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Symmetry reductions for thin film type equations

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Resumen

The lubrication equation $u_t = (u^n u_{xxx})_x$ plays an important role in the study of the interface movements. In this work we analyze the generalizations of the above equation given by $u_t = (u^n u_{xxx})_x - ku^m u_x$. By using Lie classical method the corresponding reductions are performed and some solutions are characterized.

1. Introduction

The evolution equation

$$u_t = \bigtriangledown . (u^n \bigtriangledown .(\triangle u)) = 0,$$

where u stands for the thickness of the film and n depends on the geometry of the problem, arises by analysing the evolution of a thin film of a viscous liquid, dominate by surface tension effects [10]. The one-dimensional version is

$$u_t = (u^n u_{xxx})_x, \tag{1}$$

and the modified version

$$u_t = u^n u_{xxxx} \tag{2}$$

are considered in [1] where the formation of singularities are studied. Besides, analytical approximation to the solutions are obtained in [1] and [14] for the case $0 < n << 1$ by applying perturbation theory.

In previous papers [8],[7] we have classified the classical symmetries admitted by the generalized equations

$$u_t = (f(u) u_{xxx})_x, \tag{3}$$

and

$$u_t = f(u) u_{xxxx}, \tag{4}$$

and by using symmetry reductions we found that for some particular functional forms of f the one-dimensional lubrication model admits some solutions of physical interest as similarity solutions: travelling-wave solutions, source and sink solutions, waiting time solutions and blow-up solutions. We were also able to characterize those solutions as solutions for some lower-order ODEs and moreover we obtained some particular solutions.

In some physical situations one lubrication approximation reduces the evolution equation to a long unstable wave diffusion equation.

$$u_t = (u^n u_{xxx})_x - (ku^m u_x)_x, \tag{5}$$

[2, 11].

In this work we study (5) from the point of view of the theory of symmetry reductions in partial differential equations. We obtain the classical symmetries admitted by (5), then, we use the transformations groups to reduce the equations to ordinary differential equations. Physical interpretation of these reductions and some elementary solutions are also provided.

2. Classical Lie symmetries

One of the mathematical models for diffusion processes is the fourth-order nonlinear diffusion equation (5) To apply the classical method to (5), we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$\begin{aligned}
x^* &= x + \varepsilon\xi(x,t,u) + \mathcal{O}(\varepsilon^2),\\
t^* &= t + \varepsilon\tau(x,t,u) + \mathcal{O}(\varepsilon^2),\\
u^* &= u + \varepsilon\phi(x,t,u) + \mathcal{O}(\varepsilon^2),
\end{aligned} \tag{6}$$

where ε is the group parameter.

The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi\partial_{\mathbf{x}} + \tau\partial_{\mathbf{t}} + \phi\partial_{\mathbf{u}}. \tag{7}$$

One then requires that this transformation leaves invariant the set of solutions of the equation (5). This yields an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface condition

$$\Phi \equiv \xi\frac{\partial u}{\partial x} + \tau\frac{\partial u}{\partial t} - \phi = 0. \tag{8}$$

Applying the classical method to the equation (5), with $k \neq 0$ leads to a three-parameter Lie group. Associated with this Lie group we have a Lie algebra which can be represented by the following generators:

$$\mathbf{v_1} = \partial_{\mathbf{x}}, \quad \mathbf{v_2} = \partial_{\mathbf{t}}, \quad \mathbf{v_3} = \frac{\mathbf{n}-\mathbf{m}}{\mathbf{2}}\mathbf{x}\partial_{\mathbf{x}} + (\mathbf{n}-\mathbf{2m})\mathbf{t}\partial_{\mathbf{t}} + \mathbf{u}\partial_{\mathbf{u}}$$

2.1. Optimal system

In order to construct the optimal system, following Olver, we first construct the commutator table (Table 1) and the adjoint table (Table 2) which shows the separate adjoint actions of each element in $\mathbf{v_i}$, $i = 1\ldots3$, as it acts on all other elements. This construction is done easily by summing the Lie series.

The corresponding generators of the optimal system of subalgebras are

$$\begin{aligned}&\mathbf{v_1} + \mu\mathbf{v_2},\\ &\mathbf{v_3},\\ &\lambda\mathbf{v_2} + \mathbf{v_3},\\ &\lambda\mathbf{v_1} + \mathbf{v_3},\end{aligned} \tag{9}$$

where λ and μ are arbitrary real constants.

Each generator of this optimal system (9) defines a reduction, which we now enumerate.

Tabla 1: Commutator table for the Lie algebra $\mathbf{v_i}$.

	$\mathbf{v_1}$	$\mathbf{v_2}$	$\mathbf{v_3}$
$\mathbf{v_1}$	0	0	$\frac{n-m}{2}\mathbf{v_1}$
$\mathbf{v_2}$	0	0	$(n-2m)\mathbf{v_2}$
$\mathbf{v_3}$	$-\frac{n-m}{2}\mathbf{v_1}$	$-(n-2m)\mathbf{v_2}$	0

Tabla 2: Adjoint table for the Lie algebra $\mathbf{v_i}$

Ad	$\mathbf{v_1}$	$\mathbf{v_2}$	$\mathbf{v_3}$
$\mathbf{v_1}$	$\mathbf{v_1}$	$\mathbf{v_2}$	$\mathbf{v_3} - \frac{\mathbf{n-m}}{\mathbf{2}}\epsilon\mathbf{v_1}$
$\mathbf{v_2}$	$\mathbf{v_1}$	$\mathbf{v_2}$	$\mathbf{v_3} - (\mathbf{n-2m})\epsilon\mathbf{v_2}$
$\mathbf{v_3}$	$e^{\frac{(n-m)\epsilon}{2}}\mathbf{v_1}$	$e^{(n-2m)\epsilon}\mathbf{v_2}$	0

3. Classical reductions

In the following, reductions of the equation (5) to ODE,s are obtained using the generators of the optimal system.

3.1. Reduction with the generator $\mathbf{v_1} + \mu\mathbf{v_2}$

$$z = x - \mu t, \quad u = \omega, \tag{10}$$

and the ODE

$$\mu\omega + \omega^n\omega''' - k\omega^m\omega' - c_1 = 0 \tag{11}$$

3.2. Reduction with the generator $\mathbf{v_3}$

$$z = t^{\frac{m-n}{2(n-2m)}}x, \quad u = t^{\frac{1}{n-2m}}\omega, \tag{12}$$

and the ODE

$$(n-2m)(\omega^n\omega''' - k\omega^m\omega') - z\omega - c_1 = 0 \tag{13}$$

3.3. Reduction with the generator $\lambda\mathbf{v_2} + \mathbf{v_3}$, $n = 2m$

$$z = e^{-\frac{mt}{2\lambda}}x, \quad u = e^{\frac{t}{\lambda}}\omega, \tag{14}$$

and the ODE

$$\lambda(\omega^{2m}\omega''' - k\omega^m\omega') - z\omega - c_1 = 0 \tag{15}$$

3.4. Reduction with the generator $\mathbf{v_3}$, $n = m$

$$z = x, \quad u = x^2\omega, \tag{16}$$

and the ODE

$$\omega' = \frac{4}{m}(1-\frac{2}{m})(1-\frac{1}{m})(1+\frac{2}{m})\omega^{2m+1} - k\frac{2}{m}(1+\frac{2}{m})\omega^{m+1} \tag{17}$$

3.5. Reduction with the generator $\lambda\mathbf{v_1} + \mathbf{v_3}$, $n = m$

$$z = x + \frac{\lambda}{m} ln|t|, \quad u = t^{-\frac{1}{m}\omega}, \tag{18}$$

and the ODE

$$m(\omega^m(\omega''' - k\omega') + z\omega - \lambda\omega - c_1 = 0 \tag{19}$$

4. Analysis of the reduced equations

Finally, we discuss some interpretation of the similarity variables in the above reductions and provide some particular solutions. We remark the following facts:

* Solutions of (10) satisfying the first reduction are travelling wave solutions for any arbitrary constants n and m.

* If we take in the second reduction (12) $n = m-2$, we have that the similarity solution has the form

$$u(x,t) = \frac{1}{t^{\frac{1}{2+m}}}\omega(\frac{x}{t^{\frac{1}{2+m}}}),$$

thus, if $m > -2$, $n > -4$ it is clear that $u(x,t) \to \delta(x)$ as $t \to 0$ and the similarity solution is a source solution.

* An analogue situation is found in the third reduction for the particular case $m = -2$. In fact, we have that

$$\begin{array}{lllll} \text{If } \lambda < 0 & u(x,t) \to \delta(x) & \text{as} & t \to -\infty \\ \text{If } \lambda > 0 & u(x,t) \to \delta(x), & \text{as} & t \to +\infty \end{array}$$

so, the similarity solution is in the first case a source solution while in the second case is a sink solution.

* For the fourth reduction the ODE is a first order equation that can be implicitly solved, for $k = 0$ we obtain a family of waiting-time solutions (if $n \neq 2$ or 4) given by

$$u(x,t) = \left\{ \begin{array}{ll} x^{\frac{4}{a}} \left[4(\frac{4}{a}+1)(\frac{4}{a}-1)(\frac{4}{a}-2)(t_0-t)\right]^{-\frac{1}{a}} & x \geq 0 \\ 0 & x < 0 \end{array} \right. .$$

* For $n = m$, solutions satisfying the fifth reduction are decaying travelling waves with decaying velocities.

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