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Numerical study of rotation numbers and its variationals for parametric families of circle diffeomorphisms

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Resumen

We present a numerical method to compute derivatives of the rotation number for parametric families of circle diffeomorphisms with high accuracy. Our methodology is an extension of an existing approach to compute rotation numbers, that it is based on suitable averages of the iterates of the map and Richardson extrapolation. In order to justify the method, we require the family of maps to be differentiable with respect to the parameters and the rotation number to be Diophantine. The method is used to compute the Taylor expansions of Arnold tongues with high precision.

1. Introduction

The rotation number is a very important topological invariant in the study of the dynamics associated to circle maps and, by extension, invariant curves for maps or two dimensional invariant tori for vector fields. For this reason, during the last years, several numerical methods for approximating rotation numbers have been developed. We refer to the works [1, 2, 3, 4] as examples of methods of different nature and complexity.

Recently, a new method for computing rotation numbers of circle diffeomorphisms with high precision at low computational cost has been introduced in [6]. Concretely, this method is built assuming that the circle map is conjugate to a rigid rotation in a sufficiently smooth way and, basically, it consist in averaging the iterates of this circle map together with Richardson extrapolation. From the practical point of view, it is specially suited if we are able to compute the iterates of the map with high precision, for example if we can work with a computer arithmetic having a large number of decimal digits.

The goal of this paper is to extend the method of [6] in order to compute derivatives of the rotation number with respect to parameters in parametric families of circle diffeomorphisms. Our idea is based in the same averaging-extrapolation process applied to the derivatives of the iterates of the map. To this end, we only require the family to be differentiable with respect to the parameters and the rotation number to be Diophantine. Hence, we are able to obtain accurate variational information at the same time that we approximate the rotation number. Consequently, the method allows to study parametric families of circle maps from a point of view that is not given by the previously mentioned methods.

2. Review of the method of [6] to compute rotation numbers

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the real circle and $\mathrm{Diff}_+^r(\mathbb{T})$, $r \in [0,+\infty) \cup \{\infty, \omega\}$, the group of orientation-preserving homeomorphisms of $\mathbb{T}$ of class $\mathcal{C}^r$. Given $f \in \mathrm{Diff}_+^r(\mathbb{T})$, we identify f with its lift to $\mathbb{R}$ by fixing the normalization condition $f(0) \in [0,1)$.

Definition 2.1. *Let f be the lift of an orientation-preserving homeomorphism of the circle such that $f(0) \in [0,1)$. Then the* rotation number of f *is defined as the limit*

$$\rho(f) := \lim_{|n|\to\infty} \frac{f^n(x_0) - x_0}{n},$$

that exists for all $x_0 \in \mathbb{R}$, is independent of x_0 and satisfies $\rho(f) \in [0,1)$.

Given $f \in \mathrm{Diff}_+^2(\mathbb{T})$ with $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$, Denjoy's theorem ensures that f is topologically conjugate to the rigid rotation $R_{\rho(f)}$, i.e., there exist $\eta \in \mathrm{Diff}_+^0(\mathbb{T})$ such that $f \circ \eta = \eta \circ R_{\rho(f)}$, where $R_\theta(x) = x + \theta$. In addition, if we require $\eta(0) = x_0$, for fixed $x_0 \in [0,\infty)$, then η is unique. The theoretical support of the method is provided by the regularity of η, that follows from the next result:

Theorem 2.2. *Assume that $f \in \mathrm{Diff}_+^r(\mathbb{T})$ has Diophantine rotation number $\theta = \rho(f)$ of (C,τ)-type, i.e., there exist constants $C > 0$ and $\tau \geq 1$ such that*

$$|1 - \mathrm{e}^{2\pi \mathrm{i} k\theta}|^{-1} \leq C|k|^\tau, \qquad \forall k \in \mathbb{Z}_*. \tag{1}$$

Then, if $\tau + 1 < r$, f is conjugated to $R_{\rho(f)}$ by means of a conjugacy $\eta \in \mathrm{Diff}_+^{r-\tau-\varepsilon}(\mathbb{T})$, for any $\varepsilon > 0$. Note that $\mathrm{Diff}_+^\omega(\mathbb{T}) = \mathrm{Diff}_+^{\omega-\tau-\varepsilon}(\mathbb{T})$ while the domain of analyticity is reduced.

From now on we focus in the analytic case, although finite differentiability is enough. Let us consider $f \in \mathrm{Diff}_+^\omega(\mathbb{T})$ with rotation number $\theta = \rho(f) \in \mathcal{D}$. Notice that we can write $\eta(x) = x + \xi(x)$, being ξ a 1-periodic function normalized in such a way that $\xi(0) = x_0$, for a fixed $x_0 \in [0,1)$. Now, we can write the following expression for the iterates under the lift:

$$f^n(x_0) = f^n(\eta(0)) = \eta(n\theta) = n\theta + \sum_{k\in\mathbb{Z}} \hat{\xi}_k \mathrm{e}^{2\pi \mathrm{i} kn\theta}, \qquad \forall n \in \mathbb{Z}, \tag{2}$$

where the sequence $\{\hat{\xi}_k\}_{k\in\mathbb{Z}}$ denotes the Fourier coefficients of ξ.

We introduce the following *recursive sums* for $p \in \mathbb{N}$

$$S_N^0(f) := f^N(x_0) - x_0, \qquad S_N^p(f) := \sum_{j=1}^{N} S_j^{p-1}(f). \tag{3}$$

Then, the result presented in [6] says that, under the previous hypotheses, the following sums (3) satisfy the expression (basically, the idea is to use (2) and the fact that the Fourier coefficients decay very fast due to the analyticity of η)

$$\binom{N+p}{p+1}^{-1} S_N^p(f) = \theta + \sum_{l=1}^{p} \frac{A_l^p}{N^l} + E^p(N), \tag{4}$$

where the coefficients A_l^p depend on f and p but are independent of N. Furthermore, the remainder $E^p(N)$ is uniformly bounded by an expression of order $\mathcal{O}(1/N^{p+1})$. Then, if we select a maximum number $N = 2^q$ of iterates, we obtain

$$\theta = \Theta_{q,p}(f) + \mathcal{O}(2^{-(p+1)q}), \quad \Theta_{q,p}(f) := \sum_{j=0}^{p} c_j^p \binom{N+p}{p+1}^{-1} S_N^p(f), \tag{5}$$

for some universal coefficients c_j^p which are given by

$$c_l^p = (-1)^{p-l} \frac{2^{l(l+1)/2}}{\delta(l)\delta(p-l)}, \tag{6}$$

where we define $\delta(n) := (2^n-1)(2^{n-1}-1)\cdots(2^1-1)$ for $n \geq 1$ and $\delta(0) := 1$. The operator $\Theta_{q,p}$ corresponds to the Richardson extrapolation of equation (4) of order p.

As far as the behavior of the error is concerned, by comparing (5) with 2^{q-1} and 2^q iterates, we obtain the following (heuristic) expression

$$|\theta - \Theta_{q,p}(f)| \leq \frac{\nu}{2^{p+1}} |\Theta_{q,p}(f) - \Theta_{q-1,p}(f)|, \tag{7}$$

where ν is a "safety parameter" whose role is to prevent the oscillations in the error as a function of q due to the quasi-periodic part. We take $\nu = 10$.

3. Derivatives of the rotation number with respect to parameters

Now we adapt the previous ideas in order to compute derivatives of the rotation number with respect to parameters (assuming that they exist). For a sake of simplicity, we introduce the method for one-parameter families of circle diffeomorphisms, albeit the construction can be adapted to deal with multiple parameters. Thus, consider $\mu \in I \subset \mathbb{R} \mapsto f_\mu \in \mathrm{Diff}_+^\omega(\mathbb{T})$ depending on μ in a regular way. The rotation numbers of the family $\{f_\mu\}_{\mu\in I}$ induce a function $\theta : I \to [0,1)$ given by $\theta(\mu) = \rho(f_\mu)$.

Our goal is to generalize formula (5) to approximate numerically the derivative $D_\mu^d \theta$ for any d, when it exists. For a full discussion of the construction we refer to [5]. We assume that the family $\mu \mapsto f_\mu \in \mathrm{Diff}_+^\omega(\mathbb{T})$ depends $\mathcal{C}^d$-smoothly with respect to the parameter. As before, we define the recursive sums

$$D_\mu^d S_N^0(f_\mu) := D_\mu^d (f_\mu^n(x_0) - x_0), \quad D_\mu^d S_N^p(f_\mu) := \sum_{j=0}^{N} D_\mu^d S_j^{p-1}(f_\mu).$$

Then, if $\theta(\mu_0) \in \mathcal{D}$ and $D_\mu^d \theta(\mu_0) \neq 0$, we obtain (omitting the point μ_0)

$$\binom{N+p}{p+1}^{-1} D^d_\mu S^p_N(f_\mu) = D^d_\mu \theta + \sum_{l=1}^{p-d} \frac{D^d_\mu \hat{A}^p_l}{N^l} + \mathcal{O}(1/N^{p-d+1}), \tag{8}$$

where the remainder $D^d_\mu E^p(N)$ is of order $\mathcal{O}(1/N^{p-d+1})$. Therefore, according to formula (8), we can extrapolate the d-th derivative of the rotation number as

$$D^d_\mu \theta = \Theta^d_{q,p,p-d}(f) + \mathcal{O}(2^{-(p-d+1)q}), \quad \Theta^d_{q,p,m}(f) := \sum_{j=0}^{m} c^m_j \binom{N+p}{p+1}^{-1} D^d_\mu S^p_N(f_\mu),$$

and we observe that now the extrapolation order is $p-d$ instead of p. Finally, we obtain the following heuristic formula for the extrapolation error

$$|D^d_\mu \theta - \Theta^d_{q,p,p-d}(f)| \leq \frac{\nu}{2^{p-d+1}} |\Theta^d_{q,p,p-d}(f) - \Theta^d_{q-1,p,p-d}(f)|. \tag{9}$$

Remark 3.1. *Up to this point we have assumed that $D^d_\mu \theta \neq 0$ at the computed point. However, if we know a priori that $D^r_\mu \theta = 0$ for $r = 1, \ldots, d$, then the remainder $D^d_\mu E^p(N)$ is now of order $\mathcal{O}(1/N^{p+1})$. This allows to approximate $D^d_\mu \theta$ with the same extrapolation order as the averaging order p. Indeed, we obtain*

$$0 = D^d_\mu \theta = \Theta^d_{q,p,p}(f) + \mathcal{O}(2^{-(p+1)q}).$$

4. Application to the Arnold family

As an example, we consider the Arnold family of circle maps, given by

$$\begin{array}{rccl} f_{\alpha,\varepsilon}: & \mathbb{S} & \longrightarrow & \mathbb{S} \\ & x & \longmapsto & x + 2\pi\alpha + \varepsilon \sin(x), \end{array} \tag{10}$$

where $(\alpha, \varepsilon) \in [0,1) \times [0,1)$ are parameters and $\mathbb{S} = \mathbb{R}/(2\pi\mathbb{Z})$. Notice that, for this family of maps, it is convenient to take the angles modulo 2π just for avoiding the lost of significant digits due to the factors $(2\pi)^{d-1}$ that would appear in the d-derivative of the map.

4.1. Stepping up to a Devil' staircase

Let us fix the value of $\varepsilon \in [0,1)$ and consider the one-parameter family $\{f_\alpha\}_{\alpha \in [0,1)}$ given by equation (10), i.e. $f_\alpha := f_{\alpha,\varepsilon}$. The map $\alpha \mapsto \rho(f_\alpha)$ gives rise to a "staircase" with a dense number of stairs, that is usually called a Devil' staircase.

To illustrate the behavior of the method we have computed the above staircase for $\varepsilon = 0{,}75$. The computations have been performed by taking 10^4 points of $\alpha \in [0,1)$, using 32-digit arithmetics, and a fixed averaging order $p = 8$. In addition, we estimate the error in the approximation of $\rho(f_\alpha)$ and $D_\alpha \rho(f_\alpha)$ using formulas (7) and (9), respectively. Then, we stop the computations for a tolerance of 10^{-26} and 10^{-24}, using at most $2^{22} = 4194304$ iterates.

Let us discuss the obtained results. First, we point out that only 11,4 % of the selected points have not reached the previous tolerances for 2^{22} iterates. Moreover, we observe that

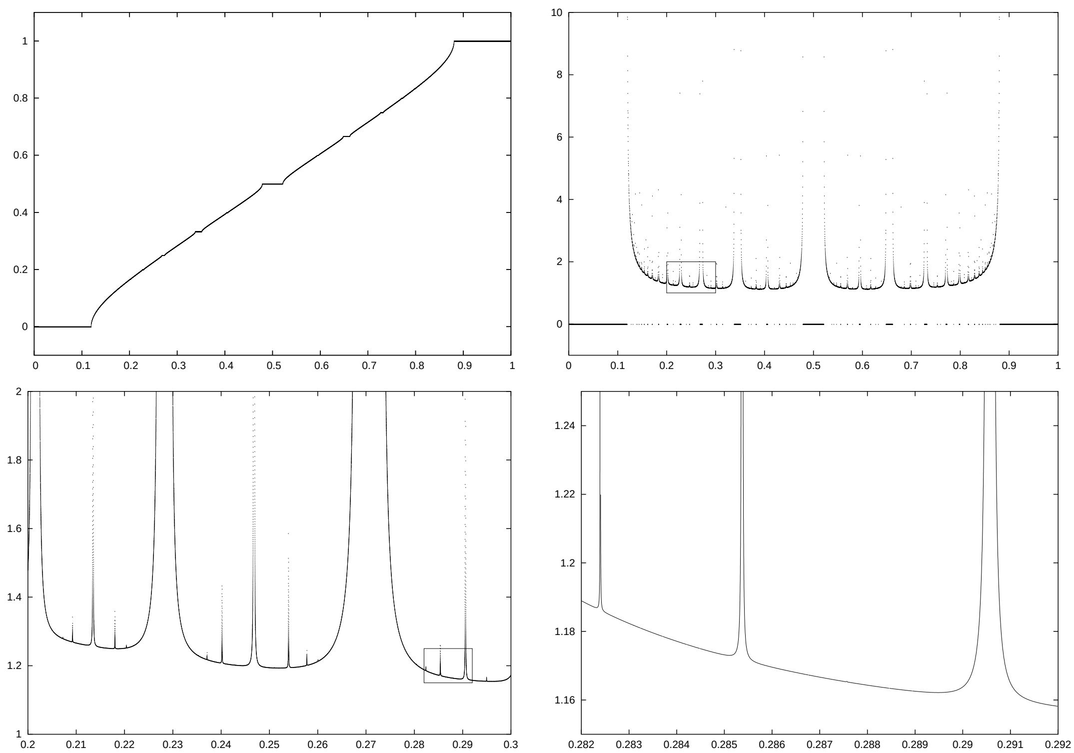

Figura 1: Devil' staircase (top-left) and its derivative (top-right) for the Arnold family with $\varepsilon = 0{,}75$. The plots in the bottom correspond to some magnifications of the derivatives top-right one.

the rotation number for 98,8 % of the points has been obtained with an error less that 10^{-20}, while the estimated error in the derivatives is less than 10^{-18} for 97,7 % on the points.

In figure 1 we show $\alpha \mapsto \rho(f_\alpha)$ and its derivative $\alpha \mapsto D_\alpha \rho(f_\alpha)$ for those points that satisfy that the estimated error is less than 10^{-18} and 10^{-16}, respectively. We recall that the rational values of the rotation number correspond to the constant intervals in the top-left plot, and note that looking at the derivative (top-right plot) we can visualize the density of the stairs better than looking at the staircase itself. We remark that both these rational rotation numbers and their vanishing derivatives have been computed as well as the Diophantine cases. Moreover, at the bottom of the same figure, we plot some magnifications of the derivative to illustrate the non-smoothness of a Devil' staircase. These magnifications have been computed by taking 10^5 and 10^6 points, respectively. We refer to [5] for more details on these computations and further discussion of the results.

4.2. Computation of the Taylor expansion of Arnold tongues

All through the rest of the section, we assume that $\theta \in \mathcal{D}$ and consider the Arnold Tongue $T_\theta = \{(\varepsilon, \alpha(\varepsilon)) : \varepsilon \in [0,1)\}$. Then, we can expand α at the origin as

$$\alpha(\varepsilon) = \theta + \frac{\alpha'(0)}{1!}\varepsilon + \frac{\alpha''(0)}{2!}\varepsilon^2 + \cdots + \frac{\alpha^{(d)}(0)}{d!}\varepsilon^d + \mathcal{O}(\varepsilon^{d+1}), \tag{11}$$

and our goal is to compute the derivatives $\alpha^{(r)}(0)$, $r \leq d$, that appear in this power series.

d	$2\pi\alpha^{(d)}(0)$	e_1
0	3.883222077450933154693731259925391915269339787692096599014776434	-
1	5.289596087298835974306750728481413682115174017433159533705768026 $\cdot 10^{-54}$	$2 \cdot 10^{-50}$
2	-1.944003667801032197325141712953470682792841985057545477738933600 $\cdot 10^{-1}$	$7 \cdot 10^{-50}$
3	6.353866339253870417285870622952031667026712174414003758743809499 $\cdot 10^{-52}$	$3 \cdot 10^{-48}$
4	9.865443989835495993231949890783720243438883460505483297079900562 $\cdot 10^{-1}$	$2 \cdot 10^{-47}$
5	4.733853534850495777271526084574485398105534790325269345544052633 $\cdot 10^{-49}$	$2 \cdot 10^{-45}$
6	-1.451874181864020963416053802229271731186248529989217665545212404 $\cdot 10^{1}$	$6 \cdot 10^{-45}$
7	-1.986768674642925514096249083525472601734104441662711304098209993 $\cdot 10^{-47}$	$7 \cdot 10^{-44}$
8	1.673363822376717001078781931538386967523434046199355922539083323 $\cdot 10^{1}$	$8 \cdot 10^{-42}$
9	-5.559060362825539878039137008326038842079877436013501651866007318 $\cdot 10^{-44}$	$2 \cdot 10^{-40}$
10	1.974679484744669888248485084754876332689468886829840384314732615 $\cdot 10^{4}$	$2 \cdot 10^{-39}$
11	4.019718902900154426125206309959051888079502318143227318836414835 $\cdot 10^{-42}$	$1 \cdot 10^{-38}$
12	3.594891944526889578314748272295019294147597687816868847742850594 $\cdot 10^{5}$	$6 \cdot 10^{-37}$
13	-4.123166034989923032518732576715313341946051550138603536248010821 $\cdot 10^{-39}$	$2 \cdot 10^{-35}$
14	2.198602821435568153883567054383394767567371744732559263055644337 $\cdot 10^{6}$	$3 \cdot 10^{-33}$
15	1.307318024754974551233761145122558811543944190022138837513637182 $\cdot 10^{-35}$	$6 \cdot 10^{-32}$
16	-4.009257214040427899940043656551946700300230713255210114705187412 $\cdot 10^{10}$	$4 \cdot 10^{-31}$
17	-6.641638995605492204184114438636683272452899190211080822408603857 $\cdot 10^{-33}$	$4 \cdot 10^{-29}$
18	-2.582559893723659427522610275977697024396910000154382754643273110 $\cdot 10^{12}$	$1 \cdot 10^{-27}$
19	-4.366235264281358239242428788236090577328510850575386329987344515 $\cdot 10^{-30}$	$2 \cdot 10^{-26}$

Tabla 1: Derivatives of $2\pi\alpha(\varepsilon)$ at the origin for $\theta = (\sqrt{5}-1)/2$. The column e_1 corresponds to the estimated error using (7).

Our idea is to use the fact that the rotation number is constant on the tongue combined with remark 3.1. To this end, we consider the one-parameter family $\{f_{\alpha(\varepsilon),\varepsilon}\}_{\varepsilon\in[0,1)}$ of circle diffeomorphisms, where the graph of α parametrizes the tongue T_θ. For this family, we have $\rho(f_{\alpha(\varepsilon),\varepsilon}) = \theta$ for any $\varepsilon \in [0,1)$, and hence, from remark 3.1 we read the expression

$$0 = \Theta^d_{q,p,p}(f_{\alpha(\varepsilon),\varepsilon}) + \mathcal{O}(2^{-(p+1)q}). \tag{12}$$

Then, let us assume that the values $\alpha'(0), \ldots, \alpha^{(d-1)}(0)$ are known, and isolate the derivative $\alpha^{(d)}(0)$ from $D^d_\varepsilon(f^n(x_0))|_{\varepsilon=0}$. Proceeding by induction with respect to n we obtain the following formula

$$D^d_\varepsilon(f^n(x_0))|_{\varepsilon=0} = 2\pi n \alpha^{(d)}(0) + g^d_n, \tag{13}$$

where $g^d := \{g^d_n\}_{n=1,\ldots,N}$ is a sequence that only requires the known derivatives $\alpha^{(r)}(0)$, for $r < d$. Concretely,

$$\begin{aligned} g^d_n &= D^{d-1}_\varepsilon(\partial_\varepsilon g(f^{n-1}(x_0))|_{\varepsilon=0} \\ &\quad + \sum_{r=1}^{d-1} \binom{d-1}{r} D^r_\varepsilon(\partial_x g(f^{n-1}(x_0))) D^{d-r}_\varepsilon(f^{n-1}(x_0))|_{\varepsilon=0} + g^d_{n-1}. \end{aligned}$$

Hence, by introducing an extrapolation operator for the previous sequence we obtain

$$\Theta^d_{q,p,p}(f)|_{\varepsilon=0} = 2\pi\alpha^{(d)}(0) + \Theta_{q,p,p}(g^d) = \mathcal{O}(2^{-(p+1)q}).$$

Therefore, the Taylor expansion (11) follows from the sequential computation of $\alpha^{(d)}(0)$ by means of the expression $2\pi\alpha^{(d)}(0) = -\Theta_{q,p,p}(g^d)$ with an error of order $\mathcal{O}(2^{-(p+1)q})$.

In particular, in table 1 we show the computations of $2\pi\alpha^{(d)}(0)$, for $0 \leq d \leq 19$, that correspond to the Arnold Tongue associated to $\theta_{1,1} = (\sqrt{5}-1)/2$. The computations are performed using 64-digit arithmetics. The implementation parameters are selected as $p = 11$ and $q = 23$.

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