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GEOMETRIC REALIZATION OF MÖBIUS TRIANGULATIONS*

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Abstract. A *Möbius triangulation* is a triangulation on the Möbius band. A *geometric realization* of a map M on a surface Σ is an embedding of Σ into a Euclidean 3-space $\mathbb{R}^3$ such that each face of M is a flat polygon. In this paper, we shall prove that every 5-connected triangulation on the Möbius band has a geometric realization. In order to prove it, we prove that if G is a 5-connected triangulation on the projective plane, then for any face f of G, the Möbius triangulation $G - f$ obtained from G by removing the interior of f has a geometric realization.

Key words. geometric realization, triangulation, Möbius band, projective plane

AMS subject classifications. 05C10, 52B70, 05C83

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1. Introduction.

Let Σ be a surface with at most one boundary component, and let M be a map on Σ. If Σ has a boundary, we suppose that some cycle of M coincides with the boundary of Σ. Such a cycle of M is called the *boundary* of M and denoted by ∂M. A vertex of M not on ∂M is called an *inner* vertex. A *k-cycle* means a cycle of length k. A *triangulation* on Σ is a map on Σ such that each face is bounded by a 3-cycle. In particular, a *Möbius triangulation* is a triangulation on the Möbius band. For an inner vertex v of a triangulation, the *link* of v is the boundary walk of the 2-cell region consisting of all faces incident to v. Throughout this paper, we suppose that the graph of a map is *simple*, i.e., with no multiple edges and no loops. For a cycle or path C in M, a *chord* of C means an edge xy of M such that $x, y \in V(C)$ but $xy \notin E(C)$. Hence C is induced in M if and only if C has no chord.

A *geometric realization* of a map M on a surface Σ is an embedding of Σ into a Euclidean 3-space $\mathbb{R}^3$ such that each face of M is a flat polygon. Steinitz's theorem states that a spherical map has a geometric realization if and only if its graph is 3-connected [10]. Moreover, Archdeacon, Bonnington, and Ellis-Monanghan proved that every toroidal triangulation has a geometric realization [1]. In general, Grünbaum conjectured that every triangulation on any orientable closed surface has a geometric realization [7], but Bokowski and Guedes de Oliveira recently showed that a triangulation by K_{12} on the orientable closed surface of genus 6 has no geometric realization [2]. (For related topics, see [5].)

Let us consider a geometric realization of a triangulation on the projective plane. Let $\mathbb{P}$ denote the projective plane throughout this paper. Since the projective plane itself is not embeddable in $\mathbb{R}^3$, no map on $\mathbb{P}$ has a geometric realization. Let G be a triangulation on $\mathbb{P}$, and let f be a face of G. Let $G-f$ denote the Möbius triangulation

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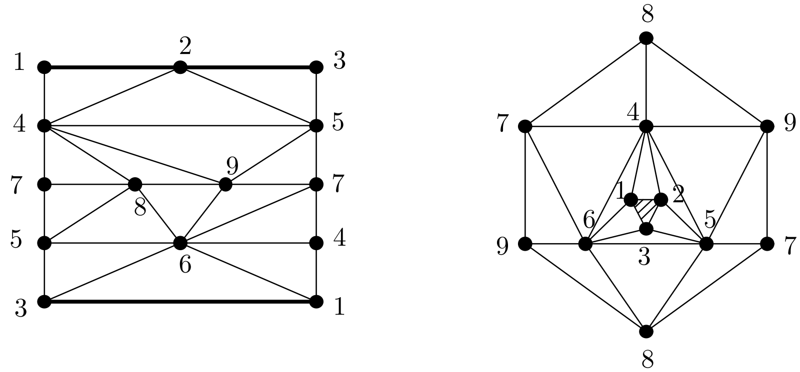

FIG. 1. *A Möbius triangulation with no geometric realization.*

obtained from G by removing the interior of f. Since the punctured surface obtained from $\mathbb{P}$ by removing a 2-cell, the *Möbius band*, is embeddable in $\mathbb{R}^3$, $G-f$ might have a geometric realization. The following is known.

THEOREM 1.1 (Bonnington and Nakamoto [3]). *Every triangulation G on the projective plane $\mathbb{P}$ has a face f such that the Möbius triangulation $G-f$ has a geometric realization.*

Brehm [4] has already found a Möbius triangulation with no geometric realization, shown in Figure 1, in which both express the same triangulation. (In Figure 1, we identify the vertices with the same label. In the right-hand side, the shaded part means the hole.) Why does Brehm's example have no geometric realization? We can prove that for each of its spatial embedding, the two disjoint 3-cycles 123 and 456 have a linking number of at least 2. (See [9] for the definition of the linking number.) However, two 3-cycles, each with an edge straight segment embedded in $\mathbb{R}^3$, have a linking number of at most 1, a contradiction. Hence, generalizing this example, we can see that if a triangulation M on the Möbius band has a boundary cycle C of length 3 and a 3-cycle C' disjoint from C which forms an annular region with C', then M never has a geometric realization.

A graph M is said to be *cyclically k-connected* if M has no separating set $S \subset V(M)$ with $|S| \leq k-1$ such that each connected component of $M-S$ has a cycle. Then the cyclical 4-connectivity of a triangulation G on $\mathbb{P}$ is necessary for a geometric realization of $G-f$ for any face f of G. We conjecture as follows that it is also sufficient.

CONJECTURE 1.2. *Let G be a triangulation on the projective plane $\mathbb{P}$. Then $G-f$ has a geometric realization for any face f of G if and only if G is cyclically 4-connected.*

In this paper, we prove the following.

THEOREM 1.3. *Let G be a 5-connected triangulation on the projective plane $\mathbb{P}$. Then $G-f$ has a geometric realization for any face f of G.*

By Theorem 1.3, a Möbius triangulation M has a geometric realization if M is obtained from a 5-connected triangulation G on $\mathbb{P}$ by removing a 2-cell.

Let M be a 5-connected Möbius triangulation with a boundary cycle $C = v_1 \cdots v_k$ of length k. Let P be the map on $\mathbb{P}$ obtained from M by pasting a 2-cell to C. If $k=3$, then P is a 5-connected triangulation on $\mathbb{P}$. If $k=4$, then P can be extended to a 5-connected triangulation on $\mathbb{P}$ by adding an edge v_1v_3 or v_2v_4. (If this is impossible, then M would have edges v_1v_3 and v_2v_4, and hence M would contain a quadrangulation isomorphic to K_4, contrary to the 5-connectivity of M.) If $k \geq 5$, then P can be extended to a 5-connected triangulation on $\mathbb{P}$ by adding a new vertex joined to all vertices on C. Hence we have the following.

COROLLARY 1.4. *Every 5-connected Möbius triangulation has a geometric realization.*

Let M be a map on a surface Σ with a boundary, and let C be the boundary cycle of M. We say that M is *internally k-connected* if M is $(k-1)$-connected and if for any vertex $v \in V(M-C)$, there are at least k disjoint paths from v to C. Clearly, if G is a 5-connected triangulation on $\mathbb{P}$, then for any $v \in V(P)$, $G - v$ can be regarded as an internally 5-connected Möbius triangulation whose boundary cycle has a length of at least 5. Hence we can relax the condition of Corollary 1.4 to prove the following.

COROLLARY 1.5. *Every internally 5-connected Möbius triangulation has a geometric realization if the boundary cycle has a length of at least 5.*

2. Split-K_5's in 5-connected triangulations.

Put a 5-cycle $C = v_1v_2v_3v_4v_5$ on $\mathbb{P}$, called the *boundary*, so that C bounds a 2-cell R on $\mathbb{P}$, where each v_i is called a *node*. (We always fix its orientation $\vec{C}$ along the numbering of the vertices.) Join v_i to v_{i+2} and v_{i+3} by edges not in R for each i. Then the resulting graph is isomorphic to K_5 in which each face except R is triangular. (See the left-hand side of Figure 2.) Consider a *splitting* (i.e., the inverse operation of an edge contraction) of v_i into two adjacent vertices, v_i and v_i', of degree 3. There are two possibilities for the splitting. When v_i and v_i' lie on C (we always suppose that v_i and v_i' appear on $\vec{C}$ in this order), $\{v_i, v_i'\}$ is called a *boundary pair* of nodes, and each of v_i and v_i' is called a *boundary split node*. (The path from v_i to v_i' on $\vec{C}$ is called the *boundary split interval* of $\{v_i, v_i'\}$.) Otherwise, $\{v_i, v_i'\}$ is called an *inner pair* of nodes, and each of v_i and v_i' is called an *inner split node*, where we always suppose that v_i lies on C. Let K be a map on $\mathbb{P}$ obtained from the above K_5 by splittings of some of v_i's. A *split-K_5* is a subdivision of K on $\mathbb{P}$. (See the right-hand side of Figure 2.)

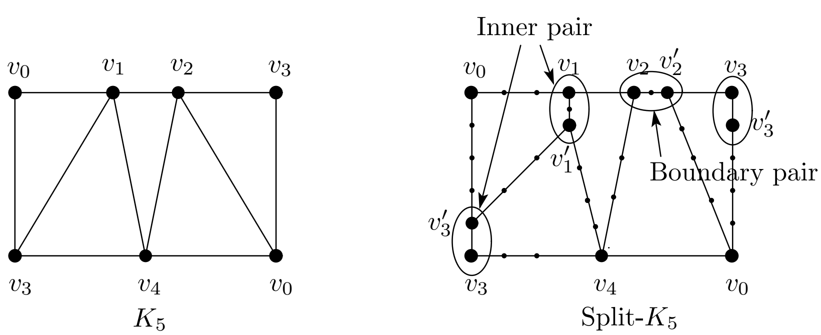

FIG. 2. *K_5 and split-K_5.*

The following is the most important claim in this paper. It guarantees that a 5-connected triangulation on $\mathbb{P}$ has a special type of a split-K_5.

LEMMA 2.1. *Let G be a 5-connected triangulation on $\mathbb{P}$, and let uvw be any face of G. Then G has a split-K_5 H such that*

(i) *the boundary ∂H of H coincides with the link of u in G.*
(ii) *H has at most one boundary pair of nodes.*
(iii) *if H has a boundary pair, then at least one of v and w is a boundary split node, but the edge vw is not contained in a boundary split interval. Otherwise, v or w is a node of H.*

In the following two sections, we give preliminaries for the proof of Lemma 2.1. In section 5, we prove Lemma 2.1.

3. Lemmas. Let G be a graph on $\mathbb{P}$, and let C be a *contractible* cycle of G, i.e., one bounding a 2-cell on $\mathbb{P}$. (A cycle or a closed curve on a surface is *essential* if it is not contractible.) Then C cuts $\mathbb{P}$ into two surfaces, one homeomorphic to an open disk and the other homeomorphic to an open Möbius band. Let $\mathrm{int}_C(G)$ denote the graph consisting of the vertices and edges lying in the disk component of C, and let $\mathrm{Int}_C(G)$ be the graph consisting of the vertices and edges lying on C and in the disk component of C. We define $\mathrm{ext}_C(G)$ and $\mathrm{Ext}_C(G)$ analogously. Note that $\mathrm{Int}_C(G)$ is not necessarily an induced subgraph of G.

Let $C = v_1v_2v_3v_4 \cdots v_k$ be a cycle. A *closed segment* $[v_i, v_j]$ is a $v_i - v_j$ path along $\vec{C}$. An *open segment* (v_i, v_j) is obtained by deleting the endvertices of the corresponding closed segment. Moreover, we use the notations $[v_i, v_j)$ and $(v_i, v_j]$, defined similarly.

Lemma 3.1. *Let G be a 5-connected triangulation on $\mathbb{P}$. Let $C = v_1v_2v_3v_4$ be a contractible 4-cycle in G. Then $\mathrm{int}_C(G)$ contains no vertices.*

Proof. Assume $v \in V(\mathrm{int}_C(G))$. Since G is 5-connected, $\mathrm{ext}_C(G)$ contains no vertices. Then we can add only two edges v_1v_3 and v_2v_4 outside C, since T is simple. Hence this contradicts that G is a triangulation. ☐

Lemma 3.2. *Let v be a vertex of a 5-connected triangulation G on $\mathbb{P}$, and let C be a contractible 5-cycle containing v in its interior. Then there exists a unique contractible 5-cycle $\overline{C}$ so that $\mathrm{Int}_{\overline{C}}(G)$ contains all contractible 5-cycles which contain v in their respective interiors.*

Proof. Let C_1 and C_2 be contractible 5-cycles containing v in their interiors, and suppose that $\mathrm{Int}_{C_1}(G)$ and $\mathrm{Int}_{C_2}(G)$ are *inclusionwise incomparable*, that is, neither $\mathrm{Int}_{C_1}(G) \subseteq \mathrm{Int}_{C_2}(G)$ nor $\mathrm{Int}_{C_1}(G) \supseteq \mathrm{Int}_{C_2}(G)$. It suffices to prove that there is a contractible 5-cycle C' such that $\mathrm{Int}_{C'}(G)$ contains both $\mathrm{Int}_{C_1}(G)$ and $\mathrm{Int}_{C_2}(G)$.

Since C_1 and C_2 are of length 5 and neither one is contained in the closed interior of the other, they intersect in exactly two vertices. These two vertices divide C_i into a segment lying in the interior of C_{3-i} and one lying in the exterior of C_{3-i}, where $i = 1, 2$. Combining the common segments and both interior segments yields a contractible cycle, which contains v in its interior. By Lemma 3.1, its length is at least 5. Combining the two exterior segments with the two common segments, we obtain a contractible cycle C' of length at most 5, since both C_1 and C_2 were 5-cycles. Since G is simple, C' contains no essential cycle, and hence it is a contractible cycle in G. Now C' has length 5 by Lemma 3.1 since it contains v in its interior. On the other hand, $\mathrm{Int}_{C'}(G)$ contains both $\mathrm{Int}_{C_1}(G)$ and $\mathrm{Int}_{C_2}(G)$, and the proof is complete. ☐

Lemma 3.3. *Let G be a 5-connected triangulation on $\mathbb{P}$, and let $C = v_1v_2v_3v_4v_5$ be a contractible 5-cycle in G. If G has no vertex in the exterior of C, then $\mathrm{Ext}_C(G)$ is isomorphic to K_5.*

Proof. We have to show that $\mathrm{ext}_C(G)$ contains every possible edge v_iv_{i+2} (in indices modulo 5). A similar argument as in the proof of Lemma 3.1 does the trick. ☐

The following lemma is an immediate consequence of 5-connectivity.

Lemma 3.4. *Let G be a 5-connected triangulation on $\mathbb{P}$, and let $v \in V(G)$. Let v' and v'' be two nonconsecutive neighbors of v. If v' and v'' have another common neighbor w which is not adjacent to v, then the cycle $vv'wv''$ is essential.*

Let D be a plane graph with boundary cycle C and each inner face triangular, and let x, y be distinct vertices of C. An *internal $x - y$ path* is a path in D joining x and y and intersecting C only at its endvertices.

LEMMA 3.5. *Let D be a triangulation on the disk with boundary cycle C, and let x, y be distinct vertices of C with $xy \notin E(C)$. Then D has an internal $x - y$ path if and only if D has no chord pq for some $p, q \in V(D) - \{x, y\}$ such that x and y are contained in distinct components of $C - \{p, q\}$.*

Proof. The sufficiency is obvious and so we consider the necessity. Suppose that C has a chord pq. By the assumption, x and y are contained in one, say D_1, of the two subgraphs D_1, D_2 such that $V(D) = V(D_1) \cup V(D_2)$ and $V(D_1) \cap V(D_2) = \{p, q\}$. In this case, we have to look for a required internal $x - y$ path in D_1. Hence in the following argument, we may suppose that D has no chord. Observe that since C is chordless, each vertex on C is adjacent to at least one vertex in $D - C$. Moreover, we can see that $\text{int}_C(D)$ is connected. (For otherwise, i.e., if $\text{int}_C(D)$ is disconnected, then there are two vertices $p', q' \in C$ such that $D - \{p', q'\}$ is disconnected. However, this is impossible since each inner face of D is triangular.) Hence we have an internal $x - y$ path in D. ☐

Let C be a contractible cycle of length at least 4 in a triangulation G. Suppose that vertices r_1, r_2, r_3, r_4 lie along C in this order, but they do not need to be consecutive along C. Let us also assume that the segments $[r_1, r_2]$, $[r_2, r_3]$, $[r_3, r_4]$, and $[r_4, r_1]$ have no chords in $\text{Int}_C(G)$. We say that $\text{Int}_C(G)$ is a 4-*patch with nodes* r_1, r_2, r_3, r_4.

We obtain the following three lemmas, carefully applying Lemma 3.5 to P.

LEMMA 3.6. *Let P be a 4-patch with nodes r_1, r_2, r_3, r_4. Assume that $r_1r_4, r_2r_3 \in E(P)$ and that u and v are vertices from (r_1, r_2) and (r_3, r_4), respectively. Then $P - \{r_1, r_2, r_3, r_4\}$ contains an $u - v$ path, or a pair of antipodal nodes are adjacent.*

LEMMA 3.7. *Let P be a 4-patch with nodes r_1, r_2, r_3, r_4. Then $P - \{r_1, r_3\}$ contains an $r_2 - r_4$ path unless $r_1r_3 \in E(P)$.*

Let P be a 4-patch with nodes r_1, r_2, r_3, r_4. An $r_2 - r_4$ *diagonal* in P is an $r_2 - r_4$ path $Q = u_1u_2u_3 \cdots u_{k-1}u_k$ ($u_1 = r_2$ and $u_k = r_4$) in $P - \{r_1, r_3\}$ if there exists indices $i < j$ such that

- (D1) the *initial segment* $u_1 \cdots u_i$ is a segment of ∂P,
- (D2) the *terminal segment* $u_j \cdots u_k$ is a segment of ∂P, and
- (D3) the *intermediate segment* $u_i \cdots u_j$ is a segment of P such that $u_i, u_j \in V(\partial P)$ and that all other vertices lie in $\text{int}(P)$.

If Q is an $r_2 - r_4$ diagonal in P, then it is also an $r_4 - r_2$ diagonal. Further, if a patch P with nodes r_1, r_2, r_3, r_4 contains an $r_2 - r_4$ path avoiding r_1 and r_3, then it also contains an $r_2 - r_4$ diagonal.

We say that an $r_2 - r_4$ diagonal Q lies *closest to* r_1 if the number of faces of P bounded by Q and the segments incident with r_1 is as small as possible.

LEMMA 3.8. *Let P be a 4-patch with nodes r_1, r_2, r_3, r_4, and let Q be the $r_2 - r_4$ diagonal closest to r_1. Let u_i and u_j be the first and last vertex of the intermediate segment of Q, respectively. Then r_1 is adjacent to $u_i, u_{i+1}, \ldots, u_{j-1}, u_j$ in P.*

4. Essential 3-linkages. A *near triangulation* R is a map on $\mathbb{P}$ with a distinguished face f such that every other face of R is triangular, and that the facial walk along f is a cycle. Suppose that the boundary cycle of f, denoted by W, has a length of at least 6. Let $v_1, v_2, v_3, v_4, v_5, v_6$ be six vertices that appear along W in this order but that do not need to be consecutive along W. An *essential* 3-*linkage* (with respect to $v_1, v_2, v_3, v_4, v_5, v_6$) is a collection L of three disjoint paths P_1, P_2, P_3 so that P_i is a $v_i - v_{i+3}$ path for $i = 1, 2, 3$. It is easy to see that $W \cup P_i$ contains some essential cycle. Let Q_1 be some minimal subpath of P_1 so that $W \cup Q_1$ still contains an essential cycle. Also Q_1, P_2, P_3 form an essential 3-linkage with possibly different endvertices. By applying the same idea on P_2 and P_3, we obtain the following lemma.

LEMMA 4.1. *Let L be an essential 3-linkage with respect to nodes $v_1, \dots, v_6$. There exists an essential 3-linkage L' so that every path in L' intersects W only at its endvertices.*

The second result has been, in greater generality, proved by Robertson and Seymour in [8]. We state it adapted to our needs.

THEOREM 4.2 (Robertson and Seymour [8]). *Let R be a near triangulation of $\mathbb{P}$ and $f = v_1v_2v_3v_4v_5v_6$ its distinguished face of length 6. Then R contains an essential 3-linkage with respect to $v_1, v_2, v_3, v_4, v_5, v_6$ if and only if*

(L1) *R contains no pair of parallel nonhomotopic edges with common endvertices;*

(L2) *R does not contain a contractible cycle C of length at most 5 whose interior contains f.*

A pair of parallel nonhomotopic edges violating (L1) forms an essential cycle of length 2. Traversing these two edges twice yields a contractible (but not simple) closed walk whose "interior" contains all faces of R. This observation enables both conditions (L1) and (L2) to be combined into a single condition, albeit with slight adaptations. For practicality, we prefer the conditions to be written separately, since they are of different flavors and have to be tackled with different approaches.

We look for essential 3-linkages in near triangulations. In the case when the length of the distinguished face exceeds 6, we first decide which six vertices are the endvertices of a linkage. The rest of this section is devoted to the proof of the following.

PROPOSITION 4.3. *Let G be a 5-connected triangulation of $\mathbb{P}$, and let v be a vertex of degree $d \geq 6$. Let $D = u_1u_2 \cdots u_d$ be the link of v in G. Then the near triangulation $R = G - v$ contains an essential 3-linkage if and only if v is not contained in the interior of a contractible cycle of length at most 5.*

Proof. Clearly a cycle containing v in its interior meets each path in an essential 3-linkage at least twice. The difficulty lies in the other direction—how to find a linkage—if v is not contained in the interior of a "short" contractible cycle.

An edge $e \in E(R)$ is said to be *essential* if the endvertices of e lie in D and $D \cup e$ contains an essential cycle. We shall split the proof of Proposition 4.3 with respect to the number of essential edges. If R contains a set of three independent essential edges, then no further proof is needed. This leaves us with the case where a maximal set of independent essential edges contains at most two edges.

Assume next that R contains a set of two independent essential edges. The four endvertices of these essential edges split the f-facial walk into four open segments. Let us choose essential edges $e = r_1r_4$ and $e' = r_3r_6$ in such a way that the union of two consecutive open segments $(r_1, r_6) \cup (r_3, r_4)$ in D contains as few vertices as possible. Suppose that (r_1, r_3) contains a vertex, say v_2, and that (r_4, r_6) contains a vertex, say v_5. Now if $r_1r_6 \in E(R) - E(D)$, then the contractible cycle $vr_4r_1r_6$ separates v_2 from v_5, and if $r_3r_4 \in E(R) - E(D)$, then the contractible cycle $vr_3r_4r_1$ separates v_2 from v_5. Neither can happen since G is 5-connected. By Lemma 3.6, we can join v_2 and v_5 by a path avoiding r_1, r_2, r_3, and r_4, and hence we can find an essential 3-linkage.

So we assume that there exists a set of two independent essential edges $e = w_1w_3$ and $e' = w_2w_4$ so that w_1, w_2, and w_3 lie consecutively along D. We may also assume that w_4 lies closer to w_3 than to w_1 along D, and that no essential edge incident with w_2 has the other endvertex in (w_3, w_4). Denote the vertices along D by $v_1, v_2, v_3, \dots, v_d$ so that $v_1 = w_1$ and $v_2 = w_2$ (also $v_3 = w_3$, but then this may not go on). Add to R the *new* edges v_1v_k, where $k = 6, \dots, d-1$, and denote the resulting near triangulation with R', with the distinguished face of size 6.

It is easy to see that R' satisfies (L1), since the newly added edges do not have their essential counterparts. Similarly, a short contractible cycle C containing the distinguished face of R' in its interior, i.e., contradicting (L2), would have to use some new edge v_1v_k, where $k \geq 6$. Now C would contain vertices v_k, w_1, w_2, and w_3, which implies that vertices v_k and w_3 have a common neighbor in R. This contradicts Lemma 3.4 since C is contractible. Hence R' contains an essential 3-linkage. Since all new edges share a common endvertex, we can, if necessary, transform the linkage into an essential 3-linkage in R.

Suppose next that there is an essential edge but we cannot find a set of two independent essential edges. Let $e = w_1w_2$ be the essential edge, and assume that the segment (w_1, w_2) is as short as possible. Since G is simple, w_1 and w_2 are not consecutive along D. Denote the vertices of D so that $w_1 = v_3$ and v_4 lies in (w_1, w_2). As (w_1, w_2) is as short as possible, we have $w_2 \neq v_1$.

As in the previous case, let R' be the near triangulation obtained by adding *new* edges v_1v_k, where $k = 6, \ldots, d-1$. We will argue that R' has an essential 3-linkage.

If R' does not satisfy (L1), then an essential edge e' must be incident with both v_1 and v_k for some k satisfying $6 \leq k \leq d$. By interlacing essential edges incident to $v_k \in [w_1, w_2] = [v_3, w_2]$, we clearly have $v_k \neq v_3$. On the other hand, v_k cannot lie in $(w_1, w_2) = (v_3, w_2)$, as two independent essential edges cannot exist, and hence $v_k = w_2$. But this contradicts 5-connectivity of G, since the 4-cycle $vv_1v_kv_3 = vv_1w_2w_1$ separates v_2 and v_4.

Next assume that R' contradicts (L2). The short cycle C contradicting (L2) can be divided into three segments: the first one between v_1 and w_1, the second between w_1 and w_2, and the third between w_2 and v_1. Their lengths are at least 2, 2, and 1, respectively, using the fact that neither v_1 and $w_1 = v_3$ nor w_1 and w_2 are consecutive along D, and the fact that C uses one of the new edges. Since the length of C is at most 5, all lower bounds are sharp. By Lemma 3.4, C must pass through v_2, and also C must pass through v_4 and $w_2 = v_5$. On the segment between w_2 and v_1 the cycle C uses exactly one edge, namely $v_1w_2 = v_1v_5$, and it also has to use one new edge. This is a contradiction, so R' satisfies both (L1) and (L2), and R' contains an essential 3-linkage. As in the previous case we can, if necessary, transform the linkage into an essential 3-linkage in R.

We are left with the case where R contains no essential edges. Even if we add new edges to the interior of f, we cannot contradict (L1), and our only concern will be meeting the condition (L2).

We proceed naively. Let us assign labels $v_1, v_2, \ldots, v_d$ to neighbors of v in the order of their indices. Add new edges of the form v_1v_k, where $k = 6, \ldots, d-1$. The newly obtained near triangulation R' may contain an essential 3-linkage, and we win. On the other hand, it may not, as we contradict (L2), and we *lose*. In this case, R' contains a short cycle C which uses a new edge v_1v_ℓ for some $\ell \in \{6, \ldots, d-1\}$.

Hence we assume that we lose for every assignment of labels $v_1, v_2, \ldots, v_d$ to the consecutive neighbors of v. Now fix an assignment of labels so that there exists a cycle C_v contradicting (L2) using a new edge v_1v_k, where k is as large as possible.

Let us denote $w_1 = v_1$, $w_2 = v_2$, $w_3 = v_3$, $w_4 = v_{k-1}$, $w_5 = v_k$, and $w_6 = v_{k+1}$. Further, let us add new edges joining w_1 to vertices of (w_6, w_1) and additional new edges joining w_3 to vertices of (w_3, w_4). We denote the newly obtained near triangulation by R_w. We claim that R_w contains an essential 3-linkage.

Assume that this is not the case, and let C_w be the obstruction according to (L2). Clearly C_w contains at least one new edge. Observe that C_w cannot contain both a new edge incident with w_1 and a new edge incident with w_3, since a segment of C_w of length at most 2 would join two nonconsecutive vertices of D. The cycle C_w cannot contain a new edge incident with w_1 since this would contradict maximality of k. Hence, C_w contains a new edge incident with w_3. Now let C' be the cycle containing the edges of C_w lying outside C_v and the edges of C_v lying outside of C_w. Then C' is a contractible cycle containing f in its interior. Let $P \subseteq C_w \cup C_v$ be the $v_1 - v_3$ path whose edges lie in the interior of C'. Since it connects two nonconsecutive vertices along f, its length is at least 3. This implies that the length of C' is at most 5, a contradiction.

Hence R_w contains an essential 3-linkage, and consequently R also contains an essential 3-linkage. This completes the proof of Proposition 4.3. □

5. Proof of Lemma 2.1. In this section, we shall prove Lemma 2.1. We begin with the following proposition.

PROPOSITION 5.1. *Let G be a 5-connected triangulation on $\mathbb{P}$, and let $u \in V(G)$. Then G has a split-K_5 H whose boundary coincides with the link of u in G.*

Proof. We will split the analysis into two cases regarding the properties of u and treat one of the two cases by referring to [6]. Let D be the link of u.

Case 1. G contains a contractible 5-cycle $C = v_1v_2v_3v_4v_5$ such that $u \in V(\mathrm{int}_C(G))$.

By Lemma 3.2, we may assume that C is the maximal 5-cycle containing u in its interior. Since G is 5-connected, there exist internally disjoint $u - v_i$ paths P_i for $i = 1, \dots, 5$.

In order to find a suitable split-K_5, we need to find a subgraph of $\mathrm{Ext}_C(G)$ which contracts to the zigzag cycle $v_1v_3v_5v_2v_4$. This task has been treated in greater generality in [6, subsection: Finding a suitable cycle minor U in G_x]. Hence we can obtain a split-K_5 H' whose boundary is C. Now let

$$H = (H' - E(C)) \cup D \cup \bigcup_{i=1}^{5} (P_i - \{v\}).$$

Then H is a split-K_5 with boundary D, in which there is no boundary pair.

Case 2. u does not lie in the interior of a contractible 5-cycle.

Then we clearly have $|D| = \deg(u) = k \geq 6$. Let f be the distinguished face of $G - v$ with boundary D. By Theorem 4.2, $G - v$ contains an essential 3-linkage $L = \{P_1, P_2, P_3\}$ with respect to $u_1, u_2, u_3, u_4, u_5, u_6$, where P_i joins u_i and u_{i+3} for $i = 1, 2, 3$. We may also assume that each P_i in L has no chord. Then L divides the near triangulation $G - v$ into three patches R_{12}, R_{23}, and R_{13}, whose nodes are (u_1, u_2, u_5, u_4), (u_2, u_3, u_6, u_5), and (u_3, u_4, u_1, u_6) lying on their boundary in this order, respectively.

We first claim that these patches contain two vertex-disjoint diagonals. Let us first prove that every two patches, say R_{12} and R_{23}, contain diagonals with disjoint endvertices. Suppose this is not the case, and let, say, u_2 be an endvertex of every possible diagonal in both R_{12} and R_{23}. By Lemma 3.7, we have $u_2u_4 \in E(R_{12})$ and $u_2u_6 \in E(R_{23})$. This contradicts the 5-connectivity of G since $\{u, u_2, u_4, u_6\}$ separates v_5 and v_1 in G. Hence we may assume that R_{12} contains a $u_1 - u_5$ diagonal D_{15} and that R_{23} contains a $u_2 - u_6$ diagonal D_{26}. We first suppose that D_{15} and D_{26} are disjoint. In this case, we can obtain a required split-K_5 H such that $H = D \cup L \cup D_{15} \cup D_{26}$.

Now consider the case when D_{15} and D_{26} share an inner vertex. Let us try to push the diagonals away: suppose that D_{15} and D_{26} are closest to u_4 and u_3, respectively. If D_{15} and D_{26} are not vertex disjoint, then the terminal segment S of D_{15} intersects the initial segment S' of D_{26} at P_2. Let w be the first vertex of S, and let w' be the last vertex of S'. Then, by Lemma 3.8, we have both $u_4w \in E(R_{12})$ and $u_3w' \in E(R_{23})$. If $w \neq w'$, then we can find a $u_2 - u_4$ diagonal in R_{12} through wu_4 and a $u_3 - u_5$ diagonal in R_{23} through u_3w'. Since they are disjoint, we are done, similarly as above.

Suppose that $w = w'$. Since $u_4w \in E(R_{12})$, we focus on the 4-patch R'_{12} with nodes u_1, u_2, w, u_4 contained in R_{12}. Note that $u_1w \notin E(R'_{12})$. (For otherwise, $\{u, u_1, w, u_3\}$ separates u_2 and u_4, since $u_3w \in E(R_{23})$. This contradicts the 5-connectivity of G.) Hence R'_{12} admits a $u_2 - u_4$ diagonal D_{24}, avoiding w and u_1, by Lemma 3.7. Let D_{35} be the $u_3 - u_5$ diagonal of R_{23} through u_3w. Then $D \cup L \cup D_{24} \cup D_{35}$ is a required split-K_5 in G since D_{24} and D_{35} are disjoint. ∎

By Proposition 5.1, a 5-connected triangulation on $\mathbb{P}$ has a split-K_5 H whose boundary coincides with the link of a specified vertex. Let $[a, b]$ denote the path in H joining two vertices a and b which is contained in the path joining two nodes in H, where $1 \leq i < j \leq 5$. Moreover, we denote $(a, b) = [a, b] - \{a, b\}$, and also use the notations $[a, b)$ and $(a, b]$ similarly.

The following claims that a boundary pair of nodes can be "moved" in a sense.

LEMMA 5.2. *Suppose that a triangulation G on $\mathbb{P}$ has a split-K_5 H with boundary C. Let $\{a', a''\}$ be a boundary pair of nodes of H, and let Q be the plane subgraph of G corresponding to a face of H with nodes a', a'', b, c. Then, for some vertex a of $[a', a'']$ in G, we can find a split-K_5 H' with boundary C such that a is a node of H' contained in neither a boundary pair nor an inner pair. Moreover, if b is contained in a boundary pair, then the number of the boundary pairs can be decreased in H'; otherwise, b might be contained in a new boundary pair of H'.*

Proof. We may suppose that a vertex y of $(a', c]$ and a vertex z of $(a', a'']$ are not adjacent in Q. (For otherwise, replacing $[a', y)$ with zy, we can regard z as a new a'.) Then, by Lemma 3.5, we can take an internal $a' - x$ path P for some x on either $(a'', b]$ or (b, c). In the former case, let $H' = H - (a'', x) \cup P$ (or $H' = H - (a'', b') \cup P$ when x is in (b, b') for an inner pair $\{b, b'\}$). See Figure 3. Then we can decrease the number of boundary split pairs. In the latter case, let $H' = H - (a'', b) \cup P$ (or $H' = K - (a'', b') \cup P$ when $\{b, b'\}$ is an inner pair), in which x might be a new boundary pair. ∎

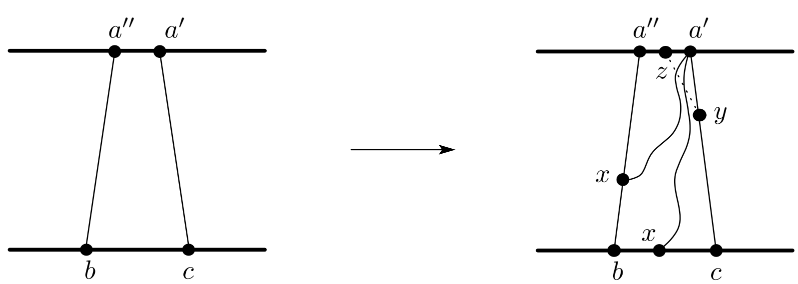

FIG. 3. *Eliminate or move a boundary split node.*

Now we shall prove Lemma 2.1.

Proof of Lemma 2.1. Let G be a 5-connected triangulation on $\mathbb{P}$, and let uvw be any face of G. By Proposition 5.1, G has a split-K_5 H whose boundary ∂H coincides with the link of u in G. Let v_1, v_2, v_3, v_4, v_5 be five nodes of H (where $\partial \vec{H}$ is fixed along the ordering of $v_1, \ldots, v_5$); some v_i's might be contained in boundary or inner pairs $\{v_i, v'_i\}$ of nodes.

We shall deform H to satisfy conditions (ii) and (iii) in the lemma. We may suppose that the edge vw is contained in $[v_1, v_2]$ so that $\vec{vw}$ is along $\partial\vec{H}$. Moreover, we may suppose that neither v_1 nor v_2 is a boundary split node. (For otherwise, we can apply Lemma 5.2 to $\{v_1, v_1'\}$ or $\{v_2, v_2'\}$.)

We first show that one of v and w can be chosen as a node in a new split-K_5. Hence we may suppose that $v \neq v_1$ and $w \neq v_2$. Let R be the plane subgraph of G corresponding to a face of H incident to $[v_1, v_2]$. Suppose that R is bounded by $[v_1, v_2]$, $[v_1, v_4]$, $[v_4, v_4']$, and $[v_2, v_4']$ of H, when $\{v_4, v_4'\}$ is a boundary split pair. (See Figure 4. Since the other two cases shown in the figure are similar, we omit the details.) Observe that there are no two vertices x and y in $[v_1, v_2]$ joined by a chord. (For otherwise, $\{x, y, u\}$ would be a 3-cut of G, contrary to the 5-connectivity of G.) Hence, by Lemma 3.5, we can find an internal path P from v to a vertex on $(v_1, v_4]$, to a vertex on $(v_4, v_4']$, or to $(v_2, v_4']$. In the first and second cases, adding P to H and deleting a segment suitably, we obtain a split-K_5 with v a node. If we do not have these cases, then there is a vertex s in $[v_1, v)$ and a vertex t in (v_4', v_2) which are adjacent in R. In this case, we must have an internal path P' from w to some vertex r of (v_4', v_2) in R. Similarly to the previous two cases, we obtain a split-K_5 with w a node.

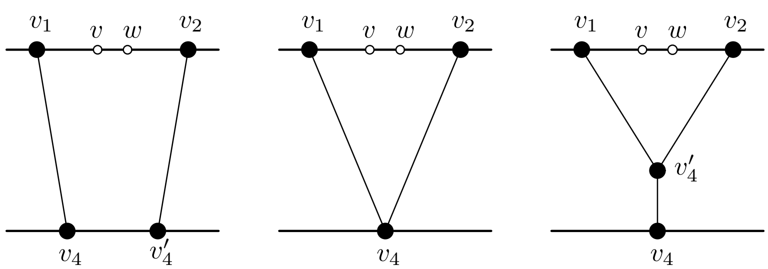

FIG. 4. *Take a path from v or w.*

We may suppose that v is a node. If v is a boundary split node, then put $v = v_1'$, and suppose that vw is contained in $[v_1', v_2]$. Otherwise, put $v = v_1$. If v_4 is contained in a boundary pair $\{v_4, v_4'\}$, then we apply Lemma 5.2 to eliminate the boundary pair $\{v_4, v_4'\}$, fixing v, or move the boundary pair toward v_2. (Otherwise, we proceed to v_2.) Then, fixing the new v_4, we apply Lemma 5.2 to $\{v_2, v_2'\}$ if $\{v_2, v_2'\}$ is a boundary split pair. Similarly, we apply Lemma 5.2 to $\{v_5, v_5'\}$ and $\{v_3, v_3'\}$ in this order if necessary. Then, the resulting split-K_5 has at most one boundary split pair containing v.

6. Proof of the theorem.

In this section, we shall prove Theorem 1.3. The main part of the proof, which is to make a geometric realization of a 5-connected triangulation G on $\mathbb{P}$ with any one face f removed, depends on the technique developed in [3].

LEMMA 6.1 (Bonnington and Nakamoto [3]). *Let T be a Möbius triangulation with boundary C. Suppose that T has a split-K_5 H with boundary C and at most one boundary pair of nodes.*

(i) *If H has no boundary pair and we let v_1, v_2, v_3, v_4, v_5 be the nodes of T lying on C in this order, then let e be the edge of $[v_1, v_2]$ incident to v_1.*

(ii) *If H has a boundary pair $\{v_1, v_1'\}$ and we let $v_1, v_1', v_2, v_3, v_4, v_5$ be the nodes of T lying on C in this order, then let e be the edge of $[v_1', v_2]$ incident to v_1'.*

Then T has a geometric realization $\hat{T}$ such that all edges on C except e can be seen from some fixed point $x \in \mathbb{R}^3$.

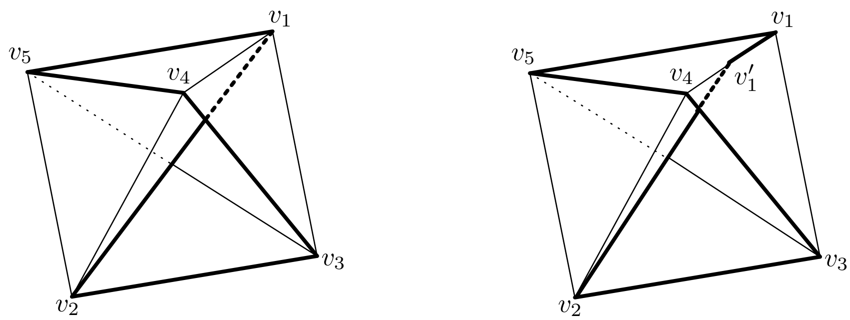

FIG. 5. *Examples of geometric realizations of T.*

Figure 5 shows examples of geometric realizations of split-K_5's satisfying Lemma 6.1. The left-hand side shows one with exactly five nodes v_1, v_2, v_3, v_4, v_5 on the boundary, and the right-hand side shows one with exactly one boundary split pair $\{v_1, v_1'\}$. (Note that a triangulation G dealt with in Lemma 6.1 might have several inner pairs of nodes.) In both parts of figure, we can see all segments on ∂H, except a side of $[v_1, v_2]$ incident to v_1 in the left-hand case and a side of $[v_1', v_2]$ incident to v_1' in the right-hand case.

Now we shall prove Theorem 1.3.

Proof of Theorem 1.3. Let G be a 5-connected triangulation on $\mathbb{P}$, and let f be any face of G bounded by uvw. Let C be the link of u. Then, by Lemma 2.1, G contains a split-K_5 H such that

(i) the boundary ∂H of H coincides with C,

(ii) H has at most one boundary split pair, and

(iii) if H has a boundary pair, then v is a boundary split node of H, but vw is not contained in a boundary split interval; otherwise, v or w is a node of H.

Consider the Möbius triangulation $G' = G - u$ with boundary C. We apply Lemma 6.1 to G' and the above H. Then we get a geometric realization $\hat{G}'$ of G' such that from some point $x \in \mathbb{R}^3$, all edges on C except vw can be seen.

First, we put the vertex u at $x \in \mathbb{R}^3$. For each edge pq of $\hat{G}'$ lying on C, let $\Delta_{pq} \in \mathbb{R}^3$ denote the triangular disk with x, p, q as its vertices. Now, for any edge $h \in E(C) - \{vw\}$, we shall fit Δ_h into the body of $\hat{G}'$, where Δ_h corresponds to a face of G incident to h and v. Since each $h \in E(C) - \{vw\}$ can be seen from $x \in \mathbb{R}^3$, the interior of Δ_h does not collide with $\hat{G}'$. Moreover, for any two distinct $h, h' \in E(C) - \{vw\}$, the interiors of Δ_h and $\Delta_{h'}$ do not collide internally, since h and h' can be seen from x simultaneously. So we get a geometric realization of $G - f$. ∎

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