mathbb{R}$ such that $x_i(\omega)(t) = 0$, then $x_i \equiv 0$.*

All of the subsequent results give qualitative information about the long-time behavior of the solutions. Let us see that, provided that we are working on a minimal set, if the system is closed, i.e. it has no inflow from the environment and has no outflow either, then the total mass is constant, all compartments out of an irreducible subset are empty and, in an irreducible subset, either all compartments are empty or all are never empty.

Theorem 2.8. *Suppose that the system is closed, i.e. $\tilde{g}_{0,i} \equiv 0$, $\tilde{I}_i \equiv 0$ for all $i \in I$. For each $c \geq 0$, there exists a minimal subset K of $\Omega \times BU$ such that $M|_K \equiv c$. Conversely, if K is a minimal subset of $\Omega \times BU$, then there exists $c \geq 0$ such that $M|_K \equiv c$. If $K = \{(\omega, x(\omega)) : \omega \in \Omega\}$ is a minimal subset of $\Omega \times BU$, then*

(i) *$x_i \equiv 0$ for all $i \in J_0$.*

(ii) *If, for some $l \in \{1, \ldots, k\}$, there exists $j_l \in J_l$ such that $x_{j_l} \equiv 0$, then $x_j \equiv 0$ for all $j \in J_l$.*

In our next result, it is seen that, provided that we are working on a minimal set, if there is no inflow from the environment, then the total mass is non-increasing, all compartments out of an irreducible subset are empty and, in any irreducible subset *with some outflow of material*, all compartments are empty. Besides, if an irreducible subset has no outflow of material, then its compartments are all empty or all never empty.

Theorem 2.9. *If $\tilde{I}_i \equiv 0$ for all $i \in I$, then*

(i) *M is non-increasing.*

(ii) *If $K = \{(\omega, x(\omega)) : \omega \in \Omega\}$ is a minimal subset of $\Omega \times BU$, then $x_i \equiv 0$ for all $i \in J_0$.*

(iii) *If $K = \{(\omega, x(\omega)) : \omega \in \Omega\}$ is a minimal subset of $\Omega \times BU$ and, for some $l \in \{1, \ldots, k\}$, there exists $j_l \in J_l$ such that $g_{0,j_l}(\omega, v) > 0$ for all $\omega \in \Omega$ and all $v \in \mathbb{R}$ with $v > 0$, then $x_j \equiv 0$ for all $j \in J_l$.*

(iv) *Suppose that, for some* $l \in \{1, \dots, k\}$, $\tilde{g}_{0,j_l} \equiv 0$ *for all* $j_l \in J_l$. *Define the* mass of J_l *as*

$$\begin{array}{rccl} M_l : & \Omega \times BU & \longrightarrow & \mathbb{R} \\ & (\omega, x) & \mapsto & \displaystyle\sum_{i \in J_l} x_i(0) + \sum_{i \in J_l} \sum_{\substack{j \in J_l \\ j \neq i}} \int_{-\infty}^{0} \left(\int_{s}^{0} g_{j,i}(\omega \cdot \tau, x_i(\tau)) d\tau \right) h_{i,j}(-s) ds. \end{array}$$

Then, for each $c \geq 0$, *there exists a minimal subset* $K = \{(\omega, x(\omega)) : \omega \in \Omega\}$ *of* $\Omega \times BU$ *such that* $x_i \equiv 0$ *for all* $i \in I \setminus J_l$ *and* $M_l|_K \equiv c$. *Conversely, if* $K = \{(\omega, x(\omega)) : \omega \in \Omega\}$ *is a minimal subset of* $\Omega \times BU$ *such that* $x_i \equiv 0$ *for all* $i \in I \setminus J_l$, *then there exists* $c \geq 0$ *such that* $M_l|_K \equiv c$. *Moreover, if there exists* $j_l \in J_l$ *such that* $x_{j_l} \equiv 0$, *then* $x_j \equiv 0$ *for all* $j \in J_l$.

Finally, in a non-closed system, i.e. a system which may have any inflow and any outflow of material, if there exists a bounded solution in an irreducible set which has *some inflow*, then all compartments of that irreducible set are never empty and there must be some outflow from the irreducible set. In particular, we have the following result.

Theorem 2.10. *Let us suppose that there exists a bounded solution of equation* $(1)_\omega$. *Let* $K = \{(\omega, x(\omega)) : \omega \in \Omega\}$ *be a minimal subset of* $\Omega \times BU$. *If, for some* $l \in \{1, \dots, k\}$, *there exist* $j_l \in J_l$ *and* $\omega_0 \in \Omega$ *such that* $I_{j_l}(\omega_0) > 0$, *then*

(i) $x_j(\omega)(t) > 0$ *for all* $j \in J_l$, *all* $t \in [0, +\infty)$ *and all* $\omega \in \Omega$.

(ii) *There exist* $j \in J_l$, $\omega_1 \in \Omega$ *and* $v_1 \in (0, +\infty)$ *such that* $g_{0,j}(\omega_1, v_1) > 0$.

In the general frame of non-closed systems, some results on the existence of bounded solutions of $(1)_\omega$ have been studied; specifically, the linear case is particularly interesting and some results in this direction have been reached.

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