XX Congreso de Ecuaciones Diferenciales y Aplicaciones
X Congreso de Matemática Aplicada
Sevilla, 24-28 septiembre 2007
(pp. 1–4)

J_2 effect and the Collision Restricted Three–Body Problem

E. Barrabés[1], J.M. Cors[2], C. Pinyol[3], J. Soler[1]

[1] *Deptartament d'Informàtica i Matemàtica Aplicada, Universitat de Girona, 17071 Girona, Spain.*
E-mails: esther.barrabes@ima.udg.es, jaume.soler@ima.udg.es.
[2] *Departament de Matemàtica Aplicada III, Universitat Politècnica de Catalunya, 08242 Manresa, Spain.*
E-mail: cors@epsem.upc.edu.
[3] *Departament d'Economia i Història Econòmica, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain. E-mail:* Conxita.Pinyol@uab.es.

Palabras clave: Collision restricted three-body problem, periodic orbits, symmetric orbits, critical inclination, continuation method.

Resumen

The existence of a new class of inclined periodic orbits of the collision restricted three–body problem is shown. The symmetric periodic solutions found are perturbations of elliptic kepler orbits and they exist only for special values of the inclination and are related to the motion of a satellite around an oblate planet.

1. Introduction

From Keplerian orbit theory, without perturbations, elliptic orbits are always confined on an planet centered plane known as orbital plane. Six parameters a, e, i, h, g and f, called the classical orbital elements, completely characterize a conic orbit (see, for example, [3]).

For Earth–orbit design, the main effect that must be included is the so called J_2 term, named after the coefficient of a planet's geopotential harmonic expansion. The J_2 term corresponds to the planet's equatorial bulge and has two important effects on the orbit. Due this extra mass the satellite reach its ascending node sooner than it would without the perturbation. Hence the node appears to move backward and the effect is called nodal regression. The second effect is the precession of the argument of perigee. J_2 causes the perigee to rotate around the orbit's normal vector. It is apparent (see, for example, pages 503–504 of [3]), that there exist a critical inclination angle, $i \simeq 63°$, such that perigee is fixed. Critically inclined elliptic orbits are very useful. The most famous of these is the Molniya orbit, which is a highly elliptic 12–hour–period orbit the Soviets originally designed to observe the northern hemisphere.

The existence of a class of inclined periodic solutions in the circular restricted three–body problem was shown by Jefferys in [6]. He showed the existence of families of elliptic orbits for any value of the eccentricity and a critical inclination. In fact, when one of the primaries is small enough, and the infinitesimal body is far away from the primaries, the small primary moves very fast and the potential looks almost like that of an oblate planet. This is why Jefferys finds a critical inclination for these orbits. The method used in [6] was the continuation method developed by Poincaré (see [9]), which is one of the most frequently used methods for proving the existence of periodic orbits.

The case dealt with in this communication (see the complete text in [2]) is different from Jefferys because the primaries move on an elliptic collision orbit along the z-axis. Heuristically speaking, however, it can be expected that far away from the primaries the potential will be similar to that of a very eccentric prolate ellipsoid, so that a J_2 effect, with its critical inclination, will exist. We show the existence of periodic solutions of Jefferys type: large semi-axis compared to the that of the primaries, arbitrary eccentricity and inclination close to critical.

The perturbed problem is degenerate due to the fast motion of the primaries, and the equations are no longer analytic when the parameter equals zero, which precludes the use of standard implicit function theorem. We overcome the difficulty by using Arenstorf's theorem, where weaker assumptions of differentiability are needed (see [1]). A planar configuration of this problem is studied in [7].

In our case, the problem has a rotational symmetry around the z-axis (which contains the colliding primaries). This symmetry would be lost if we considered elliptic non collision orbits for the primaries. See [4] and [5], where the elliptic restricted three-body problem is considered. In those papers, the periodic orbits are perturbations of the circular solutions of the Kepler problem having large radii on a plane perpendicular to that of the primaries. If the unperturbed orbit is a polar one, the precession induced by a small variation of the inclination can be used to compensate for any variation of the orbital plane due to the perturbation.

Periodic orbits in the spatial elliptic restricted three–body are also studied using double averaging in [8].

2. The Collision Restricted Three Body Problem

The Collision Restricted Three Body Problem describes the motion of a massless particle under the attraction of two primaries with equal masses, $m_1 = m_2 = 1/2$, moving on a collision elliptic orbit. In order to avoid a triple collision, we consider that the third body is far from the primaries compared to the distance between them. This fact can be introduced in the equations of motion by making the primaries very close to each other and looking for solutions of the massless particle at distance of order unity to the primaries.

Let μ be a small parameter. The distance between both primaries is given by

$$\rho = \mu(1 - \cos E_p(t)),$$

where $E_p = E_p(t)$ is the eccentric anomaly of m_1 and it is related to its mean anomaly ℓ_p

through Kepler's equation

$$E_p - \sin E_p = \ell_p, \tag{1}$$

where $\ell_p = \mu^{-3/2}t$. The period of the motion of the primaries is $T_p = 2\pi\mu^{3/2}$, so that $E_p = k\pi$ when $t = \pi k\mu^{3/2}$.

The equations of motion, in spherical coordinates, for the infinitesimal body can be written as a non autonomous Hamiltonian system depending on the parameter μ as

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2\cos^2\phi}\right) - \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{2}$$

with R_1 and R_2 given by

$$\begin{array}{rcl} R_1^2 & = & r^2 + \left(\frac{\mu}{2}\right)^2(1-\cos E_p)^2 - r\mu(1-\cos E_p)\sin\phi, \\ R_2^2 & = & r^2 + \left(\frac{\mu}{2}\right)^2(1-\cos E_p)^2 + r\mu(1-\cos E_p)\sin\phi. \end{array}$$

Notice that E_p as given by Equation (1), is a function of time t and μ, which is not defined for $\mu = 0$. So, neither the Hamiltonian (2) is defined.

Since R_1 and R_2 do not depend on θ, $\dot{p_\theta} = 0$ and the angular momentum $p_\theta = \Theta$ is constant. Thus, it can be calculated from the initial conditions and the equation for θ can be decoupled from the other equations. We call, the problem in variables (r, ϕ, p_r, p_ϕ), *reduced problem.*

3. Main result

THEOREM. *Consider the Three–Dimensional Collision Restricted Three Body Problem with masses $m_1 = m_2 = 1/2$, and primaries's semimajor axis $\mu/2$. If $\mu = k^{-2/3}$, where k is a positive integer large enough, there exist initial conditions such that the infinitesimal body moves in a symmetric periodic orbit of the* reduced problem, *of period 2π, near a Keplerian elliptic orbit. The inclination of the orbit is close to the "critical value"* $\cos i = 1/\sqrt{5}$.

Remark: all the orbits found are on an integral resonance with the motion of the primaries, i.e. the primaries undergo k complete orbits in one orbit of the infinitesimal body. If $k = p/q$ is an irreducible rational then similar arguments show that in q complete orbits of the infinitesimal the primaries undergo p complete orbits.

Agradecimientos

E. Barrabés has been supported by grant MTM2006-05849/Consolider (including a FEDER contribution), J.M. Cors by grant number MTM 2005-06098-C02-01 and by a CIRIT grant number 2001SGR00173, C. Pinyol by grant SEJ2006/00712ECO and J. Soler by grant MTM2005-07660-C02-02.

Referencias

[1] R.F. Arenstorf, *A new method of perturbation theory and its application to the satellite problem of celestial mechanics*, Journal für die Reine und Angewandte Math, 221 (1966), pp. 113–145.

[2] E. Barrabés J.M. Cors, C. Pinyol, and J. Soler, *J_2 effect and the elliptic inclined periodic orbits in the collision restricted three–body problem*, SIAM J. Appl. Dyn. Syst. (2007). To appear.

[3] Richard H. Battin, *An introduction to the mathematics and methods of astrodynamics*, AIAA Education Series, American Institute of Aeronautics and Astronautics (AIAA), Washington, DC, 1987. With a foreword by J. S. Przemieniecki.

[4] J.M. Cors, C. Pinyol, and J. Soler, *Periodic solutions in the spatial elliptic restricted three-body problem*, Phys. D, 154 (2001), pp. 195–206.

[5] ———, *Analytic continuation in the case of non-regular dependency on a small parameter with an application to celestial mechanics*, J. Differential Equations, 219 (2005), pp. 1–19.

[6] W.H. Jefferys, *A new class of periodic solutions of the three-dimensional restricted problem*, The Astronomical Journal, 71 (1966), pp. 99–102.

[7] J. Llibre & D. Pasca, *Periodic orbits of the planar collision restricted 3-body problem*, Celestial mechanics and dynamical astronomy, 96 (2006), pp. 19–29.

[8] J.F. Palacian, P. Yanguas, S. Fernandez, and M.A. Nicotra, *Searching for periodic orbits of the spatial elliptic restricted three-body problem by double averaging*, Physica D: Nonlinear Phenomena, 213 (2006), pp. 15–24.

[9] H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste. Tome III*, Gauthier-Vilars, Paris, 1899.