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A new primal-mixed finite element method for the linear elasticity problem

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Abstract

We introduced a new augmented variational formulation for the elasticity problem in the plane that involves four unknowns, namely, the displacement, the stress tensor, the strain tensor of small deformations and the pressure. We proved that this problem is well posed for appropriate values of a stabilization parameter. We also gave sufficient conditions for the well posedness of the corresponding Galerkin scheme, and detailed concrete examples of discrete spaces satisfying these conditions. We provided error estimates for these cases.

1 Introduction

Recently, J.K. Djoko and B.D. Reddy [8] proposed a new class of mixed formulations for the linear elasticity problem in the plane. The new formulations are based on the discrete EVSS (Elastic-Viscous-Split-Stress) method (see [9, 10]), and involve the displacement, the strain and the stress as unknowns. A stabilization term of the form $2\alpha\mathbf{div}(\mathbf{t}-\mathbf{e}(\mathbf{u}))$, where $\mathbf{t}$ denotes the infinitesimal strain, is added, and two different variational formulations are derived. The first one fits the abstract framework of [15, 3], and the second one has the same structure as the problem considered in [7].

Here, we introduce the infinitesimal strain and the pressure as further unknowns, and obtain a four-field variational formulation with a saddle point structure. Using the abstract theory from [5], we show that this problem is well posed. However, the bilinear form $a(\cdot,\cdot)$ is not coercive on the whole space and thus, it would be difficult to define finite element subspaces such that the corresponding Galerkin scheme is well posed. For this reason, we add a Galerkin least-squares term that renders coercive the bilinear form on the whole space. The new augmented variational formulation is showed to be well posed for appropriate values of the stabilization parameter. We also give sufficient conditions for the well posedness of the corresponding Galerkin scheme. We showed that to approximate the displacement-pressure pair, we can choose any pair of stable spaces for the Stokes problem. We consider in particular, the approximation of the pair displacement-pressure using the mixed finite element introduced in [4] and the mini-element, and provide error estimates in both cases.

2 The augmented variational formulation

Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected domain with Lipschitz-continuous boundary Γ. We consider the following problem: Given a volume force $\mathbf{f}$, determine the displacement vector field $\mathbf{u}$ and the symmetric stress tensor field $\boldsymbol{\sigma}$ of a linear elastic material occupying the region Ω, and thus satisfying the equations

$$\begin{array}{rcll} \boldsymbol{\sigma} & = & \mathcal{C}\,\mathbf{e}(\mathbf{u}) & \text{in } \Omega \\ -\,\mathbf{div}(\boldsymbol{\sigma}) & = & \mathbf{f} & \text{in } \Omega \\ \mathbf{u} & = & \mathbf{0} & \text{on } \Gamma \end{array} \tag{1}$$

We denote by $\mathbf{e}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathtt{t}}\right)$ the strain tensor of small deformations, and by $\mathcal{C}$ the elasticity tensor determined by Hooke's law, that is,

$$\mathcal{C}\,\boldsymbol{\zeta} \;:=\; \lambda\,\mathrm{tr}(\boldsymbol{\zeta})\,\mathbf{I} \;+\; 2\,\mu\,\boldsymbol{\zeta} \qquad \forall\,\boldsymbol{\zeta} \,\in\, [L^2(\Omega)]^{2\times 2}$$

where $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{2\times 2}$ and $\lambda,\ \mu > 0$ are the Lamé parameters.

In order to obtain a new variational formulation of problem (1), we introduce two auxiliary unknowns in Ω, the strain tensor of small deformations $\mathbf{t} := \mathbf{e}(\mathbf{u})$ and the pressure $p := \lambda\,\mathrm{tr}(\mathbf{t})$. Then, the constitutive law can be written as follows

$$\boldsymbol{\sigma} = \mathcal{C}\,\mathbf{t} = p\,\mathbf{I} \;+\; 2\,\mu\,\mathbf{t} \qquad \text{in } \Omega \tag{2}$$

Testing equation (2) with a function $\mathbf{s} \in [L^2(\Omega)]^{2\times 2}_{\mathtt{sym}}$, we obtain that

$$-\int_\Omega \boldsymbol{\sigma} : \mathbf{s} + \int_\Omega p\,\mathrm{tr}(\mathbf{s}) + 2\,\mu \int_\Omega \mathbf{t} : \mathbf{s} = 0 \qquad \forall\,\mathbf{s} \in [L^2(\Omega)]^{2\times 2}_{\mathtt{sym}} \tag{3}$$

On the other hand, testing the equilibrium equation with a function $\mathbf{v} \in [H^1_0(\Omega)]^2$, integrating by parts in Ω, and using the symmetry of $\boldsymbol{\sigma}$ and the boundary condition, we have that

$$\int_\Omega \boldsymbol{\sigma} : \mathbf{e}(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\mathbf{v} \in [H^1_0(\Omega)]^2 \tag{4}$$

From now on, we assume that $\mathbf{f} \in [L^2(\Omega)]^2$.

Now we remark that in case of pure homogeneous Dirichlet boundary conditions,

$$\int_\Omega p = \lambda \int_\Omega \mathrm{div}(\mathbf{u}) = \lambda \int_\Gamma \mathbf{u}\cdot\mathbf{n} = 0$$

where $\mathbf{n}$ is the unit outward normal to Γ. Therefore, we look for $p \in L^2_0(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q = 0\}$. Testing appropriately the relations defining the auxiliary unknowns $\mathbf{t}$ and p, we obtain that

$$\int_\Omega \mathbf{e}(\mathbf{u}) : \boldsymbol{\tau} - \int_\Omega \mathbf{t} : \boldsymbol{\tau} = 0 \qquad \forall\, \boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}_{\mathtt{sym}} \tag{5}$$

$$-\frac{1}{\lambda}\int_\Omega p\, q + \int_\Omega \mathrm{tr}(\mathbf{t})\, q = 0 \qquad \forall\, q \in L^2_0(\Omega) \tag{6}$$

Let us denote by $X = [L^2(\Omega)]^{2\times 2}_{\mathtt{sym}} \times [H^1_0(\Omega)]^2$ and by $M = [L^2(\Omega)]^{2\times 2}_{\mathtt{sym}} \times L^2_0(\Omega)$. Summing up equations (3) and (4), and equations (5) and (6), we arrive at the following variational formulation of problem (1): Find $((\mathbf{t},\mathbf{u}),(\boldsymbol{\sigma},p)) \in X \times M$ such that

$$\begin{array}{rcrcll} a((\mathbf{t},\mathbf{u}),(\mathbf{s},\mathbf{v})) & + & b((\mathbf{s},\mathbf{v}),(\boldsymbol{\sigma},p)) & = & \displaystyle\int_\Omega \mathbf{f}\cdot\mathbf{v} & \forall\,(\mathbf{s},\mathbf{v}) \in X \\ b((\mathbf{t},\mathbf{u}),(\boldsymbol{\tau},q)) & - & c((\boldsymbol{\sigma},p),(\boldsymbol{\tau},q)) & = & 0 & \forall\,(\boldsymbol{\tau},q) \in M \end{array} \tag{7}$$

where the bilinear forms $a : X \times X \to \mathbb{R}$, $b : X \times M \to \mathbb{R}$ and $c : M \times M \to \mathbb{R}$ are defined by

$$a((\mathbf{t},\mathbf{u}),(\mathbf{s},\mathbf{v})) = 2\,\mu \int_\Omega \mathbf{t} : \mathbf{s} \qquad c((\boldsymbol{\tau},q),(\boldsymbol{\varphi},r)) = \frac{1}{\lambda}\int_\Omega q\, r$$

$$b((\mathbf{s},\mathbf{v}),(\boldsymbol{\tau},q)) = -\int_\Omega \boldsymbol{\tau} : \mathbf{s} + \int_\Omega q\, \mathrm{tr}(\mathbf{s}) + \int_\Omega \boldsymbol{\tau} : \mathbf{e}(\mathbf{v})$$

In the following Theorem, we prove that problem (7) is well posed.

Theorem 2.1 *Problem* (7) *has a unique solution* $((\mathbf{t},\mathbf{u}),(\boldsymbol{\sigma},p)) \in X \times M$. *Moreover, there exists a constant* $C > 0$, *independent of* λ, *such that*

$$||((\mathbf{t},\mathbf{u}),(\boldsymbol{\sigma},p))||_{X\times M} \leq C\, ||\mathbf{f}||_{[L^2(\Omega)]^2}$$

Proof. Problem (7) is a saddle point problem that fits the abstract framework of [5] (see Section II.1.2). Therefore, in order to prove that this problem is well posed, with a continuous dependence constant independent of λ, we first remark that all the bilinear forms, $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$, are continuous. Now, let $B : X \to M'$ be the linear operator associated with the bilinear form $b(\cdot,\cdot)$. It is easy to see that

$$\mathrm{Ker}(B) = \left\{\, (\mathbf{s},\mathbf{v}) \in X \,:\, \mathbf{s} = \mathbf{e}(\mathbf{v}) \quad \text{in } \Omega; \quad \mathrm{div}(\mathbf{v}) = 0 \quad \text{in } \Omega \,\right\}$$

Then, by virtue of Korn's inequality, the bilinear form $a(\cdot,\cdot)$ is coercive on $\mathrm{Ker}(B)$. Indeed, given $(\mathbf{s},\mathbf{v}) \in \mathrm{Ker}(B)$, we have that

$$a((\mathbf{s},\mathbf{v}),(\mathbf{s},\mathbf{v})) = \mu\Big(||\mathbf{s}||^2_{[L^2(\Omega)]^{2\times 2}} + ||\mathbf{e}(\mathbf{v})||^2_{[L^2(\Omega)]^{2\times 2}}\Big) \geq \mu \min(1, c_K^2)||(\mathbf{s},\mathbf{v})||^2_X$$

On the other hand, it is clear that the bilinear form $c(\cdot,\cdot)$ is positive semi-definite and symmetric.

Let us prove that $b(\cdot,\cdot)$ satisfies an inf-sup condition in $X \times M$. Let $(\boldsymbol{\sigma},p) \in M$. Then, from Corollary 2.4 in [13], there exists a unique $\mathbf{w} \in [H^1_0(\Omega)]^2$ and a positive constant c_1

such that $\mathrm{div}(\mathbf{w}) = p$ in Ω and $||\mathbf{w}||_{[H^1(\Omega)]^2} \leq c_1 ||p||_{L^2(\Omega)}$. Taking $(\mathbf{s},\mathbf{v}) = (\mathbf{e}(\mathbf{w}),\mathbf{w}) \in X$, we have that

$$\sup_{\substack{(\mathbf{s},\mathbf{v})\in X\\(\mathbf{s},\mathbf{v})\neq 0}} \frac{b((\mathbf{s},\mathbf{v}),(\boldsymbol{\sigma},p))}{||(\mathbf{s},\mathbf{v})||_X} \geq \frac{b((\mathbf{e}(\mathbf{w}),\mathbf{w}),(\boldsymbol{\sigma},p))}{||(\mathbf{e}(\mathbf{w}),\mathbf{w})||_X} \geq \frac{||p||^2_{L^2(\Omega)}}{\sqrt{2}\,||\mathbf{w}||_{[H^1(\Omega)]^2}} \geq \frac{1}{c_1\sqrt{2}}||p||_{L^2(\Omega)} \tag{8}$$

On the other hand, taking $(\mathbf{s},\mathbf{v}) = (-\boldsymbol{\sigma},\mathbf{0}) \in X$ and applying the Cauchy-Schwarz inequality, we obtain that

$$\sup_{\substack{(\mathbf{s},\mathbf{v})\in X\\(\mathbf{s},\mathbf{v})\neq 0}} \frac{b((\mathbf{s},\mathbf{v}),(\boldsymbol{\sigma},p))}{||(\mathbf{s},\mathbf{v})||_X} \geq \frac{||\boldsymbol{\sigma}||^2_{[L^2(\Omega)]^{2\times 2}} - \displaystyle\int_\Omega p\,\mathrm{tr}(\boldsymbol{\sigma})}{||\boldsymbol{\sigma}||_{[L^2(\Omega)]^{2\times 2}}} \geq ||\boldsymbol{\sigma}||_{[L^2(\Omega)]^{2\times 2}} - \sqrt{2}\,||p||_{L^2(\Omega)} \tag{9}$$

Finally, we deduce from (8) and (9) that

$$\sup_{\substack{(\mathbf{s},\mathbf{v})\in X\\(\mathbf{s},\mathbf{v})\neq 0}} \frac{b((\mathbf{s},\mathbf{v}),(\boldsymbol{\sigma},p))}{||(\mathbf{s},\mathbf{v})||_X} \geq \beta\,||(\boldsymbol{\sigma},p)||_M$$

and therefore, the result follows. □

In the previous Theorem, we proved that problem (7) is well posed. However, since the bilinear form $a(\cdot,\cdot)$ is not coercive on the whole space X, it will be difficult to find finite-dimensional subspaces such that the corresponding Galerkin scheme is well posed. For this reason, we enrich the variational formulation (7) with the least-squares term

$$\alpha \int_\Omega (\mathbf{e}(\mathbf{u}) - \mathbf{t}) : (\mathbf{e}(\mathbf{v}) + \mathbf{s}) = 0 \qquad \forall\,(\mathbf{s},\mathbf{v}) \in X$$

where α is a positive parameter. This term seems superfluous at the continuous level, but it will play an important role in the discrete setting. Adding this term to the bilinear form $a(\cdot,\cdot)$, we obtain the following augmented variational formulation of problem (1): Find $((\mathbf{t},\mathbf{u}),(\boldsymbol{\sigma},p)) \in X \times M$ such that

$$\begin{array}{rclcll} \tilde{a}((\mathbf{t},\mathbf{u}),(\mathbf{s},\mathbf{v})) & + & b((\mathbf{s},\mathbf{v}),(\boldsymbol{\sigma},p)) & = & \displaystyle\int_\Omega \mathbf{f}\cdot\mathbf{v} & \forall\,(\mathbf{s},\mathbf{v}) \in X \\ b((\mathbf{t},\mathbf{u}),(\boldsymbol{\tau},q)) & - & c((\boldsymbol{\sigma},p),(\boldsymbol{\tau},q)) & = & 0 & \forall\,(\boldsymbol{\tau},q) \in M \end{array} \tag{10}$$

where the bilinear form $\tilde{a} : X \times X \to \mathbb{R}$ is given by

$$\tilde{a}((\mathbf{t},\mathbf{u}),(\mathbf{s},\mathbf{v})) = a((\mathbf{t},\mathbf{u}),(\mathbf{s},\mathbf{v})) + \alpha \int_\Omega (\mathbf{e}(\mathbf{u}) - \mathbf{t}) : (\mathbf{e}(\mathbf{v}) + \mathbf{s})$$

and $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are the bilinear forms defined above.

We remark that, by virtue of Korn's inequality, if $0 < \alpha < 2\,\mu$, then the bilinear form $\tilde{a}(\cdot,\cdot)$ is coercive on X. Indeed, given any $(\mathbf{s},\mathbf{v}) \in X$, we have that

$$\begin{array}{rcl} \tilde{a}((\mathbf{s},\mathbf{v}),(\mathbf{s},\mathbf{v})) & = & (2\mu - \alpha)\,||\mathbf{s}||^2_{[L^2(\Omega)]^{2\times 2}} + \alpha\,||\mathbf{e}(\mathbf{v})||^2_{[L^2(\Omega)]^{2\times 2}} \\ & \geq & \min(2\mu - \alpha, \alpha\, c_K^2)\,||(\mathbf{s},\mathbf{v})||^2_X \end{array} \tag{11}$$

Now, we are able to prove the following result.

Theorem 2.2 *For $\alpha \in (0, 2\mu)$, problem (10) has a unique solution $((\mathbf{t}, \mathbf{u}), (\boldsymbol{\sigma}, p)) \in X \times M$. Moreover, there exists a constant $C > 0$, independent of λ, such that*

$$||((\mathbf{t}, \mathbf{u}), (\boldsymbol{\sigma}, p))||_{X\times M} \leq C\, ||\mathbf{f}||_{[L^2(\Omega)]^2}$$

Proof. Clearly, the bilinear form $\tilde{a}(\cdot,\cdot)$ is continuous. Moreover, from (11), $\tilde{a}(\cdot,\cdot)$ is coercive on X and, in particular, on $\mathrm{Ker}(B)$. Therefore, the result follows from the proof of Theorem 2.1 and the abstract theory in [5]. □

3 The augmented primal-mixed finite element method

In this section, we consider the Galerkin scheme associated to (10) and provide sufficient conditions on the finite-dimensional subspaces that allow us to guarantee that the discrete problem is well posed. We also define explicit finite element subspaces satisfying these conditions and give the corresponding error estimates.

Let h be a positive parameter and let us consider finite-dimensional subspaces $X_h^{\mathbf{t}} \subset [L^2(\Omega)]^{2\times 2}_{\mathtt{sym}}$, $X_h^{\mathbf{u}} \subset [H_0^1(\Omega)]^2$, $M_h^{\boldsymbol{\sigma}} \subset [L^2(\Omega)]^{2\times 2}_{\mathtt{sym}}$ and $M_h^p \subset L_0^2(\Omega)$. We define $X_h := X_h^{\mathbf{t}} \times X_h^{\mathbf{u}}$ and $M_h := M_h^{\boldsymbol{\sigma}} \times M_h^p$. The Galerkin scheme associated with the augmented variational formulation (10) reads: Find $((\mathbf{t}_h, \mathbf{u}_h), (\boldsymbol{\sigma}_h, p_h)) \in X_h \times M_h$ such that

$$\begin{array}{lcllll} \tilde{a}((\mathbf{t}_h, \mathbf{u}_h), (\mathbf{s}_h, \mathbf{v}_h)) & + & b((\mathbf{s}_h, \mathbf{v}_h), (\boldsymbol{\sigma}_h, p_h)) & = & \displaystyle\int_\Omega \mathbf{f}\cdot\mathbf{v}_h & \forall\, (\mathbf{s}_h, \mathbf{v}_h) \in X_h \\ b((\mathbf{t}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, q_h)) & - & c((\boldsymbol{\sigma}_h, p_h), (\boldsymbol{\tau}_h, q_h)) & = & 0 & \forall\, (\boldsymbol{\tau}_h, q_h) \in M_h \end{array} \tag{12}$$

We can prove the following result.

Theorem 3.1 *Assume that $\alpha \in (0, 2\mu)$ and that the finite element subspaces $X_h^{\mathbf{t}}$, $X_h^{\mathbf{u}}$, $M_h^{\boldsymbol{\sigma}}$ and M_h^p are such that*

1. $\mathbf{e}(X_h^{\mathbf{u}}) \subset X_h^{\mathbf{t}}$
2. $(X_h^{\mathbf{u}}, M_h^p)$ *is a stable pair for the Stokes problem*
3. $M_h^{\boldsymbol{\sigma}} \subset X_h^{\mathbf{t}}$

Then the discrete problem (12) has a unique solution $((\mathbf{t}_h, \mathbf{u}_h), (\boldsymbol{\sigma}_h, p_h)) \in X_h \times M_h$ and there exists a positive constant C, independent of h and λ, such that

$$\begin{aligned} &||((\mathbf{t}, \mathbf{u}), (\boldsymbol{\sigma}, p)) - ((\mathbf{t}_h, \mathbf{u}_h), (\boldsymbol{\sigma}_h, p_h))||_{X\times M} \leq \\ &\qquad \leq C \inf_{((\mathbf{s}_h,\mathbf{v}_h),(\boldsymbol{\tau}_h,q_h))\in X_h\times M_h} ||((\mathbf{t}, \mathbf{u}), (\boldsymbol{\sigma}, p)) - ((\mathbf{s}_h, \mathbf{v}_h), (\boldsymbol{\tau}_h, q_h))||_{X\times M} \end{aligned} \tag{13}$$

Proof. The properties of the bilinear forms $\tilde{a}(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are satisfied in any subspaces of X and M, respectively. Therefore, it only remains to prove that the bilinear form $b(\cdot,\cdot)$ satisfies a discrete inf-sup condition in $X_h \times M_h$, that is, there exists a positive constant $\tilde{\beta}$, independent of h, such that

$$\sup_{\substack{(\mathbf{s}_h,\mathbf{v}_h)\in X_h \\ (\mathbf{s}_h,\mathbf{v}_h)\neq 0}} \frac{b((\mathbf{s}_h, \mathbf{v}_h), (\boldsymbol{\sigma}_h, p_h))}{||(\mathbf{s}_h, \mathbf{v}_h)||_X} \geq \tilde{\beta}\, ||(\boldsymbol{\sigma}_h, p_h)||_M \qquad \forall\, (\boldsymbol{\sigma}_h, p_h) \in M_h \tag{14}$$

Let $(\boldsymbol{\sigma}_h, p_h) \in M_h$. Since we assume that $\mathbf{e}(X_h^{\mathbf{u}}) \subset X_h^{\mathbf{t}}$, we can take $(\mathbf{s}_h, \mathbf{v}_h) = (\mathbf{e}(\mathbf{v}_h), \mathbf{v}_h) \in X_h$. Then, we have that

$$\sup_{\substack{(\mathbf{s}_h,\mathbf{v}_h)\in X_h\\ (\mathbf{s}_h,\mathbf{v}_h)\neq 0}} \frac{b((\mathbf{s}_h,\mathbf{v}_h),(\boldsymbol{\sigma}_h,p_h))}{||(\mathbf{s}_h,\mathbf{v}_h)||_X} \geq \sup_{\substack{\mathbf{v}_h\in X_h^{\mathbf{u}}\\ \mathbf{v}_h\neq 0}} \frac{b((\mathbf{e}(\mathbf{v}_h),\mathbf{v}_h),(\boldsymbol{\sigma}_h,p_h))}{||(\mathbf{e}(\mathbf{v}_h),\mathbf{v}_h)||_X} \geq \frac{1}{\sqrt{2}} \sup_{\substack{\mathbf{v}_h\in X_h^{\mathbf{u}}\\ \mathbf{v}_h\neq 0}} \frac{\displaystyle\int_\Omega p_h \mathrm{div}(\mathbf{v}_h)}{||\mathbf{v}_h||_{[H^1(\Omega)]^2}}$$

Now, using that $(X_h^{\mathbf{u}}, M_h^p)$ is a stable pair for the Stokes problem, there exists a positive constant γ, independent of h, such that

$$\sup_{\substack{\mathbf{w}_h\in X_h^{\mathbf{u}}\\ \mathbf{w}_h\neq 0}} \frac{\displaystyle\int_\Omega q_h \mathrm{div}(\mathbf{w}_h)}{||\mathbf{w}_h||_{[H^1(\Omega)]^2}} \geq \gamma ||q_h||_{L^2(\Omega)} \qquad \forall\, q_h \in M_h^p$$

Therefore, we deduce that

$$\sup_{\substack{(\mathbf{s}_h,\mathbf{v}_h)\in X_h\\ (\mathbf{s}_h,\mathbf{v}_h)\neq 0}} \frac{b((\mathbf{s}_h,\mathbf{v}_h),(\boldsymbol{\sigma}_h,p_h))}{||(\mathbf{s}_h,\mathbf{v}_h)||_X} \geq \frac{\gamma}{\sqrt{2}} ||p_h||_{L^2(\Omega)} \tag{15}$$

On the other hand, since $M_h^{\boldsymbol{\sigma}} \subset X_h^{\mathbf{t}}$, we can take $(\mathbf{s}_h, \mathbf{v}_h) = (-\boldsymbol{\sigma}_h, \mathbf{0}) \in X_h$. Then, applying the Cauchy-Schwarz inequality, we obtain that

$$\sup_{\substack{(\mathbf{s}_h,\mathbf{v}_h)\in X_h\\ (\mathbf{s}_h,\mathbf{v}_h)\neq 0}} \frac{b((\mathbf{s}_h,\mathbf{v}_h),(\boldsymbol{\sigma}_h,p_h))}{||(\mathbf{s}_h,\mathbf{v}_h)||_X} \geq ||\boldsymbol{\sigma}_h||_{[L^2(\Omega)]^{2\times 2}} - \sqrt{2}\, ||p_h||_{L^2(\Omega)} \tag{16}$$

Thus, inequality (14) follows from inequalities (15) and (16). $\square$

Next, we define explicit finite element subspaces satisfying the assumptions of Theorem 3.1. In what follows, we let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$ and assume that $\bar{\Omega} = \cup_{T\in\mathcal{T}_h} T$. We denote by h_T the diameter of a triangle $T \in \mathcal{T}_h$ and define the mesh size $h := \max_{T\in\mathcal{T}_h} h_T$. Given a triangle $T \in \mathcal{T}_h$, we denote by $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ its vertices, and by s_i the side opposite $\mathbf{a}_i$, for $i = 1, 2, 3$. We also let $\mathbf{n}_i$ be the unit outward normal to s_i. Finally, λ_1, λ_2 and λ_3 denote the barycentric coordinates of triangle T. In addition, given an integer $k \geq 0$, we denote by $\mathcal{P}_k(T)$ the space of polynomials in two variables defined in T of total degree at most k.

3.1 An approximation using discontinuous pressures

We first consider the pair $(X_h^{\mathbf{u}}, M_h^p)$ introduced by Bernardi and Raugel (see [4] or [13]) to solve the Stokes problem. Then, we define

$$X_h^{\mathbf{u}} = \{\mathbf{v} \in [H_0^1(\Omega) \cap \mathcal{C}^0(\bar{\Omega})]^2 \,:\, \mathbf{v}_{|_T} \in [\mathcal{P}_1(T)]^2 \oplus \langle \mathbf{n}_1\lambda_2\lambda_3, \mathbf{n}_2\lambda_3\lambda_1, \mathbf{n}_3\lambda_1\lambda_2\rangle, \quad \forall\, T \in \mathcal{T}_h\,\}$$

and

$$M_h^p = \{q \in L_0^2(\Omega) \,:\, q_{|_T} \in \mathcal{P}_0(T), \quad \forall\, T \in \mathcal{T}_h\}$$

In order to approximate the symmetric stress tensor field $\boldsymbol{\sigma}$, we define $M_h^{\boldsymbol{\sigma}}$ as the space of piecewise-constant symmetric tensors:

$$M_h^{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} \,:\, \boldsymbol{\tau}_{|T} \in S_0 \quad \forall T \in \mathcal{T}_h \right\}$$

where

$$S_0 := \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

Finally, to satisfy the assumptions of Theorem 3.1, we define the finite element subspace $X_h^{\mathbf{t}}$ as follows:

$$X_h^{\mathbf{t}} := \left\{ \mathbf{s} \in [L^2(\Omega)]^{2\times 2} \,:\, \mathbf{s}_{|T} \in S_0 \oplus \left\langle \mathbf{e}(\mathbf{n}_1\lambda_2\lambda_3), \mathbf{e}(\mathbf{n}_2\lambda_3\lambda_1), \mathbf{e}(\mathbf{n}_3\lambda_1\lambda_2) \right\rangle, \quad \forall T \in \mathcal{T}_h \right\}$$

Using the approximation properties of the subspaces involved (see [6, 13]), we deduce from (13) the following result.

Theorem 3.2 *Assume that* $\mathbf{u} \in [H_0^1(\Omega)]^2 \cap [H^2(\Omega)]^2$, $\boldsymbol{\sigma} \in [L^2(\Omega)]^{2\times 2}_{\mathtt{sym}} \cap [H^1(\Omega)]^{2\times 2}$, $\mathbf{t} \in [L^2(\Omega)]^{2\times 2}_{\mathtt{sym}} \cap [H^1(\Omega)]^{2\times 2}$ *and that* $p \in L_0^2(\Omega) \cap H^1(\Omega)$. *Then there exists a constant* $C > 0$, *independent of* h *and* λ, *such that*

$$||((\mathbf{t},\mathbf{u}),(\boldsymbol{\sigma},p)) - ((\mathbf{t}_h,\mathbf{u}_h),(\boldsymbol{\sigma}_h,p_h))||_{X\times M} \leq$$

$$\leq C\, h \left(||\mathbf{u}||_{[H^2(\Omega)]^2} + ||\boldsymbol{\sigma}||_{[H^1(\Omega)]^{2\times 2}} + ||\mathbf{t}||_{[H^1(\Omega)]^{2\times 2}} + ||p||_{H^1(\Omega)} \right)$$

3.2 An approximation using continuous pressures

Now we consider the so-called *mini-element*, introduced by Arnold et al. (see [2, 13]). In this case,

$$X_h^{\mathbf{u}} = \{ \mathbf{v} \in [H_0^1(\Omega) \cap \mathcal{C}^0(\bar{\Omega})]^2 \,:\, \mathbf{v}_{|T} \in [\mathcal{P}_1(T) \oplus \langle \lambda_1\lambda_2\lambda_3 \rangle]^2, \quad \forall T \in \mathcal{T}_h \}$$

and

$$M_h^p = \{ q \in L_0^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega}) \,:\, q_{|T} \in \mathcal{P}_1(T), \quad \forall T \in \mathcal{T}_h \}$$

To approximate the symmetric stress tensor field $\boldsymbol{\sigma}$, we consider the finite element space $M_h^{\boldsymbol{\sigma}}$ defined in the previous subsection. Then, in order to satisfy the assumptions of Theorem 3.1, we define the finite element subspace $X_h^{\mathbf{t}}$ as follows:

$$X_h^{\mathbf{t}} := \left\{ \mathbf{s} \in [L^2(\Omega)]^{2\times 2} \,:\, \mathbf{s}_{|T} \in S_0 \oplus \left\langle \mathbf{e}\begin{pmatrix} \lambda_1\lambda_2\lambda_3 \\ 0 \end{pmatrix}, \mathbf{e}\begin{pmatrix} 0 \\ \lambda_1\lambda_2\lambda_3 \end{pmatrix} \right\rangle, \quad \forall T \in \mathcal{T}_h \right\}$$

Using the approximation properties of the subspaces involved (see [6, 13]), we deduce from (13) the following result, analogous to Theorem 3.2.

Theorem 3.3 *Assume that* $\mathbf{u} \in [H_0^1(\Omega)]^2 \cap [H^2(\Omega)]^2$, $\boldsymbol{\sigma} \in [L^2(\Omega)]^{2\times 2}_{\mathtt{sym}} \cap [H^1(\Omega)]^{2\times 2}$, $\mathbf{t} \in [L^2(\Omega)]^{2\times 2}_{\mathtt{sym}} \cap [H^1(\Omega)]^{2\times 2}$ *and that* $p \in L_0^2(\Omega) \cap H^1(\Omega)$. *Then there exists a constant* $C > 0$, *independent of* h *and* λ, *such that*

$$||((\mathbf{t},\mathbf{u}),(\boldsymbol{\sigma},p)) - ((\mathbf{t}_h,\mathbf{u}_h),(\boldsymbol{\sigma}_h,p_h))||_{X\times M} \leq$$

$$\leq C\, h \left(||\mathbf{u}||_{[H^2(\Omega)]^2} + ||\boldsymbol{\sigma}||_{[H^1(\Omega)]^{2\times 2}} + ||\mathbf{t}||_{[H^1(\Omega)]^{2\times 2}} + ||p||_{H^1(\Omega)} \right)$$

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References

[1] D.N. Arnold, F. Brezzi and J. Douglas, *PEERS: A new mixed finite element for plane elasticity*, Japan J. Appl. Math., 1, 347-367, (1984).

[2] D.N. Arnold, F. Brezzi and M. Fortin, *A stable finite element for Stokes equations*, Calcolo, vol. XXI, 337-344, (1984).

[3] C. Bernardi, C. Canuto and Y. Maday, *Generalized inf-sup conditions for Chebyshev spectral approximation of the Stokes problem*, SIAM J. Numer. Anal. vol 25, 6, pp. 1237-1271, (1988).

[4] C. Bernardi and G. Raugel, *Analysis of some finite elements for the Stokes problem*, Math. Comp. 44, (1985), pp. 71-79.

[5] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, 1991.

[6] P.G. Ciarlet, *The Finite Element Methods for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[7] P. Ciarlet, J. Huang and J. Zou, *Some observations on generalized saddle-point problems*, SIAM J. Matrix Anal. Appl. vol. 25, 1, pp. 224-236, (2003).

[8] J.K. Djoko and B.D. Reddy, *An extended Hu-Washizu formulation for elasticity.* Comput. Methods Apply. Mech. Engrg. vol. 195, 39&40, pp. 6330-6346, (2006).

[9] M. Fortin, R. Guenette and R. Pierre, *Numerical analysis of the modified EVSS method*, Comput. Methods Appl. Mech. Engrg. 143 (1997) 79-95.

[10] A. Fortin, R. Guenette and R. Pierre, *On the discrete EVSS method*, Comput. Methods Appl. Mech. Engrg. 189 (2000) 121-139.

[11] G.N. Gatica, *Analysis of a new augmented mixed finite element method for linear elasticity allowing RT0-P1-P0 approximations*, ESAIM: M2AN vol. 40, 1, pp. 1-28 (2006).

[12] T.P. Barrios, G.N. Gatica, M. González and N. Heuer, *A residual based a posteriori error estimator for an augmented mixed finite element method in linear elasticity*, ESAIM: M2AN vol. 40, 5, pp. 843-869 (2006).

[13] V. Girault and P.A. Raviart, *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*, Springer Series in Computational Mathematics 5, Springer-Verlag, Berlin, 1986.

[14] R.B. Kellogg and B. Liu, *A finite element method for the compressible Stokes equations*, SIAM J. Numer. Anal. vol. 33, 2, pp. 780-788, (1996).

[15] R.A. Nicolaides, *Existence uniqueness and approximation for generalized saddle point problems*, SIAM J. Numer. Anal. vol 19, 2, pp. 349-357, (1982).