XX Congreso de Ecuaciones Diferenciales y Aplicaciones
X Congreso de Matemática Aplicada
Sevilla, 24-28 septiembre 2007
(pp. 1–7)

Null controllability results for parabolic equations in unbounded domains

Manuel González-Burgos[1], Luz de Teresa[2]

[1] *Dpto. E.D.A.N., Universidad de Sevilla, Aptdo. 1160, E-41080 Sevilla. E-mails:* manoloburgos@us.es.
[2] *Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., 04510 D.F., México. E-mail:* deteresa@matem.unam.mx.

Palabras clave: Controllability, parabolic PDE, unbounded domains.

Resumen

In this talk we present some results concerning the null controllability for a heat equation in unbounded domains. We characterize the conditions that must satisfy the weight function, introduced by Fursikov and Imanuvilov, in order to prove a global Carleman inequality for the adjoint problem and then to get a null controllability result. We give some examples of unbounded domains (Ω, ω) that satisfy these sufficient conditions. Finally, when $\Omega \setminus \overline{\omega}$ is bounded, we prove the null controllability of the semi-linear heat equation when the nonlinearity is slightly superlinear.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an unbounded connected open set with boundary $\partial\Omega$ of class $C^{0,1}$ uniformly. Let $\omega \subset \Omega$ be a nonempty open subset and assume $T > 0$. We will consider the parabolic linear problem

$$\begin{cases} y_t - \Delta y + B \cdot \nabla y + ay = v1_\omega & \text{in } Q = \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0,T), \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $B \in L^\infty(Q)^N$, $a \in L^\infty(Q)$, $y^0 \in L^2(\Omega)$, and $v \in L^2(Q)$. In (1), 1_ω denotes the characteristic function of the subset ω, $y = y(x,t)$ is the state and $v = v(x,t)$ is the control function.

We will also consider the nonlinear problem

$$\begin{cases} y_t - \Delta y + f(y, \nabla y) = v 1_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{2}$$

where $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a locally Lipschitz-continuous function, y^0 is given in an appropriate Banach space and v is a control which also acts on the system through the open subset ω.

The main goal of this paper is to analyze the null controllability properties of (1) and (2). We recall that (1) (resp., (2)) is null controllable at time T if for every $y_0 \in L^2(\Omega)$ (resp., $y_0 \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega)$) there exists a control $v \in L^2(Q)$ such that the solution y of (1) satisfies

$$y(\cdot, T) = 0 \quad \text{in} \quad \Omega, \tag{3}$$

(resp., (2) possesses a solution $y \in C^0([0,T]; L^2(\Omega))$ satisfying (3)).

There are few results on the null controllability of (1) and (2) when Ω is an unbounded set. In fact, the first result was negative: in [10] the authors considered the case $\Omega = (0, \infty) \subset \mathbb{R}$, $a \equiv 0$ and $B \equiv 0$, and proved that there are no initial data in any negative Sobolev space that may be driven to zero at a finite time by a control function acting on the boundary $\{x = 0\}$ (see also [11]). The first positive result on null controllability of (1) and (2) was obtained in [2]. There, the authors considered the semilinear heat equation (2) with a globally-Lipschitz continuous function f and proved the null controllability of the problem when the distributed control acts on a set ω satisfying that $\Omega \setminus \overline{\omega}$ is bounded. In [3] the authors studied the null controllability of system (1) when $a \equiv 0$, $B \equiv 0$, $\Omega = (0, \infty)$ and ω is an unbounded open set of the form $\omega = \bigcup_{n \geq 0} \omega_n$. Under some technical assumptions on ω_n, they proved two null controllability results: one result with control functions in $L^2(Q)$ for initial data in some weighted $L^2(\Omega)$-spaces and another one for arbitrary initial data in $L^2(\Omega)$ by using control functions in a weighted $L^2(Q)$-space. Finally, in [12] the author also considered the heat equation ($a \equiv 0$ and $B \equiv 0$) with a Dirichlet boundary condition in an unbounded domain Ω and proved several results about the null controllability of this system with distributed and boundary controls. To be precise, the author shows the lack of null controllability of the heat equation under a geometric condition on Ω and ω and also proves a positive null controllability result for unbounded product domains.

Our first result in this talk is related to the results in the previous papers but our approach is different from the approaches of the previous authors. In fact we give sufficient conditions on the weight function (and in this sense on Ω and ω) that allow to construct a global Carleman inequality for the so-called adjoint problem. As a consequence, we will be able to recover the results of [2] and prove the results of [12] about the null controllability in $L^2(\Omega)$ of system (1) when $a \in L^\infty(Q)$ and $B \in L^\infty(Q)^N$. Our second main result concerns the controllability properties of system (2) when $\Omega \setminus \overline{\omega}$ is bounded and the function f has a superlinear growth at infinity. In this case the techniques used in [4] do not apply since they use the fact that $L^q(\Omega) \subset L^p(\Omega)$ for $1 \leq p < q \leq \infty$ when Ω is of finite measure.

2. A global Carleman inequality

All along this paper we will assume that $\Omega \subset \mathbb{R}^N$ is an unbounded connected open set with boundary at least of class $C^{0,1}$ uniformly, such that

$$D_\Omega(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega). \tag{4}$$

This "non classical" assumption is needed to give examples in unbounded set that are the cartesian product of regular sets. Observe that even in the simple case in which Ω is the cartesian product of two (bounded or unbounded) intervals, Ω is only a $C^{0,1}$ set, however it is a "good one" in the sense that (4) holds.

We also assume that $\omega \subset \Omega$ is a nonempty open subset such that there exist an open set ω_0 and a positive function $\eta^0 \in C^2(\mathbb{R}^N)$ satisfying

$$\begin{cases} \omega_0 \subset \omega \subset \Omega \quad \text{with} \quad d_0 = \operatorname{dist}(\omega_0, \Omega \setminus \overline{\omega}) > 0, \\ |\nabla \eta^0| \geq \mathcal{C}_0 > 0 \text{ in } \overline{\Omega} \setminus \omega_0, \quad \dfrac{\partial \eta^0}{\partial n} \leq 0 \text{ on } \partial\Omega, \\ |\eta^0| + |\nabla \eta^0| + \displaystyle\sum_{i,j} \left| \frac{\partial^2 \eta^0}{\partial x_i \partial x_j} \right| \leq \mathcal{C}_1 \text{ in } \Omega, \end{cases} \tag{5}$$

for two positive constants $\mathcal{C}_0$ and $\mathcal{C}_1$. In the next section we will give some examples of unbounded domains and control regions that satisfy the previous hypotheses.

As said above, we first prove a global Carleman inequality for the linear problem

$$\begin{cases} -\varphi_t - \Delta\varphi = F_0 + \displaystyle\sum_{i=1}^N \frac{\partial F_i}{\partial x_i} & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(x,T) = \varphi_0(x) & \text{in } \Omega, \end{cases} \tag{6}$$

where F_0, $F_i \in L^2(Q)$ $(1 \leq i \leq N)$ and $\varphi_0 \in L^2(\Omega)$. To this end, let us introduce for $(x,t) \in Q$,

$$\alpha(x,t) = \frac{e^{2\lambda m ||\eta^0||_\infty} - e^{\lambda\left(m||\eta^0||_\infty + \eta^0(x)\right)}}{t(T-t)}, \quad \xi(x,t) = \frac{e^{\lambda\left(m||\eta^0||_\infty + \eta^0(x)\right)}}{t(T-t)}, \tag{7}$$

with s and λ positive real numbers and $m > 1$ a fixed real number. One has:

Theorem 2.1. *Under the previous assumptions, there exist three positive constants σ_1, λ_1 and C_1 only depending on $\mathcal{C}_0$, $\mathcal{C}_1$ and d_0 such that, for every $\varphi_0 \in L^2(\Omega)$ and $F_0, F_i \in L^2(Q)$ $(1 \leq i \leq N)$, the corresponding solution φ to (6) satisfies*

$$\begin{cases} s\lambda^2 \displaystyle\iint_Q e^{-2s\alpha}\xi|\nabla\varphi|^2 + s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\varphi|^2 \leq C_1 \left(s^3\lambda^4 \iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^3|\varphi|^2 \right. \\ \qquad \left. + \displaystyle\iint_Q e^{-2s\alpha}|F_0|^2 + s\lambda^2 \sum_{i=1}^N \iint_Q e^{-2s\alpha}\xi^2|F_i|^2 \right), \end{cases} \tag{8}$$

for all $s \geq s_1 = \sigma_1(T + T^2)$ and $\lambda \geq \lambda_1$.

The proof of this inequality follows the method introduced by Imanuvilov and Yamamoto in [9] (see also [6] and [5]) and will be given in [8].

3. Null controllability of the linear problem. Examples

In this Section, we will assume that $y_0 \in L^2(\Omega)$ and that (5) holds. One has,

Theorem 3.1. *There exists a control $\widehat{v} \in L^2(Q)$ such that the corresponding solution $\widehat{y}$ to system* (1) *satisfies* (3) *and*

$$||\widehat{v}||^2_{L^2(Q)} \le \exp(M(T))||y_0||^2_{L^2(\Omega)} . \tag{9}$$

with $M(T) = \left(C \left(1 + T + \frac{1}{T} + T||a||_\infty + ||a||_\infty^{2/3} + (1+T)||B||^2_\infty \right) \right)$.

In addition, let us assume that either $\partial\omega \cap \partial\Omega$ is the empty set or $\partial\omega \cap \partial\Omega$ is of class C^2 uniformly. Then, given $y^0 \in L^2(\Omega)$ there exists a new control $v \in L^2(Q) \cap L^\infty(Q)$ satisfying

$$||v||^2_{L^2(Q)} + ||v||^2_\infty \le \exp(M(T))||y_0||^2_{L^2(\Omega)} \tag{10}$$

which also gives the null controllability of system (1). *In* (9)*, C is a positive constant only depending on $\mathcal{C}_0$, $\mathcal{C}_1$ and d_0; in* (10)*, C also depends on Ω and ω.*

Sketch of the proof. The proof of this theorem is done in two steps:
First step. Since ω satisfies the assumptions in Theorem 2.1, there exists another subset ω_1 such that $\omega_0 \subset \omega_1 \subset \omega$, $\operatorname{dist}(\omega_0, \Omega \setminus \overline{\omega}_1) = d_0/2$, and $\operatorname{dist}(\omega_1, \Omega \setminus \overline{\omega}) > 0$. By using the Carleman inequality (8) (for the set ω_1) it is by now classical (e.g. [6]) to obtain an observability inequality for the adjoint equation of (1) that allows the construction of a control $\widehat{v} \in L^2(Q)$ satisfying (9), with $\operatorname{supp} \widehat{v} \subset \overline{\omega}_1 \times [0,T]$ and such that the corresponding solution $\widehat{y}$ to (1) satisfies $\widehat{y}(\cdot, T) = 0$ in Ω.
Second step. We construct a new null control $v \in L^\infty(Q) \cap L^2(Q)$ for system (1). This can be carried out by adapting the technique introduced in [1]. We proceed as follows. Let $\eta \in C^\infty([0,T])$ be such that $\eta \equiv 1$ in $[0, T/3]$, $\eta \equiv 0$ in $[2T/3, T]$, and $0 \le \eta \le 1$, $|\eta'(t)| \le C/T$ in $[0,T]$. Due to the properties of ω_1, it is possible to construct a function $\theta \in C^\infty(\mathbb{R}^N)$ such that

$$\begin{cases} \operatorname{supp} \theta \cap (\overline{\Omega} \setminus \omega) = \emptyset, \quad \theta(x) = 1 \quad \forall x \in \omega_1 , \\ 0 \le \theta(x) \le 1, \quad |D^\alpha \theta(x)| \le C(\alpha) d_0^{-|\alpha|} \quad \forall x \in \mathbb{R}^N, \ \forall \alpha \in \mathbb{N}^N. \end{cases}$$

Let $Y \in L^2(0,T; H^1_0(\Omega))$ be the weak solution of system (1) corresponding to $v \equiv 0$. Let us consider

$$y(x,t) = [1 - \theta(x)]\widehat{y}(x,t) + \theta(x)\eta(t)Y(x,t) \quad \forall (x,t) \in Q. \tag{11}$$

It is not difficult to see that y is the solution of (1) corresponding to the control

$$v = 2\nabla\theta \cdot \nabla[\widehat{y} - \eta(t)Y] + \Delta\theta\, [\widehat{y} - \eta(t)Y] - (B \cdot \nabla\theta)[\widehat{y} - \eta(t)Y] + \theta\eta'(t)Y .$$

This solution satisfies (3). In addition, by construction of θ, $\operatorname{supp} v \subset \overline{\omega} \times [0,T]$ and the local parabolic regularity allows to see that $v \in L^\infty(Q)$ (see [8]). ■

Now, it is interesting to exhibit some examples of unbounded open domains and control regions where the assumptions above hold (for the details, see [8]:

1. When $\Omega \subset \mathbb{R}^N$ is an unbounded connected open set with boundary $\partial\Omega$ of class C^2 uniformly and $\omega \subset \Omega$ satisfies that $\Omega \setminus \overline{\omega}$ is bounded.

2. When $\Omega = (0, \infty) \subset \mathbb{R}$ (resp., $\Omega = \mathbb{R}$) and one considers

$$\omega = \bigcup_{n \geq 1} (a_n, b_n), \quad (\text{resp., } \omega = \bigcup_{n \geq 1} ((a_n, b_n) \cup (a_{-n}, b_{-n}))),$$

where, for every $n \geq 1$, $0 < a_n < b_n < a_{n+1}$, $\lim a_n = \lim b_n = \infty$ (resp., $0 > a_{-n} > b_{-n} > a_{-n-1}$, $\lim a_{-n} = \lim b_{-n} = -\infty$) and

$$\begin{cases} a_{n+1} - b_n \leq M < \infty \\ b_n - a_n \geq m > 0 \end{cases} \quad (\text{resp., } \begin{cases} b_{-n} - a_{-n-1} \leq M < \infty \\ a_{-n} - b_{-n} \geq m > 0 \end{cases}).$$

3. The example we are going to present is, as far as we know, authentically new. We give an example of unbounded sets Ω and ω such that $\Omega \setminus \overline{\omega}$ is an unbounded set with infinite measure and where the observability inequality holds. In fact, let Ω be an unbounded open set uniformly of class C^2. The set ω is going to be constructed from Ω by eliminating an "infinite strip" around the boundary. That is,

$$\omega = \{x \in \Omega : \text{dist}\,(x, \partial\Omega) > \delta_0\}$$

for some $\delta_0 > 0$ that is going to be fixed later. For the choice of δ_0 we will need to recall some geometric results and properties which are inherent to uniformly C^2 unbounded sets. To this aim, we will use results appearing in the paper of Fornaro et al. [7].

We recall that a set Ω satisfies a *uniform interior sphere condition* if for each point $y_0 \in \partial\Omega$ there exists a ball B_{y_0}, depending on y_0, contained in Ω and such that $\overline{B}_{y_0} \cap \partial\Omega = \{y_0\}$, moreover the radii of these balls are bounded from below by a positive constant. The following results are extracted from [7]:

Proposition 3.2. *If $\partial\Omega$ is uniformly of class C^2, then it satisfies a uniform interior sphere condition.* ∎

Proposition 3.3. *Assume that $\partial\Omega$ is uniformly of class C^2 and let $\widehat{\delta}$ be a positive constant such that at each point of $\partial\Omega$ there exists a ball which satisfies the interior sphere condition at y_0 with radius greater or equal to $\widehat{\delta}$. Then:*

a) *For every $x \in \Omega_{\widehat{\delta}} = \{y \in \overline{\Omega} : dist\,(y, \partial\Omega) < \widehat{\delta}\}$ there exists a unique $\xi = \xi(x) \in \partial\Omega$ such that $|x - \xi| = dist\,(x, \partial\Omega)$.*

b) *Let us define the function $d(x) = dist\,(x, \partial\Omega)$, then $d \in C_b^2(\Omega_{\widehat{\delta}})$.*

c) *$\nabla d(x) = -n(\xi(x))$ for every $x \in \Omega_{\widehat{\delta}}$ and where $n(\xi)$ denotes the unit outward normal vector to $\partial\Omega$ at point ξ.* ∎

With the previous results in mind, we are now able to define δ_0 and prove that the sets Ω and ω satisfy (5) (obviously, (4) holds). In fact, let us take $\delta_0 = \widehat{\delta} - \varepsilon$, where $\widehat{\delta} > 0$ is as in Proposition 3.3 and $\varepsilon \in (0, \widehat{\delta})$, and

$$\omega_0 = \{x \in \Omega : \text{dist}\,(x, \Omega) > \widehat{\delta} - \varepsilon/2\}.$$

It is easy to see that the first statement of (5) holds with $d_0 \equiv \varepsilon/2$. On the other hand, we define

$$\tilde{\eta}_0(x) = \begin{cases} d(x) \equiv \operatorname{dist}(x, \partial\Omega) & \text{if } x \in \overline{\Omega} \setminus \omega_0\,, \\ \widehat{\delta} - \varepsilon/2 & \text{if } x \in \omega_0\,. \end{cases}$$

Let us remark that $\overline{\Omega} \setminus \omega_0 \subset \Omega_{\widehat{\delta}}$ and $\operatorname{dist}(\overline{\Omega} \setminus \omega_0, \Omega \setminus \Omega_{\widehat{\delta}}) = \varepsilon/2 > 0$. We can then consider $\rho \in C^2(\overline{\Omega})$ with $0 \le \rho(x) \le 1$ in $\overline{\Omega}$, $\rho(x) \equiv 1$ in $\overline{\Omega} \setminus \omega_0$, $\operatorname{supp} \rho \subset \Omega_{\widehat{\delta}}$ and $|D^\alpha \rho(x)| \le C/\varepsilon^{|\alpha|}$ for every $x \in \Omega$ and $\alpha \in \mathbb{N}^N$ with $|\alpha| \le 2$. Finally, if we define $\eta^0(x) = \tilde{\eta}_0(x)\rho(x)$ and taking into account Propostion 3.3 and the previous properties we readily obtain that η^0 also satisfies the two last statement of condition (5).

4. Let us suppose that $\Omega \subset \mathbb{R}^N$ is an open set which satisfies (4), ω is an open subset of Ω and (Ω, ω) satisfies (5) with constants $\mathcal{C}_0$, $\mathcal{C}_1$ and d_0. Then, if $\mathcal{O} \subset \mathbb{R}^M$ $(M \ge 1)$ is another open set with boundary of class $C^{0,1}$ uniformly, and such that the open set $\widetilde{\Omega} = \Omega \times \mathcal{O} \subset \mathbb{R}^{N+M}$ satisfies (4), then $\widetilde{\Omega}$ with $\widetilde{\omega} = \omega \times \mathcal{O}$ also satisfy assumption (5) with the same constants (and therefore, with constants which are independent of $\mathcal{O}$).

5. The previous example can be readily generalized as follows: Let us assume that $\Omega_1 \subset \mathbb{R}^N$, $\Omega_2 \subset \mathbb{R}^M$ $(N, M \ge 1)$ are two open sets satisfying condition (4) and such that for $\Omega = \Omega_1 \times \Omega_2$ condition (4) holds. Suppose that $\omega_1 \subset \Omega_1$, $\omega_2 \subset \Omega_2$ are two open subsets such that (Ω_1, ω_1) and (Ω_2, ω_2) satisfy (5). Then, condition (5) comes true for (Ω, ω) with $\omega = \omega_1 \times \omega_2$.

Remark 1. The previous examples generalize some known results on null controllability of system (1) in unbounded domains that have been studied before. Firstly, we again obtain the null controllability result of the linear system (1) established in [2] when (Ω, ω) satisfies the conditions in example 1. Secondly, by means of the examples 4 and 5, we generalize the positive results on null controllability of the heat equation obtained in [12]. Observe that our approach allows us to consider general coefficients $a \in L^\infty(Q)$ and $B \in L^\infty(Q)^N$ in system (1). ■

4. Null controllability of the nonlinear problem

Finally, when $\Omega \setminus \overline{\omega}$ is bounded, we have the following result concerning the nonlinear system (3). From now on, we consider a locally Lipschitz-continuous function $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that $f(0,0) = 0$. Under these assumptions we can decompose f as $f(s,p) = g(s,p)s + G(s,p) \cdot p$ with $g, G \in L^\infty_{loc}$. One has:

Theorem 4.1. *Let Ω be an unbounded connected open set of class C^2 uniformly and $\omega \subset \Omega$ an open subset such that $\Omega \setminus \overline{\omega}$ is bounded. Suppose that f as above satisfies*

$$\lim_{|(s,p)| \to \infty} \frac{|g(s,p)|}{\log^{3/2}(1 + |s| + |p|)} = 0, \qquad \lim_{|(s,p)| \to \infty} \frac{|G(s,p)|}{\log^{1/2}(1 + |s| + |p|)} = 0. \tag{12}$$

Then, for every $y^0 \in W^{1,\infty}(\Omega) \cap H^1_0(\Omega)$ there exists a control $v \in L^2(\Omega) \cap L^\infty(\Omega)$ such that (2) *admits a solution y that satisfies* (3).

Idea of the proof. The proof is obtained by a fixed-point argument (see [4] for the bounded case) and is a consequence of Theorem 3.1. In order to avoid the lack of compactness of the Sobolev embeddings when Ω is unbounded, we use the construction of y as in (11), and take advantage of the fact that $\operatorname{supp}(y - \eta(t)\theta(x)Y)$ is contained in a bounded set (key point in [2]). See [8] for a complete proof.

Acknowledgements

The first author was partially supported by D.G.E.S. (Spain), grant BFM2003–06446. The second authors was partially supported by C.O.N.A.C.yT. (Mexico).

References

[1] O. Bodart, M. González-Burgos, R. Pérez-García, Existence of insensitizing controls for a semilinear heat equation with a superlinear nonlinearity, Comm. PDE 29 (2004), no. 7-8, 1017–1050.

[2] V.R. Cabanillas, S.B. de Menezes, E. Zuazua, Null controllability in unbounded domains for the semilinear heat equation with nonlinearities involving gradient terms, J. Optim. Theory Appl. 110 (2001), no. 2, 245–264.

[3] P. Cannarsa, P. Martinez, J. Vancostenoble, Null controllability of the heat equation in unbounded domains by a finite measure control region, ESAIM Control Optim. Calc. Var. 10 (2004), no. 3, 381–408.

[4] A. Doubova, E. Fernández-Cara, M. González-Burgos, E. Zuazua, On the controllability of parabolic systems with a nonlinear term involving the state and the gradient, SIAM J. Control Optim. 41 (2002), no. 3, 798–819.

[5] E. Fernández-Cara, S. Guerrero, Global Carleman inequalities for parabolic systems and applications to controllability, SIAM J. Control Optim. 45 (2006), no. 4, 1399–1446.

[6] E. Fernández-Cara, E. Zuazua, The cost of approximate controllability for heat equations: The linear case, Adv. Differential Equations 5 (2000), no. 4–6, 465–514.

[7] S. Fornaro, G. Metafune, E. Priola, *Gradient estimates for Dirichlet parabolic problems in unbounded domains.* J. Differential Equations 205 (2004), no. 2, 329–353.

[8] M. González-Burgos, L. de Teresa, Some results on controllability for linear and nonlinear heat equations in unbounded domains, submited.

[9] O. Yu. Imanuvilov and M. Yamamoto, Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations, Publ. Res. Inst. Math. Sci. 39 (2003), no. 2, 227–274.

[10] S. Micu, E. Zuazua, On the lack of null-controllability of the heat equation on the half-line, Trans. Amer. Math. Soc. 353 (2001), no. 4, 1635–1659.

[11] S. Micu, E. Zuazua, On the lack of null-controllability of the heat equation on the half space, Port. Math. (N.S.) 58 (2001), no. 1, 1–24.

[12] L. Miller, On the null-controllability of the heat equation in unbounded domains, Bull. Sci. Math. 129 (2005), no. 2, 175–185.