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Locally grid graphs: classification and Tutte uniqueness☆

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Abstract

We define a locally grid graph as a graph in which the structure around each vertex is a 3×3 grid ⊞, the canonical examples being the toroidal grids $C_p \times C_q$. The paper contains two main results. First, we give a complete classification of locally grid graphs, showing that each of them has a natural embedding in the torus or in the Klein bottle. Secondly, as a continuation of the research initiated in (On graphs determined by their Tutte polynomials, Graphs Combin., to appear), we prove that $C_p \times C_q$ is uniquely determined by its Tutte polynomial, for $p, q \geqslant 6$.

Keywords: Locally grid graph; Toroidal grid; Tutte polynomial

1. Introduction

Given a fixed graph H, a connected graph G is said to be locally H if for every vertex x the subgraph induced on the set $N(x)$ of neighbours of x is isomorphic to H. For instance, the only graphs locally C_3, C_4 and C_5 are, respectively, the one-skeletons of the tetrahedron, octahedron and icosahedron. But for any $n \geqslant 6$, there are infinitely many non-isomorphic graphs which are locally C_n. As a different example, if P is the Petersen graph, then there are exactly three locally P graphs [8]. There exists by now an extensive literature on this topic. See for instance, Refs. [1,3,4,6,7,9]. We also note Ref. [11], where locally C_6 graphs appear in an unrelated problem.

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The local condition we introduce in this paper is a different one and is motivated by our study of the toroidal grids $C_p \times C_q$, where C_p is a cycle of length p. Around each vertex of a toroidal grid there is a small 3×3 grid ⊞, and we take this condition as our definition of locally grid graphs; the precise definition is slightly more technical and is given in Section 2. Observe that the locally grid condition involves not only a vertex and its neighbours, but also four vertices at distance two.

The paper contains two main results. First, we give a complete classification of locally grid graphs. They fall into several families and each of them has a natural embedding in the torus or in the Klein bottle, in which the four squares around each vertex become faces of the embedding. The key concept for obtaining the classification is that of opposite and adjacent edges: two incident edges are called adjacent if they are contained in a common square, and opposite otherwise. Starting at a given edge, we construct a walk taking each time an opposite edge. The nature of these paths gives the different cases of the classification.

Our second main result has to do with the Tutte polynomial, a two-variable polynomial $T(G;x,y)$ associated with any graph G, which contains much information on G [5]. From now on, all graphs have no isolated vertices. Call a graph G *Tutte unique* (T-unique for short) if $T(H;x,y)=T(G;x,y)$ implies $G \cong H$ for every other graph H. As explained in [12], this is a natural extension of the concept of chromatically unique graphs, defined as those graphs uniquely determined by their chromatic polynomial [10]. We also remark that Tutte polynomials and T-uniqueness have been studied more generally for matroids [2].

In Section 5 we show that the toroidal grid is T-unique for $p,q \geqslant 6$. A sketch of the proof is as follows. Assuming $T(H;x,y)=T(C_p \times C_q;x,y)$, we prove in Section 3 that H must be locally grid; by the classification theorem, H must be in one of the families defined in Section 2. To conclude the proof we show in Section 5 that any locally grid graph different from $C_p \times C_q$ has a different Tutte polynomial. The key tool is a careful counting of edge-sets of a given size and rank in locally grid graphs; this is the content of Section 4.

We conclude the paper with some remarks and open problems.

2. Classification

In this section, we define and give a complete classification of locally grid graphs.

Let $N(x)$ be the set of neighbours of a vertex x. We say that a 4-regular connected graph G is a *locally grid graph* if for every vertex $x \in V(G)$ there exists an ordering x_1,x_2,x_3,x_4 of $N(x)$ and four different vertices y_1,y_2,y_3,y_4, such that, taking the indices modulo 4,

$$N(x_i) \cap N(x_{i+1}) = \{x, y_i\},$$

$$N(x_i) \cap N(x_{i+2}) = \{x\},$$

and there are no more adjacencies among $\{x,x_1,\ldots,x_4,y_1,\ldots,y_4\}$ than those required by this condition. In particular this implies that a locally grid graph is simple (that is,

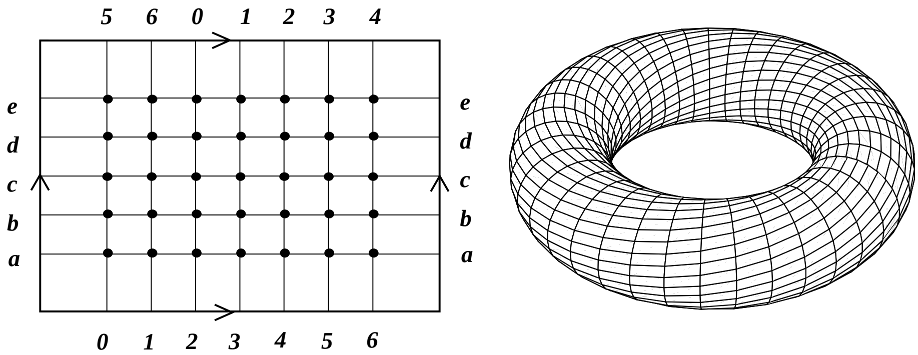

Fig. 1. Left: the torus $T^2_{7,5}$. Right: an illustration of a “twisted torus” in $\mathbb{R}^3$.

without loops or multiple edges) and triangle-free, and that every vertex belongs to exactly four squares (cycles of length 4). Note that this definition excludes the toroidal grids $C_4 \times C_q$ since with the above notation either x_1, x_3 or x_2, x_4 would have a common neighbour different from x. This restriction is due to some technical reasons that will be explained later. Notice also that locally grid graphs are two-connected.

Let $H = P_p \times P_q$ be the $p \times q$ grid, where P_l is a path with l vertices. Label the vertices of H with the elements of the abelian group $\mathbb{Z}_p \times \mathbb{Z}_q$ in the natural way. Notice that the vertices of degree four already have the locally grid property. In order to obtain a locally grid graph we add new edges suitably among the vertices of degree two and three. If we add the edges

$$\{(j,0),(j,q-1) \mid 0 \leqslant j \leqslant p-1\} \cup \{(0,j),(p-1,j) \mid 0 \leqslant j \leqslant q-1\},$$

the result is the toroidal grid $C_p \times C_q$. Other ways to do this give the following three families of graphs.

The Torus $T^{\delta}_{p,q}$. The graph $T^{\delta}_{p,q}$ is built just as the graph $C_p \times C_q$, but moving the adjacencies in the first direction δ vertices to the right. That is,

$$\begin{aligned} E(T^{\delta}_{p,q}) = E(H) &\cup \{\{(i,0),(i+\delta,q-1)\},\ 0 \leqslant i \leqslant p-1\} \\ &\cup \{\{(0,j),(p-1,j)\},\ 0 \leqslant j \leqslant q-1\}. \end{aligned}$$

Note that we can assume $\delta \leqslant p/2$. For $\delta = 0$ we obtain the toroidal grid $C_p \times C_q$; in this case we simply write $T_{p,q}$.

All these graphs are embeddable in the topological torus. See Fig. 1 for an embedding of $T^2_{7,5}$; in this figure, as in the next two, the vertices of the graph are represented by dots, and two points with the same label correspond to points that are identified in the surface.

Although we have not mentioned this explicitly, the values of p and q must be large enough to guarantee the locally grid condition. For example, for $\delta = 0$, p and q must be at least 5, but for $\delta = 1$ we can take $q = 4$ and $p = 5$. Similar observations apply to the next two families. The exact bounds are stated in Theorem 1.

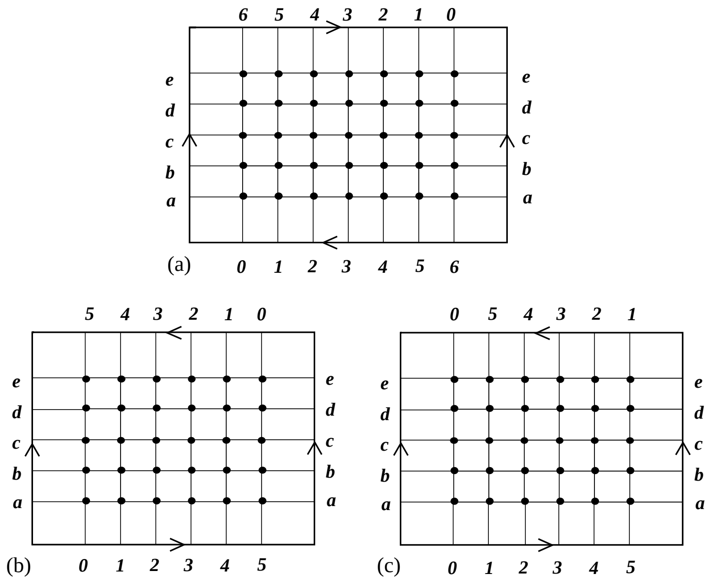

Fig. 2. Three examples of Klein bottles: (a) $K^1_{7,5}$, (b) $K^0_{6,5}$, and (c) $K^2_{6,5}$.

The Klein bottles $K^i_{p,q}$. For $i \in \{0,1,2\}$, $i \equiv p \bmod 2$, define the graph $K^i_{p,q}$ as follows. Keep the adjacencies of the second direction untouched and reverse the ones in the first direction, thus obtaining graphs embeddable in the Klein bottle. If p is odd, the edges are as follows:

$$E(K^1_{p,q}) = E(H) \cup \{\{(0,j),(p-1,j)\},\ 0 \leqslant j \leqslant q-1\}$$
$$\cup\{\{(j,0),(p-j-1,q-1)\},\ 0 \leqslant j \leqslant p-1\}.$$

The superscript 1 means that there is only one adjacency made in the usual way, namely $\{((p-1)/2,0),((p-1)/2,q-1)\}$.

If p is even, there are two cases:

$$E(K^0_{p,q}) = E(H) \cup \{\{(0,j),(p-1,j)\},\ 0 \leqslant j \leqslant q-1\}$$
$$\cup\{\{(j,0),(p-j-1,q-1)\},\ 0 \leqslant j \leqslant p-1\}$$

and

$$E(K^2_{p,q}) = E(H) \cup \{\{(0,j),(p-1,j)\},\ 0 \leqslant j \leqslant q-1\}$$
$$\cup\{\{(j,0),(p-j,q-1)\},\ 0 \leqslant j \leqslant p-1\}.$$

In the first case there are no adjacencies of the kind $\{(c,0)(c,q-1)\}$, and in the second one there are two of them.

In Fig. 2 we show embeddings of these three kinds of graphs in the Klein bottle.

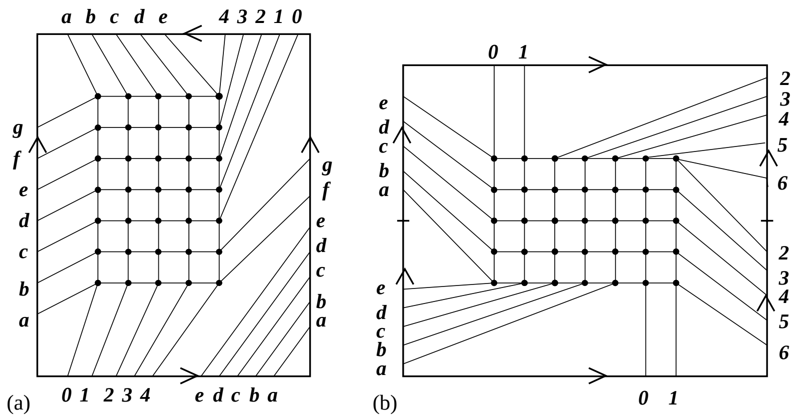

Fig. 3. Two "strange" graphs: (a) $S_{5,7}$ and (b) $S_{7,5}$.

The graphs $S_{p,q}$. Define the graph $S_{p,q}$ (S is for "strange") for $p \leqslant q$ as follows:

$$\begin{aligned} E(S_{p,q}) = E(H) &\cup \{\{(j,0),(p-1,q-p+j)\},\ 0 \leqslant j \leqslant p-1\} \\ &\cup \{\{(0,i),(i,q-1)\},\ 0 \leqslant i \leqslant p-1\} \\ &\cup \{\{(0,i),(p-1,i-p)\},\ p \leqslant i \leqslant q-1\}. \end{aligned}$$

For $q \leqslant p$, the edges of $S_{p,q}$ are as follows:

$$\begin{aligned} E(S_{p,q}) = E(H) &\cup \{\{(j,0),(0,q-1-j)\},\ 0 \leqslant j \leqslant q-1\} \\ &\cup \{\{(p-1-i,q-1),(p-1,i)\},\ 0 \leqslant i \leqslant q-1\} \\ &\cup \{\{(i,q-1),(i+q,0)\},\ 0 \leqslant i \leqslant p-q-1\}. \end{aligned}$$

Note that when $p = q$ both definitions agree. Fig. 3 shows embeddings of the two kinds of "strange" graphs in the Klein bottle. Notice that in the second one we use a different model for the Klein bottle.

It is straightforward to verify that all the graphs we have defined are locally grid and have pq vertices. The rest of the section is devoted to the proof that these families exhaust all the cases.

Theorem 1. *If G is a locally grid graph with N vertices, then one and only one of the following holds*:

(1) $G \cong T^{\delta}_{p,q}$ *with* $pq = N$, $p \geqslant 5$, $\delta \leqslant p/2$, *and* $\delta + q \geqslant 5$ *if* $q \geqslant 4$, *or* $\delta + q \geqslant 6$ *if* $q = 2, 3$, *or* $4 \leqslant \delta < p/2$, $\delta \neq p/3, p/4$ *if* $q = 1$.

(2) $G \cong K^{i}_{p,q}$ *with* $pq = N$, $p \geqslant 5$, $i \equiv p \pmod 2$ *for* $i \in \{0,1,2\}$, *and* $q \geqslant 4 + \lceil i/2 \rceil$.

(3) $G \cong S_{p,q}$ *with* $pq = N$, $p \geqslant 3$, *and* $q \geqslant 6$.

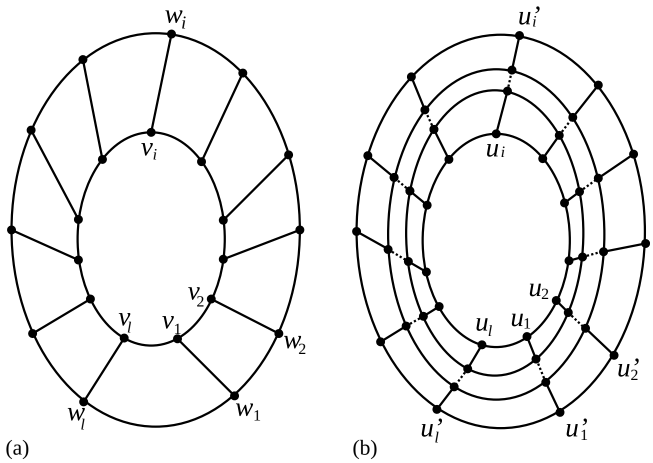

Fig. 4. (a) The cycle C and the walk $W(w_1, f)$ of length l. (b) The same structure after s steps.

Let $G=(V,E)$ be a locally grid graph. Two edges e_1, e_2 incident with a vertex v are called *adjacent* if there is a square containing them both; otherwise, they are called *opposite*. The four edges incident at v are thus classified into two pairs of opposite edges.

Given $v \in V$ and $e \in E$, e incident with v, we define the walk $W(v,e)$ as the sequence $(v_1 = v)(e_1 = e)v_2e_2 \dots v_le_lv_{l+1}$ such that e_i, e_{i+1} are opposite and v_{l+1} is the first vertex equal to v_i for some i, $1 \leqslant i < l$. In other words, we visit the vertices of the graph, starting from e, following opposite edges until we repeat some vertex. Notice that $W(v,e)$ is a circuit only if $v_{l+1} = v_1$; in this case, e_1 and e_l may be opposite (this is always the case in torus and Klein bottles), or adjacent (this can only happen in "strange" graphs).

We call a path $v_1e_1 \dots v_le_lv_{l+1}$ an *opposite path* if for every $i < l$, e_i and e_{i+1} are opposite edges. Similarly we call a cycle $v_1e_1 \dots v_le_lv_1$ an *opposite cycle* if every pair of consecutive edges is opposite. If we are not interested in the edges of the cycle or path, we simply write $v_1 \dots v_lv_1$. Two opposite cycles $v_1 \dots v_lv_1$ and $w_1 \dots w_lw_1$ of the same length are called *parallel* if $v_iw_i \in E$ and $v_iw_iw_{i+1}v_{i+1}$ is a square for all i, $1 \leqslant i \leqslant l$ (for example, those in Fig. 4a).

We need some easy, but essential, properties about opposite paths and cycles. All omitted proofs are straightforward.

Lemma 2. *If $v_1v_2v_3$ and $w_1w_2w_3$ are consecutive vertices in two opposite paths, and $v_2w_2 \in E$ is adjacent to v_1v_2, then either $v_1w_1 \in E$ or $v_1w_3 \in E$.*

Lemma 3. *If $v_1v_2v_3$ and $w_1w_2w_3$ are consecutive vertices in two opposite paths, and $v_1w_1, v_2w_2 \in E$ with $v_1w_1w_2v_2$ a square, then $v_3w_3 \in E$ and $v_2w_2w_3v_3$ is a square.*

Lemma 4. *Let $e, f \in E$ be a pair of opposite edges at the vertex v. Then every square that contains v contains either e or f but not both.*

Lemma 5. *Every opposite cycle has length at least five.*

Lemma 6. *Let $v_1 \dots v_l$ be an opposite path with $v_i \neq v_j$ for $i \neq j$. If $v_1 v_k \in E$ for some $k < l$, and $v_1 v_k$ is adjacent to $v_1 v_2$, then necessarily $v_2 v_{k+1} \in E$.*

Proof. By Lemma 2, either $v_2 v_{k-1} \in E$ or $v_2 v_{k+1} \in E$. In the first case we apply repeatedly Lemma 3 and we get $v_3 v_{k-2}, v_4 v_{k-3}, \dots, v_i v_{k-i+1}, \dots \in E$. For $i = \lfloor k/2 \rfloor$ we obtain either a triangle, which is impossible in a locally grid graph, or a contradiction to Lemma 4. Thus we conclude that $v_2 v_{k+1} \in E$. □

The proof of Theorem 1 considers different cases, depending on the nature of the walks $W(v,e)$. We choose a distinguished walk and then use it to recover the locally grid graph.

Case 1: All walks $W(v,e)$ in G are opposite cycles.

Suppose first that there exists an opposite cycle $v_1 \dots v_n v_1$ in G that contains all vertices. Since G is 4-regular, the vertex v_1 is joined to some other vertex v_{k+1}, for some k with $4 \leqslant k \leqslant n/2$. By Lemma 6 and repeated application of Lemma 3, we obtain that v_i is adjacent to v_{i+k} and v_{i-k}, where all the subindices are read modulo n. This determines all the adjacencies between vertices of G; note that if n is even and $k = n/2$ the resulting graph is not locally grid. Therefore G is isomorphic to $T^k_{n,1}$ for some k with $4 \leqslant k < n/2$, and with $k \neq n/3, n/4$ because in these cases there are triangles or every vertex is contained in five squares.

From now on, we can assume that not all vertices of G are contained in an opposite cycle. Let $C = v_1 \dots v_l v_1$ be a walk $W(v,e)$ of shortest length. We claim that every vertex in C has only two neighbours among $V(C)$. If not, we can assume that $v_1 v_{k+1} \in E$ for some k, $k \geqslant 4$. By Lemma 6 and repeated application of Lemma 3 we get that $v_{k+1} v_{2k+1} \in E$, and in general that $v_{ik+1} v_{(i+1)k+1} \in E$. Note that these edges have to form a cycle C', because of the 4-regularity of G. Since G is connected and there are vertices not in C, the cycle C' cannot include all the vertices in C. Therefore, it is shorter than C, a contradiction.

Take now a vertex $w_1 \in V - C$ such that $v_1 w_1 \in E$. Let f be the edge incident with w_1 that shares a square with $v_1 v_2$, and consider the walk $W(w_1, f)$, which is an opposite cycle of length at least l, $w_1 w_2 \dots w_r$, with $r \geqslant l$. It is an immediate consequence of Lemma 3 and the choice of f that $v_2 w_2, \dots, v_l w_l \in E$ and, more generally, $w_s v_{s'} \in E$, with $s' \equiv s \bmod l$. By Lemma 4 it follows that $w_1, \dots, w_r \notin C$. If $r = l$, then C and $W(w_1, f)$ are parallel cycles (see Fig. 4a); otherwise, the structure is as in Fig. 5a. We study each case separately.

Case 1.1: Suppose we have a set of opposite cycles $C_1, \dots, C_s$ for s with $s \geqslant 2$, and such that C_i and C_{i+1} are parallel for all i, $1 \leqslant i < s$, and $C = C_j$ for some $1 \leqslant j \leqslant s$ (like those in Fig. 4b and with vertices named as in this figure). Let us denote by $C_{1\dots s}$ a set of vertices and edges like this. Note that all vertices have degree four, except the vertices in C_1 and C_s, which have degree three. We show next that if it is not possible to add a new opposite cycle C_{s+1} parallel to C_s or to C_1, then the vertices in C_1 and C_s must be joined to complete a locally grid graph.

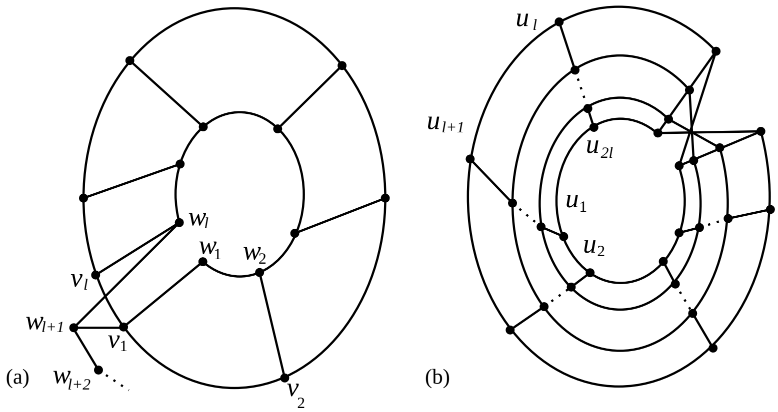

Fig. 5. (a) The cycle C and the walk $W(w_1, f)$ with length $l > r$. (b) The same structure after s steps.

Assume there is a pair of adjacent vertices $v \in C_{1\dots s}$ and $w \notin C_{1\dots s}$; we can suppose $v \in C_s$. As we did before, let f be an edge incident with w and adjacent to vw, and consider the opposite cycle $W(w, f)$. By Lemmas 2 and 3 $W(w, f)$ and C_s are parallel (in this case, the length of the walk $W(w, f)$ must be necessarily l). Since all the vertices in $C_{1\dots s}$ have degree either three or four, none of the vertices in $W(w, f)$ belongs to $C_{1\dots s}$. We can thus define $C_{s+1} = W(w, f)$.

If it is not possible to add a new opposite cycle to $C_{1\dots s}$, then the fourth neighbour of any vertex $u \in C_1$ belongs to either C_1 or C_s. Let w be the fourth neighbour of $u_1 \in C_1$. If $w \in C_1$, we apply Lemmas 6 and 3 and deduce that l must be even and that $w = u_{l/2+1}$, since we can add only one more edge to each vertex. Thus u_i is joined to $u_{i+l/2}$. The vertices in C_s must be joined in the same way. Since all the degrees are already four, our graph is completed. In this case G is a Klein bottle $K^0_{2s,l/2}$. We note here that this situation is possible unless $s = 2$; in this case we have an opposite cycle of length 4, a contradiction to Lemma 5.

If the neighbour w of u_1 belongs to C_s, $w = u'_k$, there are two possibilities, depending on whether u_2 is adjacent to u'_{k+1} or to u'_{k-1}. In the first case the resulting graph is a Torus $T^{k-1}_{l,s}$, and in the second one a Klein bottle $K^i_{l,s}$, where i depends on k and the parity of l. As in the previous paragraph, for small values of s not all ranges of k are allowed. Straightforward checking shows that for $s = 2, 3, 4$, the minimum values of k are $5, 4, 1$, respectively.

Case 1.2: We study now the case in which the length r of the walk $W(w_1, f)$ exceeds l, the length of C; the notation is as in Fig. 5a. By Lemma 3, $v_i w_{i+l} \in E$ for all i such that $l + i \leqslant r$. If $r < 2l$ then, again by Lemma 3, we obtain that $v_1 w_{2l-r+1} \in E$. This leads to a contradiction because the four neighbours of v_1 are v_2, v_l, w_1 and w_{l+1}, all of them different from w_{2l-r+1}. Therefore $r \geqslant 2l$. If $r > 2l$, then $v_1 w_{2l+1} \in E$, which is again a contradiction. We conclude that $r = 2l$ and that every vertex in C is adjacent to two vertices in $W(w_1, f)$.

Suppose that we have s cycles $C_1, \dots, C_s$, $s \geqslant 2$, such that $C_1 = C$, $C_2 = W(w_1, f)$, and C_i is parallel to C_{i+1}, for all i, $2 \leqslant i \leqslant s-1$; note that this implies that $C_2, \dots, C_s$

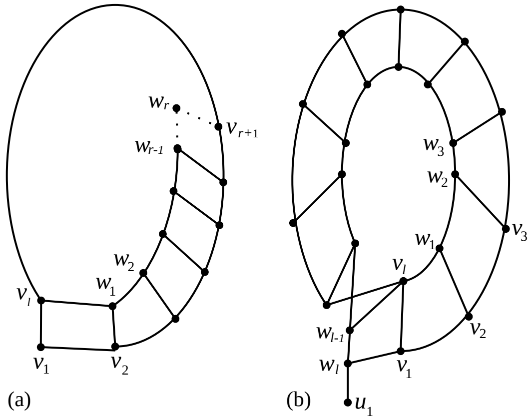

Fig. 6. Two intermediate steps in Case 2.

have length $2l$. We call this structure $M_{1\ldots s}$ (see Fig. 5b for notation). We prove next that either we can complete a locally grid graph or add another cycle of length $2l$ parallel to C_s.

If there is no vertex in $M_{1\ldots s}$ whose fourth neighbour lies outside $M_{1\ldots s}$, the neighbour of a vertex in the last cycle C_s belongs to this cycle. Let u_k be such that $u_1u_k \in E$. By Lemmas 6 and 3, we obtain that $u_{2l-k+2}u_1 \in E$. Since u_1 had already degree three, the only choice is $k = l + 1$. It is easy to see that this graph is a Klein bottle $K^1_{2s-1,l}$.

Suppose now that there exists $u'_1 \in V - M_{1\ldots s}$ such that $u_1u'_1 \in E$. Take an edge f containing u'_1 and adjacent to $u_1u'_1$, and consider the walk $W(u'_1, f) = u'_1 \ldots u'_r u'_1$. This walk is an opposite cycle of length $r \geqslant l$. If $r > l$, then we can show as before that $r = 2l$ and we can add a cycle C_{s+1} parallel to C_s. If $r = l$, by Lemma 3 $u_iu'_i, u_{l+i+1}u'_i \in E$ for $i \leqslant l$. All the vertices have degree four and thus we have completed a locally grid graph. In this case we obtain a Klein bottle $K^2_{2s,l}$. Note that this situation is not possible for $s = 2$, because in this case we would have opposite cycles of length four ($u_1u'_1u_{l+1}v_1$, for instance), contradicting Lemma 5.

We have proved that every locally grid graph such that all the walks $W(u,e)$ are opposite cycles is a Torus or a Klein bottle. The family $S_{p,q}$ arises when considering the case in which there is a walk that is not an opposite cycle, as we see next.

Case 2: Not all the walks $W(v,e)$ are opposite cycles.

Choose a vertex $v_1 \in V$ and an edge $e_1 \in E$ such that $W(v_1, e_1) = v_1e_1v_2 \ldots e_lv_1$ is a cycle, e_l and e_1 are adjacent edges, and such that $W(v_1, e_1)$ is the shortest walk with this property; call it C.

Since e_1 and e_l are adjacent at v_1, then v_2 and v_l have a second common neighbour w_1. By Lemma 4, $w_1 \notin C$. Now w_1 and v_3 have a second common neighbour, w_2, which also does not belong to C; note that if $w_2 = v_1$, then there would be a triangle in G. Suppose now that $w_1, \ldots, w_{r-1} \in V - C$ for $r < l$ are such that w_i is a common neighbour of v_{i+1} and w_{i-1} (see Fig. 6a). Vertices w_{r-1} and v_{r+1} also have a second common neighbour w_r. This vertex w_r cannot be any of the v_i for i with $2 \leqslant i \leqslant r$, since these vertices have degree three. By Lemma 4, w_r is different from v_i for i with $r+2 \leqslant i \leqslant l$. In the next claim we prove that w_r is also different from v_1.

Claim 1. *The vertex w_r is different from v_1.*

Proof. Suppose not. Then v_1 and v_{r+2} must have a common neighbour besides v_{r+1}; since v_1 already has degree four, this neighbour is either v_2 or v_l. If it is v_l, apply Lemma 6 to the opposite path $v_l v_{l-1} \dots$ and the edge $v_l v_{r+2}$, and deduce that v_{l-1} and v_{r+1} are adjacent; then $v_1 v_l v_{l-1} v_{r+1}$ is a square that contradicts Lemma 4. Hence v_{r+2} is adjacent to v_2. Now apply repeatedly Lemma 3 and get that $v_{l-r+1} v_1 \in E$. Since the neighbours of v_1 are v_2, v_l, v_{r+1} and w_{r-1}, we obtain a contradiction unless $r = l/2$. In this case, the neighbours of v_1 are $v_2, v_l, v_{l/2+1}$ and $w_{l/2-1}$, and the edges $v_1 v_l$ and $v_1 v_{l/2+1}$ are opposite. But the vertex $v_{l/2}$ is a common neighbour of v_l and $v_{l/2+1}$, and this contradicts the locally grid property at v_1. □

We define recursively w_i as the common neighbour of v_{i+1} and w_{i-1} for $i \leqslant l-2$. By Claim 1 and the remarks before it, each w_i is a new vertex. Observe that if $w_{l-2} w_1 \in E$ we would contradict the minimality of C. Therefore the second common neighbour of w_{l-2} and v_l is a new vertex w_{l-1}; so it is w_l, the other common neighbour of v_1 and w_{l-1} (see Fig. 6b). Note that $w_l v_1 v_2$ is an opposite path since $w_l v_1 v_l w_{l-1}$ is a square.

Let us focus now on the vertex w_l. Its four neighbours are v_1, w_{l-1}, u_1 and u', where the edge $w_l u_1$ is opposite to $w_{l-1} w_l$, and the edge $w_l u'$ is opposite to $w_l v_1$. We show that u_1 is different from all the vertices that have appeared previously.

Claim 2. *The vertex u_1 is different from w_i for all i, $1 \leqslant i \leqslant l$, and from v_j for all j, $1 \leqslant j \leqslant l$.*

Proof. The only cases that must be checked carefully are the following.

(1) Suppose $u_1 = w_i$ for some i, $2 \leqslant i \leqslant l-3$. Then the walk $w_i w_{i+1} \dots w_l$ $(u_1 = w_i)$ contradicts the choice of C.

(2) Suppose $u_1 = v_j$ for some $3 \leqslant j \leqslant l-2$. Applying Lemma 6 to the opposite path $w_l v_1 v_2 \dots$ and to the edge $w_l v_j$, we deduce $v_1 v_{j+1} \in E$. Now apply Lemma 3 $l-j$ times and obtain $v_{l-j} v_l \in E$. This is a contradiction since the neighbours of v_l are $v_1, v_{l-1}, w_1, w_{l-1}$. □

Let u_2 be the second common neighbour of u_1 and v_1, let u_3 be the second common neighbour of u_2 and v_2, and so on. By an argument similar to that of Claim 1, all the vertices u_i for $1 \leqslant i \leqslant l$ are new vertices, and it is immediate that $u_i v_{i-1} \in E$ and $u_l w_1 \in E$ (see Fig. 7a).

Now suppose that we have built s layers like this from C and the vertices, adjacencies and notation are as in Fig. 7b. Call this structure R_s. The vertices w_2' and u_l' have a second common neighbour, which is either a new vertex or u_1' (because all the other vertices in R_s have degree at least three).

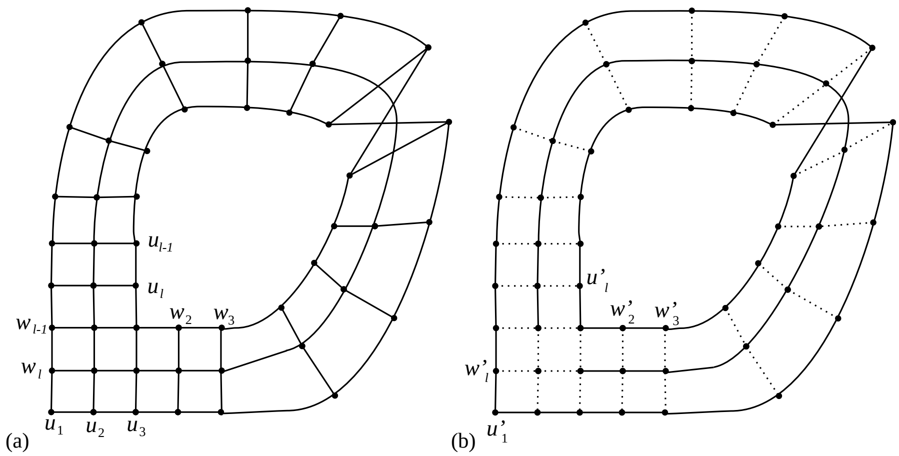

Fig. 7. (a) The first layer. (b) The same structure after s steps.

Assume first that u'_1 is the new common neighbour of w'_2 and u'_l. By Lemma 2, u'_2 has to be adjacent to a neighbour of w'_2; the only choice is w'_3, since the other neighbours have already degree four. By repeated application of Lemma 3 we obtain that $u'_i w'_{i+1} \in E$ for all $i < l$. This settles all the adjacencies; the resulting graph is $S_{2s+1,l}$. This may not seem immediate from our definition of "strange" graphs. To see that G is indeed one of $S_{p,q}$, we have to find a walk in $S_{p,q}$ that plays the role of C, that is, we have to find a non-opposite walk of shortest length. There are several walks that satisfy this property; using the same notation as in the definition of $S_{p,q}$, one possibility is $C = W((p-3, q-3), \{(p-3, q-3), (p-2, q-3)\})$. Notice that whether we have a "strange" graph of the first or second kind depends on the values of s and l, although all the pictures correspond to the case $p \leqslant q$ for simplicity.

Let us treat now the case in which the second common neighbour z_1 of u'_l and w'_2 is not u'_1. From the comments above, z_1 is clearly different from all the vertices in R_s. As we did in the case of the first layer, let z_i be the second common neighbour of z_{i-1} and w'_{i+1} for $3 \leqslant i \leqslant l-1$, and let z_l be the second common neighbour of z_{l-1} and u'_1. With a reasoning analogous to that of Claim 1, it can be proved that none of the vertices z_i belongs to R_s, and thus we arrive to the situation depicted in Fig. 8. If z'_1 is one of the previous vertices, by an argument similar to that in Claim 2 we see that the only possibilities are $z'_1 = z_1$ or $z'_1 = u'_{l-1}$. If $z'_1 = z_1$, by Lemma 3 we obtain $u'_{i-1} z_i \in E$ for all i, $1 \leqslant i \leqslant l$; the locally grid property is satisfied at each vertex and the resulting graph is $S_{2s+2,l}$. The other case is impossible, because if $z'_1 = u'_{l-1}$, then Lemma 6 applied to the opposite path $z_l u'_1 u'_2 \ldots$ implies that u'_1 is adjacent to u'_l, a vertex that already has degree 4.

The only case that remains is when z'_1 is a new vertex. Define z'_2 as the second common neighbour of z'_1 and u'_1, and recursively z'_j as the second common neighbour of z'_{j-1} and u'_{j-1}, for j with $3 \leqslant j \leqslant l$. Again as in Claim 1 all the vertices z'_j are new, $z'_j u'_{j-1} \in E$ and z'_l is joined to z_1. We have thus added a new layer to R_s.

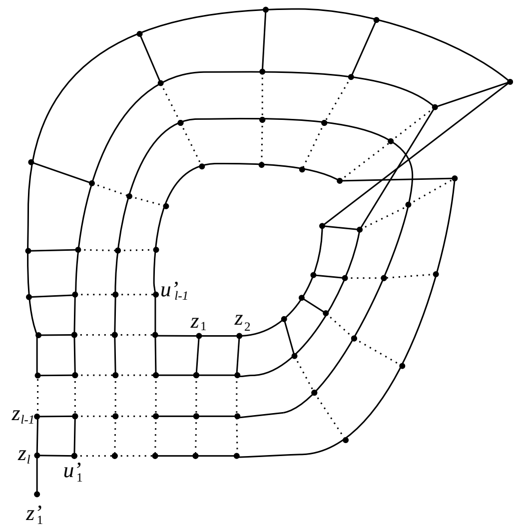

Fig. 8. In the middle of a new layer.

This concludes the proof of the classification theorem of locally grid graphs.

We can also use the walks $W(v,e)$ to prove that almost all our graphs are non-isomorphic. It is easy to see that the structure of the walks $W(v,e)$ of a locally grid graph (how many of them there are and how long they are) is invariant under isomorphism. Using this fact one sees that among all the graphs we have defined, and with a fixed number of vertices pq, the only pair that are isomorphic are $T^{\delta}_{p,q}$ and $T^{\delta'}_{p',q'}$ with $pq = p'q', (p,\delta) = q'$ and $(p',\delta') = q$. The structure of the walks $W(v,e)$ can also be used to show that the only vertex-transitive locally grid graphs are the Torus $T^{\delta}_{p,q}$ for every p,q,δ. The Klein bottles $K^{0}_{p,q}$ might be thought to be vertex-transitive, but note that among the opposite cycles of length $2q$, there are only two of them that have chords, while the remaining ones are chordless (a chord is an edge joining two non-consecutive vertices of a cycle).

Using the classification theorem we can prove several properties of locally grid graphs just by proving them for each family. One example of this is the following corollary.

Corollary 7. *The edge-connectivity of every locally grid graph is equal to* 4.

3. Tutte polynomials

We first recall the definition and basic facts about Tutte polynomials (see [5,13] for thorough surveys).

Let $G=(V,E)$ be a graph with vertex set V and edge set E (loops and multiple edges are allowed). For every subset $A \subseteq E$, its *rank* is $r(A) = n - k(G|A)$, where

$n = |V|$ and $k(G|A)$ is the number of connected components of the spanning subgraph (V,A). The *rank-size generating polynomial* is defined as:

$$R(G;x,y) = \sum_{A \subseteq E} x^{r(A)} y^{|A|}.$$

Notice that the coefficient of $x^i y^j$ in $R(G;x,y)$ counts the number of spanning subgraphs in G with rank i and j edges. The *Tutte polynomial* is given by

$$T(G;x,y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

It should be clear that both $R(G;x,y)$ and $T(G;x,y)$ contain exactly the same information about G, and so the Tutte polynomial tells us for every i and j the number of edge-sets in G with rank i and size j.

However the Tutte polynomial has several interesting properties not shared by R. In particular, it satisfies the fundamental contraction-deletion rule

$$T(G;x,y) = T(G-e;x,y) + T(G/e;x,y),$$

provided that $e \in E(G)$ is neither a bridge nor a loop, where $G-e$ and G/e denote, respectively, the result of deleting and contracting the edge e in G.

Our purpose in this section is to show that the condition of being locally grid can be captured from the knowledge of the Tutte polynomial. We need the following lemma, which follows from Theorems 2.2 and 2.4 in [12].

Lemma 8. *If G is a 2-connected simple graph and H is T-equivalent to G, then H is also simple and 2-connected. If G is a 2-connected simple graph, then the following parameters are determined by its Tutte polynomial*:

(1) *The number of vertices and edges.*
(2) *The edge-connectivity.*
(3) *The number of cycles of length three, four and five.*

Theorem 9. *Suppose a graph H is T-equivalent to a locally grid graph G that does not contain cycles of length 5. Then H is locally grid and contains no cycles of length 5.*

Proof. By the classification theorem and Corollary 7, we know that G has pq vertices for some p,q, has $2pq$ edges, and edge-connectivity equal to 4; by the previous lemma, H has these same parameters. We deduce that H has minimum degree at least 4 and hence is 4-regular. Since G is locally grid, from the previous lemma we also know that H is triangle-free and has exactly pq squares (cycles of length four). Also, since G has no C_5, neither does H. □

Claim 1. *H contains no subgraph isomorphic to $K_{2,3}$.*

Proof. The graph $K_{2,3}$ has rank 4 and size 6. Since G does not contain any subgraph with these parameters, neither does H. □

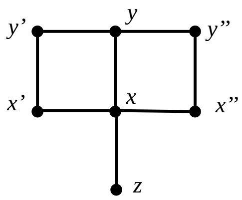

Fig. 9. Local structure around vertex x.

Claim 2. *H contains exactly $2pq$ cycles C_6 with a* long *chord, that is, a chord joining opposite vertices.*

Proof. We consider subgraphs of rank 5 and size 7. Besides C_6 with a long chord, there are two possibilities for a triangle-free graph of this kind: $K_{2,3}$ plus one edge sharing at most one endpoint with the edges in $K_{2,3}$; and a cycle C_5 plus one vertex joined to two vertices of the cycle. By Claim 1 and the fact that H has no C_5, these two possibilities are excluded in H. Hence the number of C_6 with a long chord in H is the same as in G, namely $2pq$. □

Claim 3. *Every edge of H is in exactly two squares.*

Proof. Since H is 4-regular and contains no $K_{2,3}$, an edge can be in at most 3 squares. For $i = 0, 1, 2, 3$, let n_i be the number of edges contained in exactly i squares. Then, double counting the number of pairs (e, s) such that s is a square containing the edge e, we obtain

$$4pq = 0n_0 + 1n_1 + 2n_2 + 3n_3.$$

By Claim 2, double counting the number of pairs (e, h) such that h is a C_6 having e as a long chord, we obtain

$$2pq = 0n_0 + 0n_1 + 1n_2 + 3n_3.$$

From the above two equations it follows

$$n_1 + n_2 = 2pq.$$

But since $n_0 + n_1 + n_2 + n_3 = 2pq$ we have $n_0 = n_3 = 0$ and, consequently, $n_1 = 0$ and $n_2 = 2pq$, as was to be proved. □

Observe that the union of the two squares in the previous claim must form a C_6 having the given edge as a long chord, since the other two possibilities imply the existence of either a $K_{2,3}$ or a double edge.

Finally, we check the locally grid condition. Let $x \in V(H)$, and let y be a neighbour of x. From the above claim it follows that the edge xy is in two squares $xyy'x'$ and $xyy''x'$. Let z be the fourth vertex adjacent to x. Then we have the situation depicted in Fig. 9.

Consider now a square containing xz. It must contain a second edge incident with x, and it cannot be xy since then xy would belong to three squares. Hence the two squares containing xz must be $zxx'u$ and $zxx''v$ for some u and v. Note that $\{u,v\}\cap\{y',y''\}=\emptyset$, otherwise xy would be in three squares. Also $u\neq v$, since $u=v$ would force xx' to be in three squares.

In order to finish we must show two more things. First, that there is no other edge among the vertices $\{x,x',x'',y,y',y'',z,u,v\}$; this is clear since otherwise we have a triangle, a C_5, or an edge contained in more than two squares. Secondly, that y and z do not have any common neighbour besides x, and that the same holds for x' and x''; this is because otherwise an edge would be in 3 squares.

4. Codifying and counting edge-sets

This and the following section are devoted to the proof of the T-uniqueness of $C_p\times C_q$. We show that for every locally grid graph G different from $C_p\times C_q$ and with pq vertices there is at least one coefficient $a_{i,j}$ of the rank-size generating polynomial $R(G;x,y)$ in which $C_p\times C_q$ and G differ. This coefficient is related to the topological structure of locally grid graphs.

Let $\mathscr{M}$ be the surface in which a locally grid graph G is embedded naturally, that is, $\mathscr{M}$ is a torus if $G\cong T^{\delta}_{p,q}$ and a Klein bottle if $G\cong K^{i}_{p,q},S_{p,q}$. We call a cycle C in G *contractible* if $\mathscr{M}-C$ has two connected components, one of them contractible to a point. Otherwise we call C an *essential cycle*. In other words, a contractible cycle determines a simply connected region, whereas an essential cycle does not.

Let l_G be the shortest length of an essential cycle in G (for example, $l_{C_p\times C_q}=\min\{p,q\}$ and $l_{K^0_{p,q}}=\min\{p,q+1\}$). The number of essential cycles of length l_G contributes to the coefficient a_{l_G-1,l_G} of $R(G;x,y)$, which counts the number of edge-sets with rank l_G-1 and size l_G, but clearly there are other subgraphs that also contribute to this coefficient. We prove that if G and G' are locally grid graphs of the same order, and $m\leqslant\min\{l_G,l_{G'}\}$, then G and G' have the same number of edge-sets with rank $m-1$ and size m that do not contain essential cycles. Therefore, if $l_G<l_{G'}$, then the coefficients of $x^{l_G-1}y^{l_G}$ in $R(G;x,y)$ and $R(G';x,y)$ are different, and thus G and G' are not T-equivalent.

The aim of this section is to give the basic counting tool in order to prove Lemma 10. In the next section, we calculate the quantities l_G for all locally grid graphs G and using Corollary 15 we show that $C_p\times C_q$ is a T-unique graph.

Lemma 10. *Fix $m>0$. The number of edge-sets with rank $m-1$ and size m that do not contain an essential cycle is the same for all locally grid graphs G with pq vertices and such that $m\leqslant l_G$.*

We call an edge-set $A\subseteq E(G)$ a *normal edge-set* if it does not contain any essential cycle. We prove first the lemma for connected normal edge-sets and then generalize to the case of several connected components. The proof has three main steps. Firstly we see that locally grid graphs are *locally orientable* and then use that to establish a canonical way to represent edge-sets with words over a given alphabet. By means of these words we count the number of edge-sets described above and show that it does not depend on the graph. Some of the technical details of the proof will be left to the reader.

Let G be a locally grid graph. An *orientation at a vertex* v consists of labeling the four edges incident with v bijectively with the labels N,S,E,W in such a way that the edges labelled N and S are opposite, and so are the ones labelled E and W (note that the cyclical clockwise order around v need not be N,E,S,W). By the orientation (v,e,f) we mean that v is the origin vertex, e is labelled E and f is labelled N. For $\alpha \in \{N,S,E,W\}$, we denote by α^{-1} the label opposite to α. If w is a vertex adjacent to v, then the orientation at v induces an orientation at w in the natural way: if the edge $\{v,w\}$ was labelled α from v, it is labelled α^{-1} from w, and if $xvwy$ is a square, and $\{x,v\}$ was labelled β from v, $\{w,y\}$ is also labelled β from w. In the same way, if P is a path beginning at v, we can translate the orientation to all the vertices in P. If P and P' are paths joining v and v' the orientation at v' induced by P could be different from that induced by P'. This does not happen if the union of P and P' is a contractible cycle.

Indeed, in this case the union of P and P' determines a simply connected region. This allows us to transform one path into the other one through the simply connected region by means of the following two elementary transitions and their inverses: if e,f,g,h are the four edges of a square ordered cyclically, we can change e,f,g in a path by h, or e,f by g,h, and these operations do not change the orientation at the endpoint.

Therefore, if we fix an orientation at a vertex $v \in V(A)$ of a connected normal edge-set A, then all the vertices in $V(A)$ are unambiguously oriented. With this orientation fixed, every path in A can be described by a sequence of the labels $\{N,S,E,W\}$ (see Fig. 10a). This enables us to assign coordinates to every vertex in $V(A)$. The vertex v has coordinates $(0,0)$, and $w \in V(A)$ has coordinates (i,j) if in one (and thus in every) path in A joining v to w, the number of labels E minus the number of labels W equals i, and the number of labels N minus the number of labels S equals j. It can be proved that if $|A| \leqslant l_G+2$, then different vertices have different coordinates. The proof is by induction on $|A|$ and it uses the two transitions mentioned in the last paragraph. From now on, and unless otherwise stated, we consider that all normal edge-sets have at most l_G+2 elements. Note finally that if the orientation is changed the coordinates will also change.

Now we are ready to codify the edge-sets of a locally grid graph. In order to do this, we define a set Γ of words over the alphabet $\{N,S,E,W\}$ that represents all possible connected normal edge-sets. The definition of Γ is given in the infinite square lattice L^∞ and we use the previous discussion on orientations to assign a unique word to every connected normal edge-set A. Having done this, it is quite simple to evaluate the quantities defined in Lemma 10.

Define the infinite plane square lattice L^∞ as the infinite graph having as vertices $\mathbb{Z}\times\mathbb{Z}$ and in which (i,j) is joined to $(i-1,j),(i+1,j),(i,j-1)$ and $(i,j+1)$. Let $\mathscr{S}$ be the group of the graph automorphisms of L^∞ (that is, the group generated by the translations, symmetries and rotations of the plane that map vertices to vertices). Given

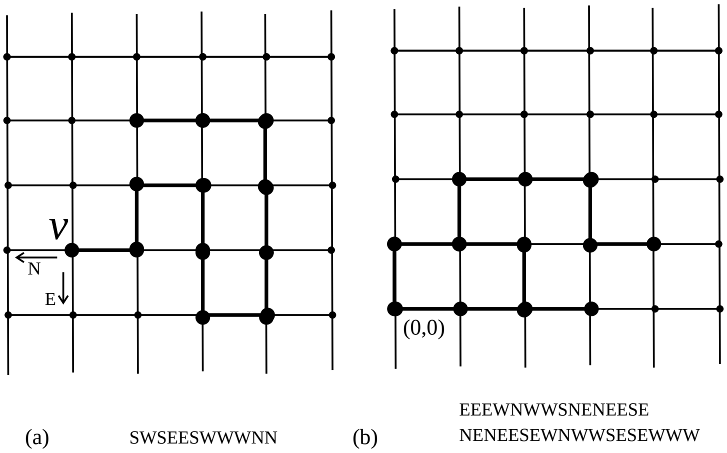

Fig. 10. (a) A path described using a sequence over $\{N,S,E,W\}$. (b) A connected subgraph and two different sequences codifying it.

B_1 and B_2 finite connected edge-sets in L^∞, we say that $B_1 \sim B_2$ if there is $\sigma \in \mathscr{S}$ such that $\sigma(B_1) = B_2$. The relation $\sim$ is an equivalence relation. Let $\Sigma(L^\infty)$ be the set of all finite connected edge-sets of L^∞ and choose $\mathscr{B}$ to be a set of representatives of $\Sigma(L^\infty)/\sim$ such that every $B \in \mathscr{B}$ contains the vertex $(0,0)$. Intuitively, this set of representatives covers all the possible "shapes" that a normal edge-set could have. Label the edge $\{(0,0),(1,0)\}$ with E and the edge $\{(0,0),(0,1)\}$ with N. Now assign to each $B \in \mathscr{B}$ a sequence $\gamma_B = \alpha_1\alpha_2\ldots\alpha_{n(B)}$ over the alphabet $\{N,S,E,W\}$ in such a way that beginning at the origin and following the instructions given by γ_B, the edges covered are exactly those of B (note that an edge might be covered more than once and that there are several choices for the sequence γ_B, but among all possibilities we choose one at random; see Fig. 10b for an example). We will refer to γ_B as the *word* of B and $\Gamma = \{\gamma_B\colon B \in \mathscr{B}\}$ is the set of all possible such words. Conversely, given $\gamma \in \Gamma$, $B(\gamma)$ will denote the edge-set in $\mathscr{B}$ such that $\gamma_{B(\gamma)} = \gamma$.

The next step consists of assigning one word from Γ to each normal connected edge-set of a locally grid graph. Given $\gamma \in \Gamma$ and a locally grid graph G, we can produce from γ a connected edge-set of $E(G)$. Take a vertex $v \in V(G)$ and two edges $e, f \in E(G)$ adjacent at v. The set $A^G(\gamma, v, e, f)$ is produced following the code given by γ from the vertex v with orientation (v, e, f). We call this set an *instance of γ in G* and the triple (v, e, f) is the *anchor* of the instance. It is easy to prove by induction on the length of the word that if the instance $A^G(\gamma, v, e, f)$ is a normal edge-set, then it has the same rank and size as $B(\gamma)$. Note that taking different anchors we might obtain the same instance of γ (see Fig. 11 for an example). The number of anchors that lead to the same edge-set depends only on the symmetries of $B(\gamma)$ and we call it $\mathrm{sym}(\gamma)$.

Now, we prove that every connected normal edge-set $A \subseteq E(G)$ is the instance of a unique word $\gamma \in \Gamma$. This is done through the following pair of lemmas.

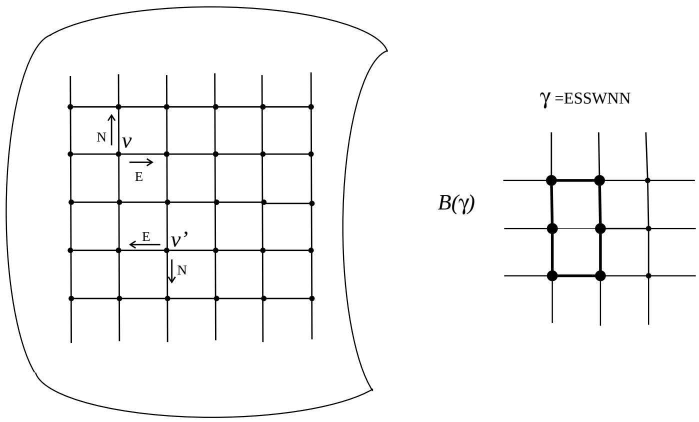

Fig. 11. Various anchors of the same word that lead to the same instance in a locally grid graph.

Lemma 11. *Given a connected normal edge-set* $A \subseteq E(G)$ *and an orientation* (v, e, f), $v \in V(A)$, *there exists a unique edge-set* $B' \subseteq E(L^\infty)$ *and a unique graph isomorphism* $\varphi : A \to B'$ *such that*:

(1) $\varphi(v) = (0, 0)$.
(2) *If the edge* $xy \in A$ *is labelled* α *from* x, *then* $\bar{\varphi}(xy)$ *is labelled* α *from* $\varphi(x)$ *according to the orientation* $((0,0), \{(0,0),(1,0)\}, \{(0,0),(0,1)\})$, *where* $\bar{\varphi}$ *is the morphism induced on edges by* φ.

Proof. The sketch of the proof is as follows. Assign coordinates to the vertices of A according to the orientation (v, e, f), as explained before (note that we need A to be normal and connected to assign coordinates unambiguously). Define $\varphi : V(A) \to V(L^\infty)$ by $\varphi(x) = (i, j)$ if the coordinates of x are (i, j), and $\bar{\varphi} : A \to E(L^\infty)$ by $\bar{\varphi}(xy) = \varphi(x)\varphi(y)$. Take $B' = \bar{\varphi}(A)$. The uniqueness of B' is proved by induction on $\max\{d(v, x), x \in V(A)\}$. □

Lemma 12. *Given a connected normal edge-set* $A \subseteq E(G)$, *there exists a unique word* $\gamma(A) \in \Gamma$ *and a (not necessarily unique) anchor* (v, e, f) *such that* $A = A^G(\gamma(A), v, e, f)$.

Proof. Fix $v' \in V(A)$ and a pair e', f' of adjacent edges at v'. Apply the previous lemma to A with orientation (v', e', f') and obtain B'. Let $B \in \mathscr{B}$ be the representative of the equivalence class of B' and let $\sigma \in \mathscr{S}$ be the automorphism of L^∞ that maps B to B'. Take v to be the vertex $\varphi^{-1}\sigma((0, 0))$. Now assume $\sigma(\{(0,0),(1,0)\})$ is labeled α from $\sigma((0,0))$. Then take as e the edge labeled α from v according to the orientation (v', e', f') (note that this edge does not necessarily belong to A). Find f analogously. Then $A = A^G(\gamma(A), v, e, f)$. To prove the uniqueness, suppose that there exists $B'' \in \mathscr{B}$ such that $A = A^G(\gamma_{B''}, v'', e'', f'')$. Then, by the definition of instance, we can find $\sigma' \in \mathscr{S}$ such that $\sigma'(B'')$ satisfies the conclusion of Lemma 11 when applied to A with orientation (v', e', f'). Hence B' and B'' are in the same equivalence class, and thus $B'' = B$. □

Lemma 12 enables us to use words to count edge-sets and to prove the following weak version of Lemma 10.

Lemma 13. *For fixed $m > 0$, the number of connected normal edge-sets with rank $m-1$ and size m is the same for all locally grid graphs G with pq vertices and such that $m \leqslant l_G$.*

Proof. The preceding lemma shows that every normal connected edge-set A is the instance of one and only one word $\gamma(A)$. As it has been mentioned before, since A is normal and connected, A and $B(\gamma(A))$ have both the same rank and size. It is not difficult to prove by induction on m that the instance of a word γ such that $r(B(\gamma)) = m-1$ and $|B(\gamma)| = m$ is a normal edge-set. Therefore, the number of normal edge-sets with rank $m-1$ and size m equals the number of distinct instances of words corresponding to edge-sets in L^∞ with rank $m-1$ and size m. For γ fix, we can choose $8pq$ different anchors from which we obtain $8pq$ instances. The number of anchors that give rise to the same instance is $\mathrm{sym}(\gamma)$, that depends only on the word and not on the graph in which we produce the instance. We denote by $\Gamma^{r,s}$ the set of all words $\gamma \in \Gamma$ such that $r(B(\gamma)) = r$ and $|B(\gamma)| = s$. Then the number of connected normal edge-sets with rank $m-1$ and size m is

$$\sum_{\gamma \in \Gamma^{m-1,m}} \frac{8pq}{\mathrm{sym}(\gamma)},$$

which does not depend on the graph G. $\square$

Our next aim is to prove the non-connected version of this lemma.

Lemma 14. *For fixed $m > 0$ and $n > 1$, the number of normal edge-sets with rank $m-1$, size m, and n connected components is the same for all locally grid graphs G with pq vertices and such that $m \leqslant l_G$.*

Proof. Let $A_1, \ldots, A_n$ be the connected components of a normal edge-set $A \subseteq E(G)$ and denote by γ_i the word corresponding to A_i. Denote by $\mathscr{M}(\Gamma)$ the family of all multisets of Γ and define

$$\Gamma_n^{r,s} = \left\{ \{\gamma_1, \ldots, \gamma_n\} \in \mathscr{M}(\Gamma)\colon \sum_1^n r(B(\gamma_i)) = r, \sum_1^n |B(\gamma_i)| = s \right\}.$$

We choose an ordering for every multiset $\tilde{\gamma}$ and view it as an ordered tuple when necessary. We assign to A the multiset from $\Gamma_n^{m-1,m}$ that corresponds to the words of the connected components of A. An *anchor* for $\tilde{\gamma} \in \Gamma_n^{m-1,m}$ is a tuple $x = (v_1, e_1, f_1, \ldots, v_n, e_n, f_n)$ such that $v_i \in V(G)$ and e_i and f_i are adjacent edges at v_i (note that the triples (v_i, e_i, f_i) may not be pairwise different). The *instance* of $\tilde{\gamma}$ with anchor x in a locally grid graph G is the edge-set $\bigcup A^G(\gamma_i, v_i, e_i, f_i)$. We say that an instance of $\tilde{\gamma}$ is an *overlapping instance at* (i,j) if $A^G(\gamma_i, v_i, e_i, f_i) \cup A^G(\gamma_j, v_j, e_j, f_j)$ is connected; note that an instance overlapping at (i,j) might overlap at other pairs too. With these definitions it should be clear that the number of normal edge-sets as in the statement of the lemma is the sum over all words $\tilde{\gamma}\in\Gamma_n^{m-1,m}$ of the number of instances of $\tilde{\gamma}$ non-overlapping at any pair. The only thing that remains to be proved is that this last quantity, call it $C_{\tilde{\gamma}}^G$, does not really depend on G.

Let $\mathscr{X}^G$ be the set of all possible anchors in G for $\tilde{\gamma}$ (with an abuse of notation we omit the reference to $\tilde{\gamma}$ in $\mathscr{X}^G$). Define $\mathscr{X}_{ij}^G, 1\leqslant i<j\leqslant n$, as the set of all anchors in $\mathscr{X}^G$ that give rise to an instance overlapping at (i,j). As an application of the Principle of Inclusion and Exclusion we obtain the following expression for $C_{\tilde{\gamma}}^G$:

$$C_{\tilde{\gamma}}^G=\frac{\sum_{I\subseteq\{(i,j):1\leqslant i<j\leqslant n\}}(-1)^{|I|}|\mathscr{X}_I^G|}{\mathrm{sym}(\gamma_1)\cdots\mathrm{sym}(\gamma_n)\mathrm{sym}(\tilde{\gamma})},$$

where $\mathscr{X}_I^G=\bigcap_{(i,j)\in I}\mathscr{X}_{ij}^G$, $\mathscr{X}_\emptyset^G=\mathscr{X}^G$, and $\mathrm{sym}(\tilde{\gamma})$ is the number of permutations $\pi\in\mathscr{S}_n$ such that $\pi(\tilde{\gamma})=\tilde{\gamma}$. The lemma follows now from the next claim.

Claim. *$|\mathscr{X}_I^G|$ does not depend on G.*

Proof. We can view I as the set of edges of a graph H_I with vertex set $\{1,\ldots,n\}$. We prove first the case in which H_I is connected. This means that all the instances of the anchors in $\mathscr{X}_I^G$ are connected edge-sets in G.

Fix an orientation (v,e,f) in G and define $\mathscr{X}_I^{(v,e,f)}$ as the subset of $\mathscr{X}_I^G$ consisting of all the anchors that begin with (v,e,f). It is easy to see that $|\mathscr{X}_I^G|=8pq|\mathscr{X}_I^{(v,e,f)}|$. Define now the sets of anchors in the infinite plane lattice $\mathscr{X}^\infty,\mathscr{X}_{ij}^\infty$ and $\mathscr{X}_I^\infty$ analogously to $\mathscr{X}^G,\mathscr{X}_{ij}^G$ and $\mathscr{X}_I^G$. Define also $\mathscr{X}_I^{\text{origin}}$ as the subset of $\mathscr{X}_I^\infty$ in which anchors begin by $((0,0),\{(0,0),(1,0)\},\{(0,0),(0,1)\})$. Since $\tilde{\gamma}$ consists of n words adding up to rank $m-1$ and size m, for $m\leqslant l_G$, the instance of one of the components γ_i of $\tilde{\gamma}$ contains a cycle of length less than l_G. This implies that the instances of $\tilde{\gamma}$ with anchor in $\mathscr{X}_I^{(v,e,f)}$ do not contain essential cycles, and hence are normal connected edge-sets. Thus we can apply Lemma 11 to prove that there exists a bijection between $\mathscr{X}_I^{(v,e,f)}$ and $\mathscr{X}_I^{\text{origin}}$. Therefore, if H_I is connected then $|\mathscr{X}_I^G|=8pq|\mathscr{X}_I^{\text{origin}}|$ for every locally grid graph G with pq vertices.

If H_I is not connected, let $V_1,\ldots,V_s\subseteq\{1,\ldots,n\}$ be the vertices of its connected components and let $I_1,\ldots,I_s\subseteq I$ be the edge-sets of these components (note that some I_k might be empty). Then

$$|\mathscr{X}_I^G|=|\mathscr{X}_{I_1}^G|\cdots|\mathscr{X}_{I_s}^G|,$$

where the anchors in $\mathscr{X}_{I_k}^G$ refer only to the words $\gamma_i\in\tilde{\gamma}$ for $i\in V_k$ and not to the whole of $\tilde{\gamma}$. Since all the H_{I_k} are now connected, the argument above shows that none of $|\mathscr{X}_{I_k}^G|$ depends on G, and therefore neither does $|\mathscr{X}_I^G|$, and the claim is proved.

This finishes the proof of the lemma. □

Lemmas 13 and 14 together imply Lemma 10 and the following corollary.

Corollary 15. *Let G,G' be a pair of locally grid graphs with pq vertices. If $l_G\neq l_{G'}$, then $T(G;x,y)\neq T(G';x,y)$. If $l_G=l_{G'}$ but G and G' do not have the same number of shortest essential cycles, then also $T(G;x,y)\neq T(G';x,y)$.*

Proof. Suppose that $l_G < l_{G'}$. By Lemma 10, the number of normal edge-sets of rank $l_G - 1$ and size l_G is the same in G and G'. Since there are essential cycles of length l_G in G, but not in G', the coefficient of $x^{l_G-1}y^{l_G}$ in the rank-size generating polynomial is greater in G than in G', and thus their Tutte polynomials are different. The second statement follows in a similar way. □

A careful revision of the proof of Lemma 10 shows that in some special cases it is also possible to count the number of normal edge-sets with size greater than l_G. We say that a word γ' contains the word γ if $B(\gamma)$ is a subgraph of $B(\gamma')$. Denote by $N(\gamma, G, m, r)$ the number of normal edge-sets in G with rank r and size m, and such that its word γ' contains γ. We say that an edge-set $A \subseteq E(G)$ is a *forbidden edge-set for* γ if it contains an essential cycle and a subset $B \subseteq A$ such that B is normal and has γ as its word.

Corollary 16. *Let $\gamma \in \Gamma$ be such that $B(\gamma)$ contains at least one cycle. Then the quantity $N(\gamma, G, m, r)$ is the same for all locally grid graphs G with pq vertices, no forbidden edge-set for γ of size m, and such that $l_G \geqslant m - 2$.*

5. Tutte uniqueness

In the previous sections we have assembled all the necessary machinery in order to prove the T-uniqueness of products of cycles. In the light of Corollary 15, we first examine the length and number of the shortest essential cycles in all locally grid graphs:

Lemma 17. *If G is a locally grid graph, then the length l_G of the shortest essential cycle, and the number of those cycles, or a lower bound on this number, are given in the following table.*

G	l_G	# of essential cycles	
$C_p \times C_q$	$\min\{p, q\}$	q	if $p < q$
		$2p$	if $p = q$
		p	if $q < p$
$T^{\delta}_{p,q}$	$\min\{p, q+\delta\}$	q	if $p < q + \delta$
		$q + p\binom{q+\delta-1}{\delta}$	if $p = q + \delta$
		$p\binom{q+\delta-1}{\delta}$	if $q + \delta < p$
$K^0_{p,q}$	$\min\{p, q+1\}$	q	if $p < q + 1$
		$5q$	if $p = q + 1$
		$4q$	if $q + 1 < p$
$K^1_{p,q}$	$\min\{p, q\}$	q	if $p < q$
		$q + 1$	if $p = q$
		1	if $q < p$
$K^2_{p,q}$	$\min\{p, q\}$	q	if $p < q$
		$q + 2$	if $p = q$
		2	if $q < p$
$S_{p,q}$	$\min\{2p, q\}$	$\# \geqslant 2^{q-1}$	if $p \leqslant q \leqslant 2p$
		$\# \geqslant q$	if $2p \leqslant q$
		2^q	if $q \leqslant p$

Proof. All locally grid graphs can be obtained by adding some edges to a (p,q)-grid H. Let us call any of these edges an *exterior edge*. Clearly every essential cycle must contain some exterior edge. Essential cycles of shortest length are obtained by joining the two ends of an exterior edge by a path contained in the grid H. For every family of locally grid graphs, we study the lengths of such paths. We repeatedly use the fact that the length of a shortest path between the points $(0,0)$ and (a,b) in a grid is $a+b$, and that there are $\binom{a+b}{a}$ such paths.

1. $T^{\delta}_{p,q}$: There is only one shortest path determined by each of the q exterior edges of the form $\{(0,j),(p-1,j)\}$, and the resulting cycle has length p. Each of the p edges of the form $\{(i,0),(i+\delta,q-1)\}$ determines $\binom{q+\delta-1}{\delta}$ shortest cycles of length $q+\delta$.
2. $K^{i}_{p,q}$: The q edges of the form $\{(0,j),(p-1,j)\}$ give rise to the same number of essential cycles as in the previous case.
   If $i=0$, among the edges of the form $\{(j,0),(p-j-1,q-1\}$, the shortest essential cycle is determined by any of the following four edges:

   $$\{(0,0),(p-1,q-1)\},\{(p-1,0),(0,q-1)\},$$

   $$\{(p/2,0),(p/2-1,q-1)\},\{(p/2-1,0),(p/2,q-1)\}.$$

   Any of these gives rise to q cycles of length $q+1$.
   If $i=1$, the edge $\{((p-1)/2,0),((p-1)/2,q-1)\}$ is, among the "twisted" edges, the one that determines a shortest cycle; in this case the cycle has length q. Similarly, if $i=2$ there are two essential cycles of length q.
3. $S_{p,q}$: We treat first the case $p \leqslant q$. Each of the $(q-p)$ exterior edges of the form $\{(0,i),(p-1,i-p)\}$ determines $\binom{2p-1}{p}$ cycles of length $2p$. If $2p \leqslant q$, then $(q-p)\binom{2p-1}{p} > q$, and the bound in the table follows. The edges of the form $\{(0,i),(i,q-1)\}$ or $\{(i,0),(p-1,q-p+i)\}$ give rise to $\binom{q-1}{i}$ cycles of length q each. We have thus to evaluate the quantity

   $$2\sum_{i=0}^{p-1}\binom{q-1}{i} = \sum_{i=0}^{p-1}\binom{q-1}{i} + \sum_{i=0}^{p-1}\binom{q-1}{q-1-i}.$$

   If $q \leqslant 2p$, then $q-p \leqslant p$ and hence the expression above contains at least once all the binomials of the form $\binom{q-1}{j}$ for j with $0 \leqslant j \leqslant q-1$. Therefore the number of cycles of length q is at least 2^{q-1}.

   Let us study now the case $q<p$. Each of the q edges of the form $\{(j,0),(0,q-1-j)\}$ determines $\binom{q-1}{j}$ cycles of length q. These quantities add up to 2^{q-1} cycles. Since the edges of the form $\{(p-1-i,q-1),(p-1,i)\}$ behave in the same way, we have a total of 2^{q} essential cycles of length q. The essential cycles determined by the edges $\{(i,q-1),(i+q,0)\}$ have length $2q$, and thus they are never the shortest ones. □

Now, we can formulate our main result:

Theorem 18. *The graph $C_p \times C_q$ is T-unique for $p,q \geqslant 6$.*

Proof. Let $p,q \geqslant 6$ be fixed integers, and let G be a graph T-equivalent to $C_p \times C_q$. By Theorem 9 we know that G is locally grid, and by Theorem 1 we know that G is one of $T^{\delta}_{p',q'}, K^{i}_{p',q'}, S_{p',q'}$, with $p'q' = pq$.

In order to show that G is necessarily $C_p \times C_q$, we make use of Corollary 15 and Lemma 17. Thus, it only remains to distinguish the cases in which the length and number of shortest essential cycles can agree with the length and number of shortest essential cycles in $T_{p,q}$. We assume that $p \leqslant q$.

Case I: $T^{\delta}_{p,q}$, $\delta > 0$, $p < q$.

Our aim is to show that $T_{p,q}$ has more edge-sets with rank $q-1$ and size q than $T^{\delta}_{p,q}$. Recall that H is the $p \times q$ grid used to define locally grid graphs. For every r with $0 \leqslant r \leqslant q-1$, denote by E_r the set of edges that join a vertex at height r in H with a vertex at height $r+1$. Let A be an edge-set of size q in either $T_{p,q}$ or $T^{\delta}_{p,q}$. If there exists some r such that A does not contain any edge in E_r, define $s(A)$ as

$$s(A) = \min\{r | A \cap E_r = \emptyset\}.$$

Observe that if $A \subseteq E(T^{\delta}_{p,q})$ the minimum $s(A)$ always exists, whereas there are some essential cycles of length q in $T_{p,q}$ that contain one edge of each set E_r. To prove that $T_{p,q}$ has more edge-sets with rank $q-1$ and size q than $T^{\delta}_{p,q}$ it is enough to find a bijection φ_r between $\{A \subseteq E(T_{p,q}) | r(A) = q-1, |A| = q, s(A) = r\}$ and $\{A \subseteq E(T^{\delta}_{p,q}) | r(A) = q-1, |A| = q, s(A) = r\}$.

Define $\bar{\varphi}_r$ from $E(T_{p,q}) - E_r$ to $E(T^{\delta}_{p,q}) - E_r$ as

$$\{(x,y),(x',y')\} \to \begin{cases} \{(x,y),(x',y')\} & \text{if } r+1 \leqslant y,y' \leqslant q-1, \\ \{(x-\delta,y),(x'-\delta,y')\} & \text{otherwise.} \end{cases}$$

It is straightforward to show that $\bar{\varphi}_r$ is a bijection. To obtain the desired bijection φ_r define $\varphi_r(A)$ as $\bigcup_{e \in A} \bar{\varphi}_r(e)$.

Case II: $K^{i}_{p,q}$, $p < q$.

This case is solved by an argument similar to the one in the previous case.

Case III: $K^{0}_{p',q'}$, with $p = q'+1$, $q = 4q'$, $p' = 4(q'+1)$.

In this case we prove that in $T_{p,q}$ there are more edge-sets with rank $q'+2$ and size $q'+3$ than in $K^{0}_{p',q'}$. These edge-sets can be classified into three groups:

(1) Normal edge-sets.
(2) Sets containing an essential cycle of length $q'+1$ and two other edges.
(3) Essential cycles of length $q'+3$.

We show that the number of edge-sets in each of these groups is greater in $T_{p,q}$ than in $K^{0}_{p',q'}$, therefore proving that $R(T_{p,q};x,y) \neq R(K^{0}_{p',q'};x,y)$.

(1) We apply Corollary 16 to count normal edge-sets. The only possible forbidden edge-sets are the ones in $K^{0}_{p',q'}$ with rank $q'+2$ and size $q'+3$ as shown in

Fig. 12a. Therefore, the number of normal edge-sets having rank $q'+2$ and size $q'+3$, and containing a cycle of length at least six, is the same in $K^0_{p',q'}$ as in $T_{p,q}$. It only remains to prove that the number of normal edge-sets having rank $q'+2$ and size $q'+3$, and containing a cycle of length four, is smaller in $K^0_{p',q'}$ than in $T_{p,q}$.

Again by Corollary 16, the number of edge sets with rank $q'+1$, size $q'+2$, and containing a square is the same in both graphs, call it $s_{q'+1}$. Add one edge to each of these sets in order to obtain edge-sets with size $q'+3$ and rank at most $q'+2$. There are three possibilities depending on which edge we are adding. The resulting set can be of one of the following types.

A. A normal edge-set with rank $q'+2$.
B. A normal edge-set containing two contractible cycles and hence having rank $q'+1$.
C. An edge-set containing an essential cycle of length $q'+1$ and a contractible cycle of length four.

Call $\mathscr{A}(G)$, $\mathscr{B}(G)$ and $\mathscr{C}(G)$, $G\in\{T_{p,q},K^0_{p',q'}\}$, the families of all edge-sets in G that belong to the groups A, B, C, respectively. We have to prove that $|\mathscr{A}(T_{p,q})|>|\mathscr{A}(K^0_{p',q'})|$. We have the following equality.

$$s_{q'+1}(2pq-q'-2)=|\mathscr{A}(G)|(q'-1)+\sum_{B\in\mathscr{B}(G)}(q'+3-\delta(B))$$
$$+|\mathscr{C}(G)|(q'-1),$$

where by $\delta(B)$ we denote the number of edges of B which do not belong to all cycles of length four in B.

Note that $\mathscr{C}(T_{p,q})$ is empty whereas the sets in Fig. 12a belong to $\mathscr{C}(K^0_{p',q'})$. Applying Corollary 16 several times we get that

$$\sum_{B\in\mathscr{B}(T_{p,q})}(q'+3-\delta(B))=\sum_{B\in\mathscr{B}(K^0_{p',q'})}(q'+3-\delta(B)).$$

Therefore $|\mathscr{A}(T_{p,q})|$ must be greater than $|\mathscr{A}(K^0_{p',q'})|$.

(2) In $T_{p,q}$ every essential cycle of length $q'+1$ plus two edges has rank $q'+2$. This is not true for all essential cycles in $K^0_{p',q'}$ (see Fig. 12a). Since by hypothesis the number of essential cycles is the same, the number of edge-sets in this case is greater in $T_{p,q}$ than in $K^0_{p',q'}$.

(3) There is one single possibility in $T_{p,q}$ for essential cycles of length $q'+3$ (see Fig. 12b) and we have $2q\binom{p}{2}$ of them. The two possibilities in $K^0_{p',q'}$ correspond to paths depicted in Fig. 13. Since we are assuming that $p'=4(q'+1)=4p$, and $p\geqslant 6$, in both cases there are four different exterior edges that can be used to produce an essential cycle of length $q'+3$. The total number of such cycles is then $4q'\binom{q'}{2}+4\binom{q'+2}{3}$, where the factor 4 stands for the possible exterior edges, and the remaining factors correspond to the possible choices of the vertical steps. Using the relationships $p=q'+1, q=4q'$ we again see that the first quantity is greater than the second.

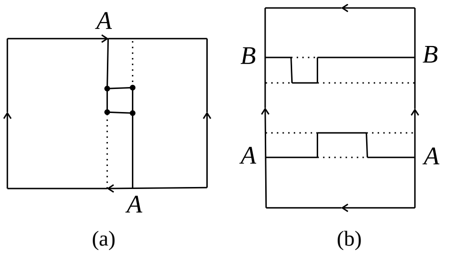

Fig. 12. (a) A set of rank $q'+1$ and size $q'+3$ containing an essential cycle. (b) Essential cycles of length $q'+3$ in $T_{p,q}$.

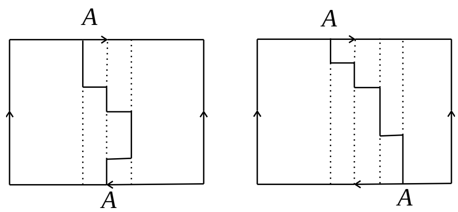

Fig. 13. Essential cycles of length $q'+3$ in $K^0_{p',q'}$.

Case IV: $S_{p',q'}$ with $p'=q=2^p$, $q'=p$.

In this case we prove that $T_{p,q}$ has more edge-sets with rank $q'+1$ and size $q'+2$ that $S_{p',q'}$. The proof uses the same ideas as in Case III and we omit it for the sake of brevity. □

6. Concluding remarks

We have shown that $C_p \times C_q$ is T-unique for $p,q \geqslant 6$. Our technique does not apply to the small cases $p=3,4,5$ (since either they are not locally grid according to our definition, or they contain cycles of length 5), and these cases would require more ad hoc arguments. Also, it appears that all locally grid graphs are T-unique, but this would need cross-checking any two of them as in Section 5.

An interesting open problem is to prove T-uniqueness for products of more than two cycles. The approach taken here seems infeasible and new techniques would be required in this case.

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Note added in proof

After this paper was accepted for publication, the authors have learnt of a paper by Thomassen [14] containing a result essentially equivalent to our Theorem 1. The terminology in [14] differs from ours: the author defines a “quadrilateral tiling of the torus or the Klein bottle” in a way which, up to technical details, is equivalent to our definition of a locally grid graph. Theorem 4.1 of [14] is a classification theorem very similar to our Theorem 1. However, after a careful checking of the possible cases, we have realized that our family $S_{p,q}$ with $q < p$ does not appear in the list of quadrilateral tilings given in Theorem 4.1 [14]. Nevertheless it appears implicitly in the proof. The argument of the proof is essentially the same as in Case 2 of our proof of Theorem 1. In both proofs we have a structure somewhat similar to a Möbius strip (Fig. 7b), and we successively add squares to it until we run out of vertices. What determines whether the resulting graph $S_{p,q}$ has $p \leqslant q$ or $q < p$ is the number of layers we add relative to the length of the initial cycle. The proof of Theorem 4.1 [14] does not consider the case that gives $q < p$, that is, when the number of layers is greater than the length of the first cycle.

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