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Standing Waves for Some Systems of Coupled Nonlinear Schrödinger Equations

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Abstract

We deal with a class of systems of NLS equations, proving the existence of bound and ground states provided the coupling parameter is small, respectively, large.

1. Introduction

It is well known that coupled NLS equations arise in nonlinear Optics. For example, if $\mathbf{E}(x,z)$ denotes the complex envelope of an Electric field, planar stationary light beams propagating in the z direction in a non-linear medium are described, up to rescaling, by a nonlinear Schrödinger (NLS) equation like

$$\mathrm{i}\,\mathbf{E}_z + \mathbf{E}_{xx} + \kappa|\mathbf{E}|^2\mathbf{E} = 0,$$

where i denotes the imaginary unit and subscripts denote derivatives. In the sequel the constant κ is assumed to be *positive*, corresponding to the fact that the medium is *self-focusing*. Without loss of generality we will put $\kappa = 1$. If $\mathbf{E}$ is the sum of two right- and left-hand polarized waves a_1E_1 and a_2E_2, $a_j \in \mathbb{R}$, the preceding equation gives rise to the following system of NLS equations for E_j, $j = 1, 2$ (see e.g. [1, 13, 14])

$$\begin{cases} \mathrm{i}\,(E_1)_z + (E_1)_{xx} + (a_1^2|E_1|^2 + a_2^2|E_2|^2)E_1 &= 0, \\ \mathrm{i}\,(E_2)_z + (E_2)_{xx} + (a_1^2|E_1|^2 + a_2^2|E_2|^2)E_2 &= 0. \end{cases} \tag{1.1}$$

We will look for *standing waves*, namely for solutions to (1.1) of the form $E_j(z,x) = e^{\mathrm{i}\lambda_j z}u_j(x)$, where $\lambda_j > 0$ and $u_j(x)$ are real valued functions which solve the system

$$\begin{cases} -(u_1)_{xx} + \lambda_1 u_1 &= (a_1^2u_1^2 + a_2^2u_2^2)u_1, \\ -(u_2)_{xx} + \lambda_2 u_2 &= (a_1^2u_1^2 + a_2^2u_2^2)u_2. \end{cases} \tag{1.2}$$

If we take the coupling factor β as a parameter and let the coefficients of u_j^3 be different, say $\mu_j > 0$, (1.2) becomes

$$\begin{cases} -u_1'' + \lambda_1 u_1 &= \mu_1 u_1^3 + \beta u_2^2 u_1, \\ -u_2'' + \lambda_2 u_2 &= \mu_2 u_2^3 + \beta u_1^2 u_2. \end{cases} \tag{1.3}$$

Most of the papers on NLS systems deal with the existence of specific explicit solutions, see e.g. [8], or with results based on numerical arguments. Recently, some more general rigorous achievements have been obtained, see [6, 11, 15]. We mainly deal with systems of two equations like

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 &= \mu_1 u_1^3 + \beta u_2^2 u_1, \quad u_1 \in W^{1,2}(\mathbb{R}^n), \\ -\Delta u_2 + \lambda_2 u_2 &= \mu_2 u_2^3 + \beta u_1^2 u_2, \quad u_2 \in W^{1,2}(\mathbb{R}^n), \end{cases} \tag{1.4}$$

where $n = 2, 3$, $\lambda_j, \mu_j > 0$, $j = 1, 2$, and $\beta \in \mathbb{R}$.

Roughly, we will show that there exist $\Lambda' \geq \Lambda > 0$, depending upon λ_j, μ_j, such that (1.3) has a radially symmetric solution $(u_1, u_2) \in W^{1,2}(\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$, with $u_1, u_2 > 0$, provided $\beta \in (0, \Lambda) \cup (\Lambda', +\infty)$. Moreover, for $\beta > \Lambda'$, these solutions are ground states, in the sense that they have minimal energy and their Morse index is 1. It is worth pointing out that for any β (1.4) has a pair of *semi-trivial* solutions having one component equal to zero. These solutions have the form $(U_1, 0)$, $(0, U_2)$ where U_j is the positive radial solution of $-\Delta u + \lambda_j u = \mu_j u^3$, $u \in W^{1,2}(\mathbb{R}^n)$. Of course, we look for solutions different from the preceding ones. On the other hand, the presence of $(U_1, 0)$ and $(0, U_2)$ can be usefully exploited to prove our existence results. Actually, the main idea is to show that the Morse index of $(U_1, 0)$ and $(0, U_2)$ changes with β: for $\beta < \Lambda$ small their index is 1, while for $\beta > \Lambda'$ their index is greater or equal than 2. This fact, jointly an appropriate use of the method of natural constraint, allows us to prove the existence of bound and ground states as outlined before.

The paper contains 4 more sections. In Section 2 we introduce notation and give the definition of bound and ground state. Sections 3 and 4 contain, respectively, some preliminary material on the method of the *natural constraint* and the key lemmas for getting the main existence results, which are stated and proved in Section 5.

A complete version of this paper can be seen in [3], where also some further results and extensions to systems with more than two equations are discussed.

2. Notation and Preliminary Definitions

Let us introduce the following notation

- $E = W^{1,2}(\mathbb{R}^n)$, the standard Sobolev space, endowed with scalar product and norm

$$(u \mid v)_j = \int_{\mathbb{R}^n} [\nabla u \cdot \nabla v + \lambda_j uv] dx, \qquad \|u\|_j^2 = (u \mid u)_j, \quad j = 1, 2;$$

- $\mathbb{E} = E \times E$; the elements in $\mathbb{E}$ will be denoted by $\mathbf{u} = (u_1, u_2)$; as a norm in $\mathbb{E}$ we will take $\|\mathbf{u}\|^2 = \|u_1\|_1^2 + \|u_2\|_2^2$;
- we set $\mathbf{0} = (0, 0)$, for $\mathbf{u} \in \mathbb{E}$, the notation $\mathbf{u} \geq \mathbf{0}$, resp. $\mathbf{u} > \mathbf{0}$, means that $u_j \geq 0$, resp. $u_j > 0$, for all $j = 1, 2$;

- H denotes the space of *radially symmetric* functions in E, and $\mathbb{H} = H \times H$.

For $u \in E$, resp. $\mathbf{u} \in \mathbb{E}$, we set

$$\begin{aligned} I_j(u) &= \tfrac{1}{2}\int_{\mathbb{R}^n}(|\nabla u|^2 + \lambda_j u^2)dx - \tfrac{1}{4}\,\mu_j \int_{\mathbb{R}^n} u^4 dx, \\ F(\mathbf{u}) &= \tfrac{1}{4}\int_{\mathbb{R}^n}\left(\mu_1 u_1^4 + \mu_2 u_2^4\right)dx, \quad G(\mathbf{u}) = G(u_1,u_2) = \tfrac{1}{2}\int_{\mathbb{R}^n} u_1^2 u_2^2 dx, \\ \Phi(\mathbf{u}) &= \Phi(u_1,u_2) = I_1(u_1) + I_2(u_2) - \beta\, G(u_1,u_2) \\ &= \tfrac{1}{2}\|\mathbf{u}\|^2 - F(\mathbf{u}) - \beta\, G(\mathbf{u}). \end{aligned}$$

Let us remark that F and G make sense because $E \hookrightarrow L^4(\mathbb{R}^n)$ for $n = 2, 3$ Any critical point $\mathbf{u} \in \mathbb{E}$ of Φ gives rise to a solution of (1.4). If $\mathbf{u} \neq \mathbf{0}$ we say that such a critical point (solution) is non-trivial. We say that a solution $\mathbf{u}$ of (1.4) is *positive* if $\mathbf{u} > \mathbf{0}$.

Among non-trivial solutions of (1.4), we shall distinguish the *bound states* from the *ground states*.

Definition 2.1 *We say that* $\mathbf{u} \in \mathbb{E}$ *is a non-trivial bound state of (1.4) if* $\mathbf{u}$ *is a non-trivial critical point of* Φ*. A positive bound state* $\mathbf{u} > \mathbf{0}$ *such that its energy is minimal among all the non-trivial bound states, namely*

$$\Phi(\mathbf{u}) = \text{mín}\{\Phi(\mathbf{v}) : \mathbf{v} \in \mathbb{E} \setminus \{\mathbf{0}\},\ \Phi'(\mathbf{v}) = 0\}, \tag{2.1}$$

is called a ground state of (1.4).

About the definition of ground states, a remark is in order.

3. The Natural Constraint

In order to find critical points of Φ, let us set $\Psi(\mathbf{u}) = (\Phi'(\mathbf{u}) \mid \mathbf{u}) = \|\mathbf{u}\|^2 - 4F(\mathbf{u}) - 4G(\mathbf{u})$, and introduce the so called Nehari manifold:

$$\mathcal{M} = \{\mathbf{u} \in \mathbb{H} \setminus \{\mathbf{0}\} : \Psi(\mathbf{u}) = 0\}.$$

Plainly, $\mathcal{M}$ contains all the non-trivial critical points of Φ on $\mathbb{H}$. Let us recall, for the reader convenience, some well known facts. First of all, for any $\mathbf{v} \in \mathbb{H} \setminus \{\mathbf{0}\}$ one has that

$$t\mathbf{v} \in \mathcal{M} \quad \Longleftrightarrow \quad t^2\|\mathbf{v}\|^2 = t^4\left[4F(\mathbf{v}) + 4\beta G(\mathbf{v})\right].$$

As a consequemce, for all $\mathbf{v} \in \mathbb{H} \setminus \{\mathbf{0}\}$, there exists a unique $t > 0$ such that $t\mathbf{v} \in \mathcal{M}$. Moreover, since F, G are homogeneous with degree 4, that $\exists\, \rho > 0$ such that

$$\|\mathbf{u}\|^2 \geq \rho, \qquad \forall\, \mathbf{u} \in \mathcal{M}. \tag{3.1}$$

Furthermore, from (3.1) it follows that

$$(\Psi'(\mathbf{u}) \mid \mathbf{u}) = -2\|\mathbf{u}\|^2 < 0, \qquad \forall\, \mathbf{u} \in \mathcal{M}. \tag{3.2}$$

From (3.1) and (3.2) we infer that $\mathcal{M}$ is a smooth complete manifold of codimension 1 in $\mathbb{E}$. Moreover, we can state the following Proposition.

Proposition 3.1 $\mathbf{u} \in \mathbb{H}$ *is a non-trivial critical point of* Φ *if and only if* $\mathbf{u} \in \mathcal{M}$ *and is a constrained critical point of* Φ *on* $\mathcal{M}$*.*

Because of this, $\mathcal{M}$ is called a *natural constraint* for Φ. A remarkable advantage of working on the Nehari manifold is that Φ is bounded from below on $\mathcal{M}$.

Concerning the Palais-Smale (PS) condition, the following Lemma holds.

Lemma 3.2 Φ *satisfies the (PS) condition on* $\mathcal{M}$*.*

A proof of this result can be seen in [3], so we omit the details because of the extension.

Remark 3.3 From the preceding arguments it follows immediately that $\text{mín}\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathcal{M}\}$ is achieved giving rise to a non-negative solution of (1.4). However, such an existence result is useless without any further specification. Actually, for every $\beta \in \mathbb{R}$, (1.4) already possesses two explicit solutions given by

$$\mathbf{u}_1 = (U_1, 0), \quad \mathbf{u}_2 = (0, U_2),$$

where U_j is radial positive and satisfies $-\Delta u + \lambda_j u = \mu_j u^3$. In other words, to find a non-trivial existence result, one has to find solutions having *both the components* not identically zero. ■

4. Evaluating the Morse index of $\mathbf{u}_j$

The aim of the following arguments is to show that there exist non-negative solutions of (1.4) different from $\mathbf{u}_j$, $j = 1, 2$. First, let us remark that if we let U denote the unique positive radial solution of $-\Delta u + u = u^3$, there holds

$$U_j(x) = \sqrt{\frac{\lambda_j}{\mu_j}}\, U(\sqrt{\lambda_j}\, x), \quad j = 1, 2.$$

Next, we set

$$\gamma_1^2 = \inf_{\varphi \in H \setminus \{0\}} \frac{\|\varphi\|_2^2}{\int U_1^2 \varphi^2}, \qquad \gamma_2^2 = \inf_{\varphi \in H \setminus \{0\}} \frac{\|\varphi\|_1^2}{\int U_2^2 \varphi^2},$$

and $\Lambda = \text{mín}\{\gamma_1^2, \gamma_2^2\}$, $\qquad \Lambda' = \text{máx}\{\gamma_1^2, \gamma_2^2\}$.
The next Proposition shows that the nature of $\mathbf{u}_j$ changes in dependence of β, Λ, Λ'.

Proposition 4.1 *(i)* $\forall\, \beta < \Lambda$, $\mathbf{u}_j$, $j = 1, 2$, *are strict local minima of* Φ *on* $\mathcal{M}$*.*
(ii) If $\beta > \Lambda'$ *then* $\mathbf{u}_j$ *are saddle points of* Φ *on* $\mathcal{M}$*. In particular,* $\inf_{\mathcal{M}} \Phi < \text{mín}\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$*.*

To prove this Proposition, we will evaluate the Morse index of $\mathbf{u}_j$, as critical points of Φ constrained on $\mathcal{M}$. Let $D^2\Phi_{\mathcal{M}}(\mathbf{u}_j)$ denote the second derivative of Φ constrained on $\mathcal{M}$. Since $\Phi'(\mathbf{u}_j) = 0$, then one has that

$$D^2\Phi_{\mathcal{M}}(\mathbf{u}_j)[\mathbf{h}]^2 = \Phi''(\mathbf{u}_j)[\mathbf{h}]^2, \qquad \forall\, \mathbf{h} \in T_{\mathbf{u}_j}\mathcal{M}. \tag{4.1}$$

Similarly, if $\mathcal{N}_j$ denotes the Nehari manifolds relative to I_j, $j = 1, 2$,

$$\mathcal{N}_j = \{u \in H \setminus \{0\} : (I_j'(u)|u) = 0\} = \{u \in H \setminus \{0\} : \|u\|_j^2 = \mu_j \int u^4\},$$

then, from the fact that $I_j'(U_j) = 0$ it follows

$$D^2(I_j)_{\mathcal{N}_j}(U_j)[h]^2 = I_j''(U_j)[h]^2, \qquad \forall\, h \in T_{U_j}\mathcal{N}_j. \tag{4.2}$$

Notice that U_j is the minimum of I_j on $\mathcal{N}_j$ and thus, using also (4.2), one has that $\exists\, c_j > 0$ such that

$$I_j''(U_j)[h_j]^2 \geq c_j \|h_j\|_j^2, \qquad j = 1, 2. \tag{4.3}$$

The next lemma shows the relationship between $T_{\mathbf{u}_j}\mathcal{M}$ and $T_{U_j}\mathcal{N}_j$.

Lemma 4.2 *There holds:* $\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_j}\mathcal{M} \Leftrightarrow h_j \in T_{U_j}\mathcal{N}_j, \quad j = 1, 2.$

Proof. One has that $h_j \in T_{U_j}\mathcal{N}_j$ iff $(U_j \mid \phi)_j = 2\mu_j \int U_j^3 \phi$, while $\mathbf{h} \in T_{\mathbf{u}}\mathcal{M}$ iff

$$(u_1 \mid h_1)_1 + (u_2 \mid h_2)_2 = 2 \int_{\mathbb{R}^n} (\mu_1 u_1^3 h_1 + \mu_2 u_2^3 h_2) + \beta \int_{\mathbb{R}^n} (u_1 h_1 u_2^2 + u_1^2 u_2 h_2).$$

Thus $\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_j}\mathcal{M}$, iff $(U_j \mid h_j)_j = 2\mu_j \int U_j^3 h_j$. ■ Because of the extension, we omit the details, see [3] for a proof.

Remark 4.3 What we have really proved is that $\mathbf{u}_j$ is a minimum, resp. a saddle point, provided $\beta < \gamma_j^2$, resp. $\beta > \gamma_j^2$, $j = 1, 2$. ■

5. Existence Results

According to Proposition 3.1, in order to find a non-trivial solution of (1.4) it suffices to find a critical point of Φ constrained on $\mathcal{M}$. The following lemma is a direct consequence of Proposition 4.1 and Lemma 3.2.

Lemma 5.1 *(i) If $\beta < \Lambda$, then Φ has a Mountain-Pass critical point $\mathbf{u}^*$ on $\mathcal{M}$, and there holds $\Phi(\mathbf{u}^*) > \text{máx}\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$.*

(ii) If $\beta > \Lambda'$ then Φ has a positive global minimum $\widetilde{\mathbf{u}}$ on $\mathcal{M}$, and there holds $\Phi(\widetilde{\mathbf{u}}) < \text{mín}\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$.

Proof. (i) Proposition 4.1-(i) and Lemma 3.2 allow us to apply the Mountain Pass theorem to Φ on $\mathcal{M}$, yielding a critical point $\mathbf{u}^*$ of Φ. By the Mountain Pass Theorem, it also follows that $\Phi(\mathbf{u}^*) > \text{máx}\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$.

(ii) By Lemma 3.2 the $\text{ínf}_{\mathcal{M}} \Phi$ is achieved at some $\widetilde{\mathbf{u}} > 0$. Moreover, if $\beta > \Lambda'$, Proposition 4.1-(ii) implies $\Phi(\mathbf{u}^*) < \text{mín}\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$. ■

Remark 5.2 In order to prove the preceding Lemma, it would be enough that only one among $\mathbf{u}_j$ is a minimum or a saddle. For example, if $\Phi(\mathbf{u}_1) < \Phi(\mathbf{u}_2)$ to prove (i) it suffices that the $\mathbf{u}_2$ is a minimum. According to Remark 4.3, this is the case provided $\beta < \gamma_2^2$. Unfortunately, a straight calculation shows that if $\Phi(\mathbf{u}_1) < \Phi(\mathbf{u}_2)$ then $\gamma_2^2 < \gamma_1^2$. Hence $\mathbf{u}_1$ is a minimum as well. Same remark holds for the case (ii). ■

We are now in position to state our general existence results.

5.1. Existence of ground states

Concerning ground states, our main result is the following

Theorem 5.3 *If* $\beta > \Lambda'$ *then* (1.4) *has a (positive) radial ground state* $\widetilde{\mathbf{u}}$.

Proof. Lemma 5.1-(ii) yields a critical point $\widetilde{\mathbf{u}} \in \mathcal{M}$ which is a non-trivial solution of (1.4). To complete the proof we have to show that $\widetilde{\mathbf{u}} > \mathbf{0}$ and is a ground state in the sense of Definition 2.1. To prove these facts, we argue as follows. Since $|\widetilde{\mathbf{u}}| = (|\widetilde{u}_1|, |\widetilde{u}_2|)$ also belongs to $\mathcal{M}$ and $\Phi(|\widetilde{\mathbf{u}}|) = \Phi(\widetilde{\mathbf{u}}) = \text{mín}\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathcal{M}\}$, we can assume that $\widetilde{\mathbf{u}} \geq \mathbf{0}$. By the maximum principle, $\widetilde{\mathbf{u}} > \mathbf{0}$. It remains to prove that

$$\Phi(\widetilde{\mathbf{u}}) = \text{mín}\{\Phi(\mathbf{v}) : \mathbf{v} \in \mathbb{E} \setminus \{\mathbf{0}\},\, \Phi'(\mathbf{v}) = 0\}. \tag{5.1}$$

By contradiction, let $\widetilde{\mathbf{v}} \in \mathbb{E}$ be a non-trivial critical point of Φ such that

$$\Phi(\widetilde{\mathbf{v}}) < \Phi(\widetilde{\mathbf{u}}) = \text{mín}\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathcal{M}\}. \tag{5.2}$$

Setting $\mathbf{w} = |\widetilde{\mathbf{v}}|$ there holds

$$\Phi(\mathbf{w}) = \Phi(\widetilde{\mathbf{v}}), \qquad \Psi(\mathbf{w}) = \Psi(\widetilde{\mathbf{v}}). \tag{5.3}$$

Let $\mathbf{w}^\star \in \mathbb{H} \setminus \{\mathbf{0}\}$ denote the Schwartz symmetric function associated to $\mathbf{w}$. Using the properties of Schwartz symmetrization, the second of (5.3) and the fact that $\widetilde{\mathbf{v}}$ is a critical point of Φ, we get $\Psi(\mathbf{w}) = \Psi(\widetilde{\mathbf{v}}) = 0$ and there exists a unique $t \in (0,1]$ such that $t\,\mathbf{w}^\star \in \mathcal{M}$. Moreover,

$$\Phi(t\,\mathbf{w}^\star) = \tfrac{1}{4}\,t^2\|\mathbf{w}^\star\|^2 \leq \tfrac{1}{4}\,\|\mathbf{w}\|^2 = \Phi(\mathbf{w}).$$

This, the first of (5.3) and (5.2) yield

$$\Phi(t\,\mathbf{w}^\star) \leq \Phi(\mathbf{w}) = \Phi(\widetilde{\mathbf{v}}) < \Phi(\widetilde{\mathbf{u}}) = \text{mín}\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathcal{M}\},$$

which is a contradiction, since $t\,\mathbf{w}^\star \in \mathcal{M}$. This shows that (5.1) holds and completes the proof of Theorem 5.3. ■

5.2. Existence of bound states

Concerning the existence of positive bound states, the following result holds.

Theorem 5.4 *If* $\beta < \Lambda$, *then* (1.4) *has a radial bound state* $\mathbf{u}^*$ *such that* $\mathbf{u}^* \neq \mathbf{u}_j$, $j = 1, 2$. *Furthermore, if* $\beta \in (0, \Lambda)$, *then* $\mathbf{u}^* > 0$.

Proof. If $\beta < \Lambda$, a straight application of Lemma 5.1-(i) yields a non-trivial solution $\mathbf{u}^* \in \mathcal{M}$ of (1.4), which corresponds to a mountain-Pass critical point of Φ on $\mathcal{M}$. Moreover, $\Phi(\mathbf{u}^*) > \text{máx}\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$ implies that $\mathbf{u}^* \neq \mathbf{u}_j$, $j = 1, 2$.

To show that $\mathbf{u}^* > \mathbf{0}$ provided $\beta \in (0, \Lambda)$, let us introduce the functional

$$\Phi^+(\mathbf{u}) = \tfrac{1}{2}\|\mathbf{u}\|^2 - F(\mathbf{u}^+) - \beta\, G(\mathbf{u}^+),$$

where $\mathbf{u}^+ = (u_1^+, u_2^+)$ and $u^+ = \text{máx}\{u, 0\}$. Consider the corresponding Nehari manifold

$$\mathcal{M}^+ = \{\mathbf{u} \in \mathbb{H} \setminus \{\mathbf{0}\} : (\nabla\Phi^+(\mathbf{u}) \mid \mathbf{u}) = 0\}.$$

Repeating with minor changes the arguments carried out in Section 3, one readily shows that what is proved in such a section, still holds with Φ and $\mathcal{M}$ substituted by Φ^+ and $\mathcal{M}^+$. In particular, Proposition 3.1 and Lemma 3.2 hold true for Φ^+ and $\mathcal{M}^+$. On the other hand, Proposition 4.1-(i) cannot be proved as before, because Φ^+ is not C^2. To circumvent this difficulty, we argue as follows.

Consider an ε-neighborhood $V_\varepsilon \subset \mathcal{M}$ of $\mathbf{u}_1$. For each $\mathbf{u} \in V_\varepsilon$ there exists $T(\mathbf{u}) > 0$ such that $T(\mathbf{u})\mathbf{u} \in \mathcal{M}^+$. Actually $T(\mathbf{u})$ satisfies

$$\|\mathbf{u}\|^2 = 4\,T^2(\mathbf{u})\left[F(\mathbf{u}^+) + \beta\,G(\mathbf{u}^+)\right],$$

and since $\|\mathbf{u}\|^2 = 4\,[F(\mathbf{u}) + \beta\,G(\mathbf{u})]$, we get

$$[F(\mathbf{u}) + \beta\,G(\mathbf{u})] = T^2(\mathbf{u})\left[F(\mathbf{u}^+) + \beta\,G(\mathbf{u}^+)\right]. \tag{5.4}$$

Let us point out that $F(\mathbf{u}^+) + \beta\,G(\mathbf{u}^+) \le F(\mathbf{u}) + \beta\,G(\mathbf{u})$ and this implies that $T(\mathbf{u}) \ge 1$. Moreover, since $\text{lím}_{\mathbf{u}\to\mathbf{u}_1} F(\mathbf{u}^+) + \beta\,G(\mathbf{u}^+) = F(\mathbf{u}_1) > 0$ it follows that there exist $\varepsilon > 0$ and $c > 0$ such that

$$F(\mathbf{u}^+) + \beta\,G(\mathbf{u}^+) \ge c, \qquad \forall\,\mathbf{u} \in V_\varepsilon.$$

This and (5.4) imply that the map $\mathbf{u} \to T(\mathbf{u})\mathbf{u}$ is a homeomorphism, locally near $\mathbf{u}_1$. In particular, there are ε-neighborhoods $V_\varepsilon \subset \mathcal{M}$, $W_\varepsilon \subset \mathcal{M}^+$ of $\mathbf{u}_1$ such that for all $\mathbf{v} \in W_\varepsilon$, there exists $\mathbf{u} \in V_\varepsilon$ such that $\mathbf{v} = T(\mathbf{u})\mathbf{u}$. Finally, from $\Phi^+(\mathbf{v}) = \frac{1}{4}\|\mathbf{v}\|^2$, see (**??**), and the fact that $T(\mathbf{u}) \ge 1$, we infer

$$\Phi^+(\mathbf{v}) = \tfrac{1}{4}\|\mathbf{v}\|^2 = \tfrac{1}{4}T^2(\mathbf{u})\|\mathbf{u}\|^2 \ge \tfrac{1}{4}\|\mathbf{u}\|^2 = \Phi(\mathbf{u}).$$

Since, according to Proposition 3.1, $\mathbf{u}_1$ is a local minimum of Φ on $\mathcal{M}$, and thus

$$\Phi^+(\mathbf{v}) \ge \Phi(\mathbf{u}) \ge \Phi(\mathbf{u}_1) = \Phi^+(\mathbf{u}_1), \quad \forall\,\mathbf{v} \in W_\varepsilon,$$

proving that $\mathbf{u}_1$ is a local strict minimum for Φ^+ on $\mathcal{M}^+$. A similar proof can be carried out for $\mathbf{u}_2$.

From the preceding arguments, it follows that Φ^+ has a Mountain Pass critical point $\mathbf{u}^* \in \mathcal{M}^+$, which gives rise to a solution of

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 &= \mu_1 (u_1^+)^3 + \beta (u_2^+)^2 u_1^+, \quad u_1 \in W^{1,2}(\mathbb{R}^n), \\ -\Delta u_2 + \lambda_2 u_2 &= \mu_2 (u_2^+)^3 + \beta (u_1^+)^2 u_2^+, \quad u_2 \in W^{1,2}(\mathbb{R}^n). \end{cases} \tag{5.5}$$

In particular, one finds that $u_j \ge 0$. In addition, since $\mathbf{u}^*$ is a Mountain-Pass critical point, one has that $\Phi^+(\mathbf{u}^*) > \text{máx}\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$. Let us also remark that $\mathbf{u}^* \in \mathcal{M}^+$ implies that $\mathbf{u}^* \ne \mathbf{0}$ and hence $u_2^* \equiv 0$ implies that $u_1^* \not\equiv 0$. From $\Phi'(u_1^*, 0) = 0$ it follows that u_1^* is a non-trivial solution of

$$-\Delta u + \lambda_1 u = \mu_1 u_+^3, \qquad u \in H.$$

Since $u_1^* \ge 0$ and $u_1^* \not\equiv 0$, then $u_1^* = U_1$, namely $\mathbf{u}^* = (U_1, 0) = \mathbf{u}_1$. This is in contradiction to $\Phi^+(\mathbf{u}^*) > \Phi(\mathbf{u}_1)$, proving that $u_2^+ \not\equiv 0$. A similar argument proves that $u_1^* \not\equiv 0$. Since both u_1^* and u_2^* are $\not\equiv 0$, using the maximum principle we get $u_1^* > 0$ and $u_2^* > 0$. ∎

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