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Rebuilding convex sets in graphs

José Cáceres[a,1], Alberto Márquez[b,2], Ortrud R. Oellermann[c,3], María Luz Puertas[a,4]

[a] *Department of Statistics and Applied Mathematics, University of Almeria, Spain*
[b] *Department of Applied Mathematics I, University of Seville, Spain*
[c] *Department of Mathematics and Statistics, The University of Winnipeg, 515 Portage Avenue, Winnipeg, Manitoba, Canada R3B 2E9*

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Abstract

The usual distance between pairs of vertices in a graph naturally gives rise to the notion of an interval between a pair of vertices in a graph. This in turn allows us to extend the notions of convex sets, convex hull, and extreme points in Euclidean space to the vertex set of a graph. The extreme vertices of a graph are known to be precisely the simplicial vertices, i.e., the vertices whose neighborhoods are complete graphs. It is known that the class of graphs with the Minkowski–Krein–Milman property, i.e., the property that every convex set is the convex hull of its extreme points, is precisely the class of chordal graphs without induced 3-fans. We define a vertex to be a contour vertex if the eccentricity of every neighbor is at most as large as that of the vertex. In this paper we show that every convex set of vertices in a graph is the convex hull of the collection of its contour vertices. We characterize those graphs for which every convex set has the property that its contour vertices coincide with its extreme points. A set of vertices in a graph is a geodetic set if the union of the intervals between pairs of vertices in the set, taken over all pairs in the set, is the entire vertex set. We show that the contour vertices in distance hereditary graphs form a geodetic set.

Keywords: Eccentricity; Contour vertex; Distance hereditary graph; Convex hull; Geodetic set

E-mail address: o.oellermann@uwinnipeg.ca (O. Oellermann).
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1. Introduction

The study of abstract convexity began in the early fifties with the search for an axiom system that defines a convex set and in some way generalizes the classical concept of a Euclidean convex set. Numerous contributions to this topic have been made. An extensive survey of this subject can be found in [20].

Among the wide variety of structures that have been studied under abstract convexity are metric spaces, ordered sets or lattices and graphs, the last being the focus of this paper. We now give a brief introduction to abstract convexity as it pertains to graphs. Let V be a finite set and $\mathscr{M}$ a finite collection of subsets of V. Then $\mathscr{M}$ is an *alignment* of V if and only if $\mathscr{M}$ is closed under intersection and contains both V and the empty set. If $\mathscr{M}$ is an alignment of V, then the elements of $\mathscr{M}$ are called *convex sets* and the pair $(V, \mathscr{M})$ is called an *aligned space*. If $S \subseteq V$, then the convex hull of S, denoted by $\mathrm{CH}(S)$, is the smallest convex set that contains S. Suppose $X \in \mathscr{M}$. Then, $x \in X$ is an *extreme point* for X if $X - \{x\} \in \mathscr{M}$. The collection of all extreme points of X is denoted by $\mathrm{ex}(X)$. A *convex geometry* on a finite set is an aligned space with the additional property that every convex set is the convex hull of its extreme points. This property is referred to as the *Minkowski–Krein–Milman property*. Several abstract convexities associated with the vertex set of a graph are well known (see [10]). Their study is of interest in computational geometry and has some direct applications to other areas such as, for example, game theory (see [4]).

For graph terminology we follow [14]; except that we use vertex instead of point and edge instead of line. All graphs considered here are connected, finite, simple, unweighted and undirected. The *distance* between a pair of vertices u, v of G is the length of a shortest u–v path in G and is denoted by $d_G(u, v)$ or $d(u, v)$ if G is clear from context. The *interval* between a pair u, v of vertices in a graph G is the collection of all vertices that lie on some shortest u–v path in G and is denoted by $I_G[u, v]$ or $I[u, v]$ if G is understood. Intervals in graphs have been studied extensively (see [2,17,18]) and play an important role in the study of several classes of graphs such as the Ptolemaic graphs (see [16]) or block graphs. A subset S of vertices of a graph is said to be g*-convex* if it contains the interval between every pair of vertices in S. It is not difficult to see that the collection of all g-convex sets is an alignment of V. We thus refer to the g-convex sets simply as convex sets. A vertex in a graph is *simplicial* if its neighborhood induces a complete subgraph. It can readily be seen that p is an extreme point for a convex set S if and only if p is simplicial in the subgraph induced by S. It is true, in general, that the convex hull of the extreme points of a convex set S is contained in S, but equality holds only in special cases. In [10] it is shown that a graph has the Minkowski–Krein–Milman property if and only if it has no induced cycles of length bigger than 3 and has no induced 3-fan (see Fig. 1). For another more recent and excellent reference text containing material on graph convexity see [6].

If a graph G has the Minkowski–Krein–Milman property and S is a convex set of $V(G)$, then we can rebuild the set S from its extreme vertices using the convex hull operation. Since this cannot be done with every graph, using only the extreme vertices of a given convex set S, it is natural to ask if it is possible to extend the set of extreme vertices of S to a set that allows us to rebuild S using the vertices in this extended set and the convex hull operation. In Section 2 we answer this question in the affirmative using the collection of ‘contour vertices’ of a set. To this end, let S be a set of vertices in a graph G and recall that

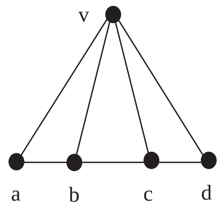

Fig. 1. A 3-fan.

the *eccentricity in* S of a vertex $u \in S$ is given by $\mathrm{ecc}_S(u) = \max\{d(u, v) : v \in S\}$ and a vertex $v \in S$ for which $d(u, v) = \mathrm{ecc}_S(u)$ is called an *eccentric vertex* for u in S. In case $S = V(G)$, we denote $\mathrm{ecc}_S(u)$ by $\mathrm{ecc}(u)$. A vertex $u \in S$ is said to be a *contour vertex* of S if $\mathrm{ecc}_S(u) \geqslant \mathrm{ecc}_S(v)$ for every neighbor v of u in S. The set of all contour vertices of S is called the *contour* set of S and is denoted by $\mathrm{Ct}(S)$. If $S = V(G)$, the subgraph induced by the contour set of S is called the *contour* of G and is denoted by $\mathrm{Ct}(G)$. In Section 3 we establish structural properties of contour vertices and characterize those graphs that are the contour of some other graph using a construction similar to the one used in [3].

In order to find the convex hull of a set S one begins by taking the union of the intervals between pairs of vertices of S, taken over all pairs of vertices in S. We denote this set by $I_G[S]$ or $I[S]$, i.e., $I[S] = \bigcup_{\{u,v\} \subseteq S} I[u, v]$ and call it the *geodetic closure* of S. One then repeats this procedure with the new set and continues until, for the first time, one reaches a set T for which the geodetic closure is the set itself , i.e., $T = I[T]$. This then is the convex hull of S. If this procedure only has to be performed once, we say that the set S is a *geodetic set* for its convex hull. In general a subset S of a convex set T is a *geodetic set* for T if $I[S] = T$. The notion of a geodetic set for the vertex set of a graph was first defined in [7].

In Section 4 we focus on geodetic sets in 'distance hereditary graphs'. We first discuss here how these graphs are related to the graphs with the Minkowski–Krein–Milman property and how the results of Section 4 extend results known for the last class. Howorka [15] defined a connected graph G to be *distance hereditary* if for every connected induced subgraph H of G and every two vertices u, v in H, $d_H(u, v) = d_G(u, v)$. In the same paper several characterizations for this class of graphs are given. We state here only one of these which we will use in this paper.

Theorem 1. *A connected graph G is distance hereditary if and only if every cycle in G of length at least* 5 *has a pair of crossing chords.*

Further useful characterizations for this class of graphs were established in [1,9,13]. Apart from having elegant characterizations, distance hereditary graphs possess other useful properties. It is a class of graphs for which several NP-hard problems have polynomial solutions. For example the Steiner problem for graphs, which is known to be NP-hard (see [11]), can be solved in polynomial time in distance hereditary graphs (see [5,8,12]). Moreover, these graphs are Steiner distance hereditary as was shown in [9]; i.e., the Steiner distance of a set of vertices is the same, in any connected induced subgraph that contains it, as it is in the graph itself.

The class of distance hereditary graphs also properly contains the graphs that possess the Minkowski–Krein–Milman property since a graph is chordal without an induced 3-fan if and only if it is a distance hereditary graph without an induced 4-cycle. It was shown in [10] that in a chordal graph every non-simplicial vertex lies on a chordless path between two simplicial vertices. If G is a chordless graph without an induced 3-fan, then G is distance hereditary and thus every induced path is necessarily a shortest path. Hence the simplicial vertices for a convex set S in a graph with the Minkowski–Krein–Milman property is a geodetic set for S. In Section 4 we show that the contour vertices of a distance hereditary graph form a geodetic set for the graph. In [19] it shown that the contour vertices can be used to find minimum Steiner geodetic sets for distance hereditary graphs.

2. The contour set of a graph

In this section we will show that the contour set of a convex set S of vertices in a graph G can be used to rebuild the set by finding its convex hull, in the same way that extreme vertices are used in chordal graphs in [10]. Moreover, we characterize those graphs having the property that the extreme vertices and the contour vertices of every convex set coincide. First we show that the contour set of G contains all the extreme vertices.

Lemma 2. *Let G be a graph and $S \subseteq V(G)$. Then $\mathrm{Ct}(S)$ contains all extreme vertices of S.*

Proof. Let $u \in S$ be an extreme vertex for S. Then u is a simplicial vertex for S. We now show that u is a contour vertex of S. Let v be a neighbor of u in S and $v_e \in S$ an eccentric vertex for v in S, i.e., $d(v, v_e) = \mathrm{ecc}_S(v)$. Suppose that $d(u, v_e) = d(v, v_e) - 1$ and let P be a shortest u–v_e path. Then the vertex following u on P, say w, is not v. Since u is simplicial, v and w must be adjacent. However, then $d(u, v_e) \geqslant d(v, v_e)$, a contradiction. So $\mathrm{ecc}_S(u) \geqslant d(u, v_e) \geqslant d(v, v_e) = \mathrm{ecc}_S(v)$ and therefore u is a contour vertex for S. $\square$

The relationship between contour vertices and extreme vertices is even closer for the class of distance hereditary graphs without induced 4-cycles. The next result is a characterization of contour vertices in graphs with the Minkowski–Krein–Milman property that resembles the characterization of simplicial vertices.

Proposition 3. *Let G be a distance hereditary graph without induced 4-cycles. A vertex $x \in V(G)$ is a contour vertex for G if and only if each neighbor v of x which is on a shortest path between x and some eccentric vertex for x satisfies $N[x] \subseteq N[v]$.*

Proof. If G is complete, the result is immediate. Suppose now that G is not complete. Then no contour vertex of G can have eccentricity 1. So if x is a contour vertex and x_e is an eccentric vertex for x, then $d(x, x_e) \geqslant 2$. Let $P : (x=)y_0 y_1 \ldots y_k (=x_e)$ be a shortest x–x_e path. Suppose $u \neq y_1$ is a neighbor of x. Then u is not on P and uP cannot be a shortest u–x_e path; otherwise, $\mathrm{ecc}(u) > \mathrm{ecc}(x)$ which is not possible since x is a contour vertex. Since G is distance hereditary, the subgraph induced by u and the vertices of P contains a shortest u–x_e path. Hence there is a chord between u and some vertex on P whose distance from x is less than or equal to 2. If uy_1 is a chord, then $u \in N(y_1)$ as desired. If u is a neighbor of y_2, then the 4-cycle xy_1y_2ux must have a chord. So uy_1 is an edge and again $u \in N(y_1)$.

Conversely, suppose that x has the property that each of its neighbors v which is on a shortest path between x and some eccentric vertex for x satisfies $N[x] \subseteq N[v]$. Suppose x has a neighbor y such that $\mathrm{ecc}(x) < \mathrm{ecc}(y)$. Then x lies on a shortest path P between y and an eccentric vertex y_e for y. So y_e is also an eccentric vertex for x. By our hypothesis y is a neighbor of the vertex adjacent to x in $P - y$. This is not possible as P is a shortest y–y_e path. So $\mathrm{ecc}(y) \leqslant \mathrm{ecc}(x)$ and x is a contour vertex for G. □

Remark 4. The above result does not hold for all chordal graphs. Take for example the 3-fan of Fig. 1. For this graph both the neighbors, of either one of the two simplicial vertices, lie on some shortest path to an eccentric vertex but their closed neighborhoods are not equal. However, the converse of the above result holds for all connected graphs G, i.e., if a vertex $x \in V(G)$ has the property that for each neighbor v of x which is on a shortest path between x and some eccentric vertex for x, $N[x] \subseteq N[v]$, then x is a contour vertex.

The following result shows that the convex hull of the contour set of a convex set of vertices in a graph is the entire set, without any restriction on the graph. So this result is similar to the Minkowski–Krein–Milman property and holds for all graphs.

Theorem 5. *Let G be a graph and S a convex subset of vertices. Then* $S = \mathrm{CH}(\mathrm{Ct}(S))$.

Proof. Suppose, to the contrary, that $S \neq \mathrm{CH}(\mathrm{Ct}(S))$. Since S is a convex set, $\mathrm{CH}(\mathrm{Ct}(S)) \subseteq S$. So, by our assumption, $S - \mathrm{CH}(\mathrm{Ct}(S)) \neq \emptyset$. Let $u \in S - \mathrm{CH}(\mathrm{Ct}(S))$ be such that $\mathrm{ecc}(u) \geqslant \mathrm{ecc}(v)$ for all $v \in S - \mathrm{CH}(\mathrm{Ct}(S))$. Since $u \notin \mathrm{Ct}(S)$, there exists a neighbor v of u in S such that $\mathrm{ecc}_S(v) > \mathrm{ecc}_S(u)$ and, by our choice of u, the vertex v belongs to $\mathrm{CH}(\mathrm{Ct}(S))$.

Let $v_e \in S$ be an eccentric vertex for v in S, i.e., $d(v, v_e) = \mathrm{ecc}_S(v)$. Note that in this case $\mathrm{ecc}_S(v_e) \geqslant \mathrm{ecc}_S(v) > \mathrm{ecc}_S(u)$ and $v_e \in \mathrm{CH}(\mathrm{Ct}(S))$. Therefore $d(u, v_e) \leqslant \mathrm{ecc}_S(u) < \mathrm{ecc}_S(v) = d(v, v_e)$ and so $d(u, v_e) + 1 \leqslant d(v, v_e)$.

Let P be a shortest v_e–u path in S. Then P followed by the edge uv, is a v_e–v path whose length is $d(u, v_e) + 1 \leqslant d(v, v_e)$. So it is a shortest path between v_e and v that contains u. This contradicts the fact that $u \notin \mathrm{CH}(\mathrm{Ct}(S))$. □

In graphs with the Minkowski–Krein–Milman property, the set of extreme vertices for a convex set S is minimal in the sense that any extreme point of S is not in the convex hull of a subset of S that does not contain it. Unfortunately the contour set does not share this property in general as can be seen in the example of Fig. 1. In this case the contour is the set $\{a, b, c, d\}$, but $\mathrm{CH}(\{a, b, d\}) = \mathrm{CH}(\{a, b, c, d\})$.

However, there are examples where the contour vertices are a minimal set in a similar way that extreme vertices are. In the graph of Fig. 2 with $S = V(G)$, the contour set is $\mathrm{Ct}(S) = \{a, b, c, d\}$ and the convex hull of any proper subset of $\mathrm{Ct}(S)$ is a proper subset of S.

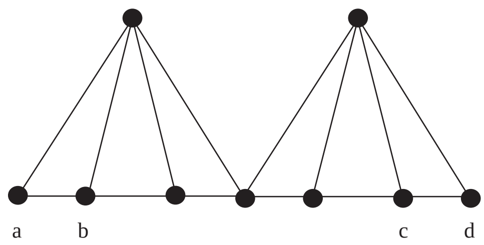

Fig. 2. Graph with a minimal contour.

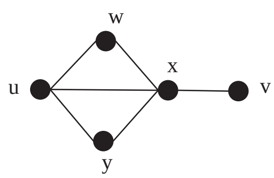

Fig. 3. A dart.

We now characterize those connected graphs for which every convex set has the property that its contour vertices coincide with its extreme points.

Theorem 6. *A connected graph G has the property that* $\mathrm{Ct}(S) = \mathrm{ex}(S)$ *for all convex sets S of vertices of G if and only if G has the Minkowski–Krein–Milman property and does not contain a dart as induced subgraph* (*see Fig.* 3).

Proof. Suppose G has the property that $\mathrm{Ct}(S)=\mathrm{ex}(S)$ for all convex sets S of vertices of G. Let S be any convex set of G. Then we know that S is the convex hull of its contour vertices. Since $\mathrm{Ct}(S)=\mathrm{ex}(S)$ it follows that S is also the convex hull of its extreme vertices. Hence G has the Minkowski–Krein–Milman property. Therefore G is chordal without induced 3-fans. Hence G is distance hereditary without induced 4-cycles. Suppose G has a dart as induced subgraph. Label the vertices of such a dart as in Fig. 3. Let X be the vertices in $I[u, v]-\{u, v\}$. Then the subgraph $\langle X\rangle$ induced by X is complete; otherwise, G has an induced 4-cycle, contradicting the fact that G is chordal. Also if $x' \in X$, then $\langle\{u, w, y, x', v\}\rangle$ is a connected subgraph of G and since G is distance hereditary it contains a shortest w–v path as well as a shortest y–v path. Hence wx', yx' are edges of G. So w and y are adjacent to every vertex in X. Since $d(w, y) = 2$, it follows that $\langle I[y, w] - \{w, y\}\rangle$ is a complete graph. Suppose $I[w, y]$ contains vertices not in $X \cup \{u\}$, say $u' \in I[w, y] - (X \cup \{u\})$. Then $d(u', v) = 2$. Since G contains no induced 4-cycles$\langle I[u', v] - \{u', v\}\rangle$ is complete. If $I[u', v] \neq X$, then there is some vertex r such that $ur \notin E(G)$ but $u'r, rv \in E(G)$. Hence $uu'rv$ is an induced u–v path of length 3. This contradicts the fact that G is distance hereditary. Thus $S = \mathrm{CH}(\{u, v, w, y, x\}) = I[w, y] \cup I[u, v]$. Hence all vertices of S except those in X are contour vertices of S. This contradicts the hypothesis since u is a contour vertex of S that is not an extreme point of S, i.e., u is not simplicial.

For the converse, suppose G has the Minkowski–Krein–Milman property and does not contain a dart as induced subgraph. Suppose S is a convex set that has a contour vertex u that is not simplicial. Then $\langle S\rangle$ is not complete and u is adjacent with a pair w, y of non-adjacent vertices. Let $X = N(u) \cap I[u, v]$. Since G has the Minkowski–Krein–Milman property it can be shown that $\langle X\rangle$ is complete. So w and y cannot both belong to $I[u, v]$. If $r \in (N(u) - I[u, v])$, then r must be adjacent to every vertex in X since u is a contour vertex of G and since G is distance hereditary. So neither w nor y belongs to X. If $uu_1u_2 \ldots u_e = v$ is a shortest u–v path in G, then $e \geqslant 2$ and neither w nor y is adjacent with u_2. Hence $\langle\{u, w, y, u_1, u_2\}\rangle$ is isomorphic to a dart, contrary to hypothesis. Thus $\mathrm{Ct}(S) = \mathrm{ex}(S)$. $\square$

Characterizing graphs G for which $\mathrm{Ct}(G) = \mathrm{ex}(G)$ appears much more difficult. Any connected graph H is an induced subgraph of a graph G with this property. To see this, take $|V(H)|$ pairwise vertex disjoint, non-trivial cliques and pair off each vertex of H in a one-to-one manner with one of these cliques. Now identify each vertex of H with exactly one vertex in the clique that it has been paired off with and let G be the resulting graph. Then G has the property that $\mathrm{Ct}(G) = \mathrm{ex}(G)$. It follows that those graphs for which the contour set and the collection of extreme points coincide can have induced cycles of arbitrarily large order. However, not every graph with this property can be constructed in this manner. Take for example the graph obtained from the 6-cycle $v_1, v_2, \ldots, v_6, v_1$ by joining a leaf u_1 to v_1 and a leaf u_4 to v_4. Then the resulting graph G has the property that $\mathrm{Ct}(G) = \mathrm{ex}(G)$ but G is not obtained by the above construction.

3. Graphs with a given contour set

In this section we characterize those graphs which are the contour of some other graph. The obvious relationship between contour and peripheral vertices allow us to use the construction used in [3] to also characterize those graphs that are the contour of some graph.

Lemma 7. *Let G be a connected graph and C a component of its contour. Then all vertices in C have the same eccentricity.*

The following result tells us which graphs are not the contour of any graph.

Proposition 8. *If H is a connected, non-complete graph with radius* 1, *then H is not the contour of any graph.*

Proof. Let H be a connected, non-complete graph with radius 1. Then some vertex $u \in V(H)$ is a neighbor of every other vertex in H. Since H is not complete there are two non-adjacent vertices v and v'. Suppose there exists a graph G such that H is the subgraph induced by its contour. Then G must be connected, since H is connected. So, using Lemma 7, every vertex in H has the same eccentricity, say k. Note that $\mathrm{ecc}(v) \geqslant 2$, because $d_G(v, v') = 2$, so $k \geqslant 2$.

Let $w \in V(G)$ be such that $\mathrm{ecc}(u) = d(u, w) = k$. Then $w \notin \mathrm{Ct}(G)$, since $k \geqslant 2$. So there exists a neighbor w_1 of w such that $\mathrm{ecc}(w_1) > \mathrm{ecc}(w) \geqslant k$. Again $w_1 \notin \mathrm{Ct}(G)$,

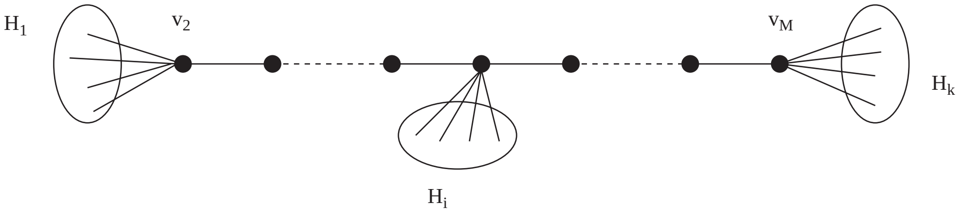

Fig. 4. A disconnected contour set.

because its eccentricity is bigger than k. So there exists a neighbor w_2 of w_1 such that $\mathrm{ecc}(w_2) > \mathrm{ecc}(w_1) > k$. This process cannot continue indefinitely since G is a finite graph. However, then the last vertex picked should be a contour vertex. Since its eccentricity is bigger than k we have a contradiction. $\square$

Suppose that H is a graph with radius greater than 1. We now describe a graph G such that its contour is H, using the construction given in [3]. Let G be the join of H and K_1. Then every vertex of H has eccentricity 2 and the vertex of $G - V(H)$ has eccentricity 1. Hence the vertices of H are precisely the contour vertices of G.

A slightly different construction allows us to obtain a graph with given disconnected contour set such that the eccentricities of the vertices in every component are given numbers at least 2. More precisely, let H be a disconnected graph with components, $H_1, H_2, \ldots, H_k$. Let $n_1, n_2, \ldots, n_k$ be k natural numbers such that $n_1 = n_k = \max\{n_1, n_2, \ldots, n_k\}$ and $M = \max\{n_1, n_2, \ldots, n_k\} \leqslant 2\min\{n_1, n_2, \ldots, n_k\} = 2m$. Note that these are natural restrictions, because M will be the diameter of the graph G and m will be greater than or equal to the radius. Then there exists a connected graph G such that H is the contour of G and the eccentricity of every vertex in each component H_i of H is equal to n_i. To construct such a graph G we begin with the path $v_1 v_2 \ldots v_{M+1}$ of order $M + 1$. Now replace v_1 by H_1 and v_{M+1} by H_k so that all vertices in H_1 are neighbors of v_2 and all vertices in H_k are neighbors of v_M.

Now, for each i, $2 \leqslant i \leqslant k-1$, there exists a vertex v_{n_i} on the path such that its eccentricity is $n_i - 1$. We now add H_i to the graph and join all the vertices of H_i to v_{n_i} (see Fig. 4). Then $\mathrm{ecc}(u_i) = n_i$ for all $u_i \in H_i$, and $\mathrm{Ct}(G) = H$.

4. Contour sets and geodetic sets in distance hereditary graphs

In this section we show that the contour vertices of a distance hereditary graph form a geodetic set. It is not difficult to see that a set S of vertices is a convex set of a distance hereditary graph G if and only if S induces a connected graph and is the union of vertices in blocks of G minus any collection of simplicial vertices from the subgraph induced by these blocks. The results of this section thus show that the contour vertices of all convex sets in distance hereditary graphs are geodetic sets for such sets. As pointed out in the introduction this may be viewed as an extension of the result which states that the simplicial vertices of convex sets in graphs with the Minkowski–Krein–Milman property are a geodetic set for the convex set.

The next result shows that if G is a distance hereditary graph, then every vertex has an eccentric vertex that is a contour vertex. Moreover, if G satisfies the Minkowski–Krein–Milman property and if x is a vertex of G, with $\mathrm{ecc}(x) \geqslant 2$, then every eccentric vertex of x must be a contour vertex.

Lemma 9. (1) *If G is a distance hereditary graph and $x \in V(G)$, then there is an eccentric vertex for x that is a contour vertex.*

(2) *Let G be a distance hereditary graph without induced* 4-*cycles. If $x \in V(G)$ is such that* $\mathrm{ecc}(x) \geqslant 2$, *then each eccentric vertex of x is a contour vertex of G.*

Proof. (1) The result holds for all distance hereditary graphs with diameter at most 2. Suppose thus that $\mathrm{diam}(G) \geqslant 3$. Among all eccentric vertices for x let x_e be one with maximum eccentricity. Let $P : x = v_0 v_1 \ldots v_k = x_e$ be a shortest x–x_e path. We show x_e is a contour vertex. If this is not the case, then x_e is adjacent with some vertex u whose eccentricity exceeds that of x_e. Thus $\mathrm{ecc}(u) > \mathrm{ecc}(x_e) \geqslant \mathrm{ecc}(x) = k$. So $\mathrm{ecc}(u) \geqslant 3$. We may assume u lies on P and that $u = v_{k-1}$. Suppose u_e is an eccentric vertex for u. Then there is a shortest u–u_e path Q that contains x_e. Suppose $Q : u = u_0 u_1 \ldots u_t = u_e$ where $u_0 = v_{k-1}$. Then u_2 is not on P. Also clearly $uu_2 \notin E(G)$. The only vertex on P that u_2 may be adjacent to is v_{k-2}. Indeed $u_2 v_{k-2}$ is an edge; otherwise, $\mathrm{ecc}(x) \geqslant d(x, u_2) > k$. Since $\mathrm{ecc}(v_{k-1}) \geqslant 3$, u_2 must be adjacent to a vertex not on P. If u_3 is adjacent with a vertex of P it can only be adjacent with v_{k-3}; otherwise, either $d(v_{k-1}, u_3) \neq 3$ or $d(x, v_k) \neq k$. However, if $u_3 v_{k-3}$ is an edge, we have a 6-cycle $v_{k-3} v_{k-2} v_{k-1} v_k u_2 u_3$without crossing chords, which is not possible in a distance hereditary graph. So $u_3 v_{k-3} \notin E(G)$ and $d(v_0, u_3) = k$. By our choice of $x_e = v_k$, $3 \leqslant \mathrm{ecc}(u_3) \leqslant \mathrm{ecc}(v_k) < \mathrm{ecc}(v_{k-1})$. So $\mathrm{ecc}(v_{k-1}) \geqslant 4$ and hence u_4 is not on P. As before we can argue that the only vertex of P that u_4 is possibly adjacent to is v_{k-4}. If $u_4 v_{k-4} \in E(G)$, then as before we obtain a 6-cycle $v_{k-4} v_{k-3} v_{k-2} u_2 u_3 u_4 v_{k-4}$ which has no crossing chords. So $u_4 v_{k-4} \notin E(G)$. However, then $d(x, u_4) = k + 1 > \mathrm{ecc}(x)$ which is not possible. So $u \neq v_{k-1}$. Note u is not adjacent with v_{k-i} for $i \geqslant 3$; otherwise, $d(x, x_e) < k$, which is not possible. Also $v_{k-2} u \in E(G)$; otherwise, we have a contradiction to our choice of $x_e = v_k$. If $v_{k-1} u \in E(G)$, then u_2 is not on P. Note that in this case u_2 is adjacent with at most one vertex on P, namely, v_{k-2}. So u_3 is not on P. For $i \geqslant 3$, u_i cannot be on P since in this case either $d(x, v_k) < k$ or $d(u, u_i) < i$, both of which cannot happen. Moreover, the only vertex on P that u_i can be adjacent with (if any) is v_{k-i}; otherwise, as in the previous case, we have a contradiction. If$u_3 v_{k-3} \in E(G)$, then $v_{k-3} v_{k-2} v_{k-1} v_k u_2 u_3 v_{k-3}$ is a 6-cycle without crossing chords. So $u_3 v_{k-3} \notin E(G)$. In this case $u_2 v_{k-3} \in E(G)$ and $d(x, u_3) = k$. So by our choice of $x_e = v_k$, $\mathrm{ecc}(u_3) \leqslant \mathrm{ecc}(v_k) < \mathrm{ecc}(u) \leqslant \mathrm{ecc}(u_t)$. So $3 < t$. If $u_4 v_{k-4} \in E(G)$, then $v_{k-4} v_{k-3} v_{k-2} u_2 u_3 u_4 v_{k-4}$ is a 6-cycle without crossing chords. So $u_4 v_{k-4} \notin E(G)$. But then $d(x, u_4) = k + 1$ contradicting the fact that $\mathrm{ecc}(x) = k$. Hence we may assume $v_{k-1} u \notin E(G)$.

If $u_2 \neq v_{k-1}$, i.e., u_2 is not on P, then at least one of $u_2 v_{k-2}$ and $u_2 v_{k-1}$ is an edge of G. If $u_2 v_{k-2} \notin E(G)$, we have a 5-cycle with out crossing chords. So assume $u_2 v_{k-2} \in E(G)$. For $i \geqslant 3$ we may argue, as before, that u_i is not on P and that u_i is adjacent to at most one vertex on P, namely, v_{k-i}. But then, as in the previous case, we have a contradiction to the fact that $\mathrm{ecc}(v) = k$. So we may assume $v_{k-1} = u_2$. Clearly, $u_3 \neq v_{k-2}$ since $d(u, u_3) = 3 \neq d(u, v_{k-2})$. Indeed, for $i \geqslant 3$, u_i is not on P and u_i is adjacent with at most

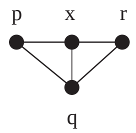

Fig. 5. Not all eccentric vertices are contour points.

one vertex of P, namely, v_{k-i}. If $u_3v_{k-3} \in E(G)$ we obtain a 6-cycle without crossing chords. So $d(x, u_3) = k$. Hence $\mathrm{ecc}(u_3) \leqslant \mathrm{ecc}(x_e) \leqslant \mathrm{ecc}(u) \leqslant \mathrm{ecc}(u_t)$. So again $t > 3$. As before we now obtain a contradiction.

(2) Let $x \in V(G)$ be such that $\mathrm{ecc}(x) \geqslant 2$ and let x_e be an eccentric vertex for x. Suppose $P : x = y_0y_1 \ldots y_k = x_e$ is a shortest x–x_e path. Assume x_e is not a contour vertex of G. Let $u \in N(x_e)$ be such that $\mathrm{ecc}(u) > \mathrm{ecc}(x_e)$. Let u_e be an eccentric vertex for u. Then there is a shortest u–u_e path $Q : u = v_0v_1v_2 \ldots v_l = u_e$ that contains x_e. So $x_e = v_1$. Then uy_{k-1} and v_2y_{k-1} are edges since x_e is an eccentric vertex of x. The path $v_3v_2y_{k-1}y_{k-2} \ldots y_0(=x)$ has length $\mathrm{ecc}(x) + 1$. So it must have a chord. If v_3y_{k-1} is a chord, then there is a 5-cycle, namely $v_3y_{k-1}v_0v_1v_2v_3$ without crossing chords, which is not possible. If $y_{k-2}v_2$, then there is again a 5-cycle without crossing chords unless $y_{k-3}u$ is an edge. But in this case $v_0v_1v_2y_{k-2}v_0$ is an induced 4-cycle. So x_e must be a contour vertex of G. $\square$

Remark 10. The condition on the vertex eccentricity in Lemma 9(2) is necessary as is shown in the following example. In the graph in Fig. 5, the vertex x has eccentricity 1, but q is an eccentric vertex of x which is not a contour vertex.

To establish the main result of this section we use the following notation. If $Q : u_0u_1 \ldots u_t$ is a path, then the *reversal* of Q is the path $u_tu_{t-1} \ldots u_0$.

Theorem 11. *Let G be a distance hereditary graph. Then* $\mathrm{Ct}(G)$ *is a geodetic set for G.*

Proof. It suffices to show that if $v \in V(G) - \mathrm{Ct}(G)$, then $v \in I[\mathrm{Ct}(G)]$. Since v is not a contour vertex of G, there is some neighbor u^1 of G such that $\mathrm{ecc}(u^1) > \mathrm{ecc}(v)$. If u^1 is not a contour of G, then u^1 has a neighbor u^2 such that $\mathrm{ecc}(u^2) > \mathrm{ecc}(u^1)$. We continue constructing a sequence $u^1, u^2, \ldots$ of vertices such that $\mathrm{ecc}(u^1) < \mathrm{ecc}(u^2) < \ldots$. Since the graph is finite the sequence terminates with some vertex u^t which has the property that its eccentricity is at least as large as that of its neighbors. Such a vertex must necessarily be a contour vertex. By Lemma 9(1) we know that u^t has an eccentric vertex u^t_e that belongs to the contour of G. Since $\mathrm{ecc}(u^t) > \mathrm{ecc}(u^{t-1})$, it follows that u^t_e is also an eccentric vertex for u^{t-1} and thus u^tu^{t-1} followed by a shortest u^{t-1}–u^t_e is a shortest u^t–u^t_e path that contains u^{t-1}. Continuing in this manner we see that the path $u^tu^{t-1} \ldots u^1v$ followed by a shortest v–u^t_e is a shortest u^t–u^t_e path that contains v. Since u^t and u^t_e are both contour vertices the result now follows. $\square$

The graph of Fig. 6 shows that Theorem 11 does not hold for graphs in general. Note that the contour set of this graph G is $\mathrm{Ct}(G) = \{v_2, v_5, w\}$ and $v_1 \notin I[\mathrm{Ct}(G)]$.

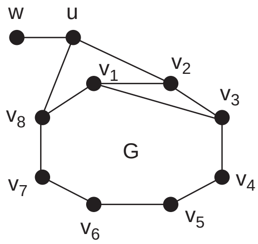

Fig. 6. A graph whose contour set is not geodetic.

Indeed, if we replace v_1 by a clique of arbitrarily large order and join every vertex in this clique with v_2, v_3 and v_8, we see that the ratio $|I[\mathrm{Ct}(G)]|/|V(G)|$ can be made arbitrarily small.

5. Closing remarks

As we mentioned in the introduction, the process of taking geodetic closures starting from a set S of vertices can be repeated to obtain a sequence $S_0, S_1, \ldots$ of sets where $S_0 = S$, $S_1 = I[S]$, $S_2 = I[S_1] \ldots$. Since $V(G)$ is finite, the process terminates with some smallest r for which $S_r = S_{r+1}$. The set S_r is then the convex hull of S and r is called the *geodetic iteration number*, $\mathrm{gin}(S)$, of S. In the graph G of Fig. 6, $\mathrm{gin}(\mathrm{Ct}(G)) = 2$. It remains an open problem to determine if $\mathrm{gin}(\mathrm{Ct}(G))$ can be larger than 2 and indeed if $\mathrm{gin}(\mathrm{Ct}(G))$ can be arbitrarily large. However, we do believe that there are other classes of perfect graphs for which the geodetic iteration number of the set of contour vertices is 1. In particular we believe that chordal graphs and house, hole, domino free graphs (see [6] for definitions) have this property. Indeed this may be true for all perfect graphs. Finding characterizations of contour vertices, for these and other classes of graphs, similar to the one given in Proposition 3 for distance hereditary graphs with out induced 4-cycles also remains and open problem.

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