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Discrete Mathematics 254 (2002) 63–74
DISCRETE MATHEMATICS
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Diagonal flips in outer-triangulations on closed surfaces

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Received 8 February 2000; received in revised form 10 October 2000; accepted 7 May 2001

Abstract

We show that any two outer-triangulations on the same closed surface can be transformed into each other by a sequence of diagonal flips, up to isotopy, if they have a sufficiently large and equal number of vertices.

Keywords: Triangulation; Outer-triangulation; Diagonal flip

1. Introduction

In this paper, we always suppose that a graph G is embedded in a closed surface, and the vertex set, edge set and face set of G are denoted by $V(G)$, $E(G)$ and $F(G)$, respectively. In addition, all embeddings considered here will be 2-cell embeddings. A k-*cycle* means a cycle of length k. A closed curve ℓ on a closed surface F^2 is said to be 1-*sided* if the tubular neighborhood of ℓ is homeomorphic to a Möbius band, and 2-*sided* otherwise.

A *triangulation* on a closed surface F^2 is a simple graph on F^2 such that each face is bounded by a 3-cycle, and any two faces share at the most one edge. A *diagonal flip* in a triangulation G is to replace a diagonal ac with bd in the quadrilateral $abcd$

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[1] Partially supported by Proyecto DGICYT PB96-1374 and Proyecto PAI FQM-0164.

PII: S0012-365X(01)00353-3

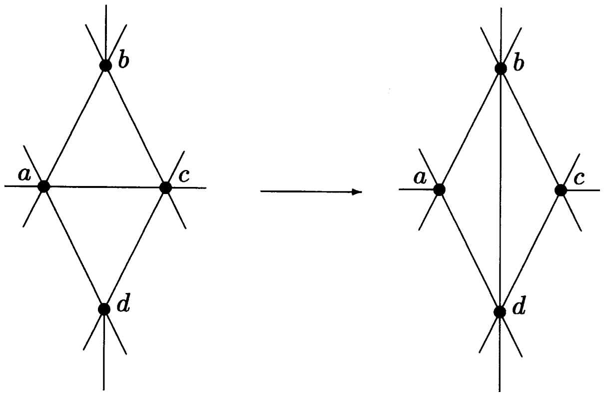

Fig. 1. A diagonal flip.

formed by two faces abc and acd sharing the edge ac (see Fig. 1). As we want to keep the graph simple whenever we carry out diagonal flips, we do not allow this operation if G already has the edge bd (since, in this case, the diagonal flip would create multiple edges between b and d).

Let G and G' be two triangulations on the same closed surface. We say that G and G' are *equivalent* to each other if there exists a sequence of triangulations $H_0,\dots,H_k$ such that

(i) $G=H_0$ and $G'=H_k$,
(ii) H_{i+1} is obtained from H_i by one diagonal flip, for $i=0,\dots,k-1$.

There are many papers concerning diagonal flips in triangulations. It has been shown that for the sphere [24], the projective plane [21], the torus [7] and the Klein bottle [21], any two triangulations with the same number of vertices are equivalent to each other, up to homeomorphism. Negami has generalized these theorems, as follows.

Theorem 1 (Negami [16]). *For any closed surface F^2, there exists a natural number $M(F^2)$ such that any two triangulations G and G' on F^2 with $|V(G)|=|V(G')|\geqslant M(F^2)$ are equivalent to each other, up to homeomorphism.*

This theorem has been extended for triangulations with specified properties [4]. Moreover, a series of theorems have been improved to hold under the condition "up to isotopy" [14]. Note that for the sphere and the projective plane, up to homeomorphism" and "up to isotopy" are equivalent. Recently, Negami [18] has given an upper bound for the minimum value $M(F^2)$ by a linear function with respect to the Euler characteristic of F^2. In fact, he has shown that the linear function gives that for the corresponding value $M'(F^2)$ for the isotopy version. Moreover, many other researches have been derived from Theorem 1 [10,15,17,19,20].

A very important case with many practical applications is that of triangulations of polygons in the plane (or the sphere). These triangulations agree with maximal outerplanar graphs. Thus, we shall generalize this concept to other surfaces. A *triangulation* on a closed surface F^2 with *boundary cycle* C (or simply *boundary*) is an embedding of a simple graph on F^2 containing C such that

(i) there is a specific face bounded by C, called the *outer face*, and
(ii) all other faces are bounded by 3-cycles.

We say that the vertices and the edges of a triangulation are *outer* if they lie on C, and *inner* otherwise. An *outer-triangulation* G is a triangulation with boundary which has no inner vertices.

As well as ordinary triangulations, we can define diagonal flips for outer-triangulations. However, as any flippable edge must be shared by two triangular faces, we cannot apply diagonal flips for outer edges. Mimicking Negami's argument in [18], one will be able to show that for any two triangulations on a closed surface with the boundary of the same length can be transformed into each other by a sequence of diagonal flips if they have the same and sufficiently large number of inner vertices. However, if we restrict the number of inner vertices, the problem seems to be far more difficult.

In this paper, we focus on outer-triangulations on surfaces. It is easy to show that any two outer-triangulations on the sphere with the same number of vertices are equivalent to each other, up to isotopy. Moreover, the same fact has been shown for the projective plane [8], the torus [5] and Klein bottle [6]. (The cases for the torus and the Klein bottle have been solved, under the condition up to homeomorphism".) In any case, the arguments used in those papers strongly depend on the topology of these individual surfaces and they cannot be applied, in general, for other surfaces. In this paper, we will show the isotopy version of a general result, as follows.

Theorem 2. *For any closed surface F^2, there exists a natural number $N(F^2)$ such that any two outer-triangulations G and G' on F^2 with $|V(G)|=|V(G')|\geqslant N(F^2)$ are equivalent to each other, up to isotopy.*

2. Diagonal flips in outer-pseudo-triangulations

Although we have to keep any triangulation simple whenever we carry out a diagonal flip, in order to prove our main result, we will neglect the simpleness of outer-triangulations meanwhile, as follows.

An *outer-pseudo-triangulation* on a closed surface F^2 is a pseudograph (loops and multiple edges are allowed) on F^2 such that there exists a specific face, called the *outer face*, bounded by the cycle in which all the vertices appear, and other faces are bounded by closed walks of length 3. A diagonal flip of an inner edge of an outer-pseudo-triangulation is also defined in the same way as previously, but the diagonal flips need not preserve the simpleness of graphs.

The following theorem can be proved in the same way as in [18]. An inner edge e in an outer-pseudo-triangulation is called *self-incident* if there is a triangular face f such that e appears twice on the boundary walk of f.

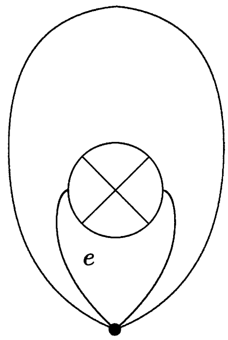

Fig. 2. Self-incident edge e.

Lemma 3. *Let P be an outer-pseudo-triangulation on a closed surface F^2 and let e be a self-incident edge of P. Then, e is a 1-sided loop whose two ends of e are consecutive in the rotation around the same vertex.*

Fig. 2 illustrates a self-incident edge e explained in the above lemma, where $\otimes$ expresses a crosscap.

Proof. Suppose that e appears twice on the boundary 3-cycle $v_1e_1v_2e_2v_3e_3$ of a face f, where $v_i \in V(P)$ and $e_i \in E(P)$ for $i=1,2,3$. We may suppose $e_1=e_2=e$. If $v_1 \neq v_2$, then v_2 has degree 1. The vertex v_2 of degree 1 cannot appear on the boundary of P, a contradiction. Thus, we have $v_1=v_2$ and hence e is a loop incident to $v_1=v_2$. Since $e_1=e_2$, the right-hand neighborhood along e and its left-hand neighborhood are traced consecutively along the boundary walk of f. Therefore, e is 1-sided and the two ends are consecutive in the rotation around v_2. $\square$

Theorem 4. *Let P and P' be two outer-pseudo-triangulations on a closed surface F^2 with the same number of vertices. Then, they can be transformed into each other, up to isotopy, by a sequence of diagonal flips. Furthermore, this sequence of diagonal flips does not switch any self-incident edge.*

Note that by Lemma 3, if an outer-pseudo-triangulation P has a self-incident edge e, then the graph obtained from P by flipping e is obviously isomorphic to P itself. Thus, the last sentence of the following theorem is trivial.

The original theorem concerning psuedo-triangulations T and T' (i.e., pseudographs with each face triangular) has been proved, as follows. First, draw T and T' on F^2 simultaneously so that $V(T)$ and $V(T')$ completely coincide. (Thus, each intersecting point of T and T' is either a vertex of them or a crossing of an edge of T and an edge of T' at their middle points.) The author of [18] proceeded to fix T on F^2 and apply diagonal flips for T' to eliminate the crossings of edges. In this case, each edge e' of T' passes through the interior of several triangular faces of T, or coincides with some edge e of T.

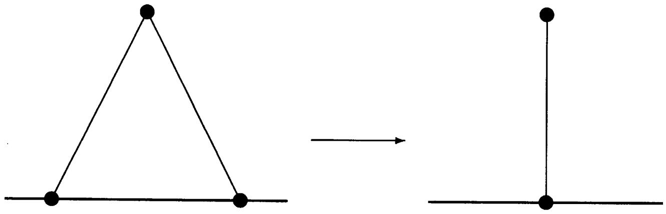

Fig. 3. Contraction of an edge.

In case of outer-pseudo-triangulations P and P', fix P and P' on F^2 so that their vertices coincide along their boundaries. Then, we get the same situation that every edge e' of P' passes through the interior of several triangular faces of P, or coincides some edge e of P. Moreover, every diagonal flip in P' preserves this condition. Thus, the same argument follows (see [18] for the details).

3. Irreducible outer-triangulations

Consider an ordinary triangulation G on a closed surface F^2. Let abc and acd be two faces sharing an edge $e = ac$ in G. The *contraction* of e (or *contracting* e) is to identify the end-vertices a and c of e and replace the multiple edges $\{ab, cb\}$ and $\{ad, cd\}$ by two single edges, respectively. We say that e is *contractible* if the graph obtained from G by contracting e is simple. We also say that a triangulation G is *contractible* to a triangulation T if G can be transformed into T by a sequence of edge contractions. A triangulation with no contractible edge is said to be *irreducible*. For the sphere, the projective plane, the torus and the Klein bottle, the complete lists of irreducible triangulations have been determined in [23,1,11,12], respectively. It is well-known that any closed surface admits only finitely many irreducible triangulations, up to homeomorphism. This fact also follows from the affirmative solution of Wagner's conjecture proved by Robertson and Seymour [22]. There are several papers proving directly the finiteness of irreducible triangulations [2,3,9], by bounding the number of vertices of them. The following result gives the best bound for it in the present.

Theorem 5 (Nakamoto and Ota [13]). *Let F^2 be a non-spherical closed surface with Euler characteristic $\chi(F^2) < 2$, and r be the Euler genus of F^2 (i.e., $r = 2 - \chi(F^2)$). If G is an irreducible triangulation of F^2, then $|V(G)| \leqslant 171r - 72$.*

Contraction of an edge in an outer-triangulation is defined only for outer edges. See Fig. 3. (If we contract an inner edge, then the boundary of the outer face will be deformed into a closed walk which is not a cycle.) We say that an outer edge e is *contractible* if its contraction yields a simple graph. We also say that an outer-triangulation G is *contractible* to an outer-triangulation T if G can be transformed into T by a sequence of contractions of outer edges. An outer-triangulation G with no contractible outer edge is said to be *irreducible*.

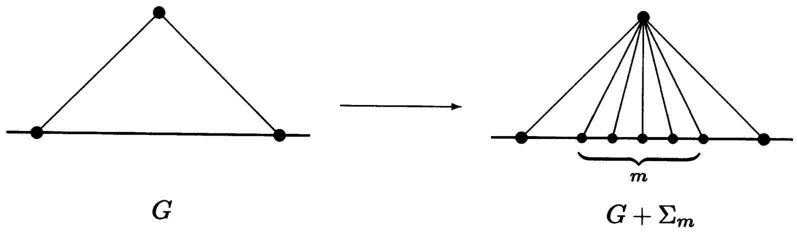

Fig. 4. Inserting m vertices of degree 3.

Lemma 6. *For any closed surface, there exist only finitely many irreducible outer-triangulations, up to homeomorphism.*

In order to prove Lemma 6, we need the following lemma shown in [6].

Lemma 7. *Let G be an outer-triangulation on a closed surface F^2. Let $\tilde{G}$ be the triangulation on F^2 obtained from G by adding a vertex in the outer face of G and joining it to all vertices of G. Then, the outer-triangulation G is irreducible if and only if the triangulation $\tilde{G}$ is irreducible.*

Now we prove Lemma 6.

Proof of Lemma 6. Combining Theorem 5 and Lemma 7, if G is an irreducible outer-triangulation on a closed surface F^2, then $|V(G)| \leqslant (171r - 72) - 1 = 171r - 71$. Thus, the proposition follows. □

Let G be an outer-triangulation on a closed surface F^2 and let xyz be a face of G such that xy is an outer edge. Subdividing xy by a single vertex v and adding an edge vz, we obtain an outer-triangulation G' with $|V(G')| = |V(G)| + 1$. In this case, we say that G' is obtained from G by *inserting* a vertex of degree 3 on xy. Lemmas 8 and 9 have already been proved in [5,6].

Lemma 8. *Let G be an outer-triangulation on a closed surface F^2 with boundary C, and let $e, e' \in E(C)$. Let G_e (resp., $G_{e'}$) be the outer-triangulation on F^2 obtained from G by inserting a vertex of degree 3 on e (resp., e'). Then, G_e and $G_{e'}$ are equivalent to each other, up to isotopy.*

Let $G + \Sigma_m$ denote the outer-triangulation obtained from G by adding m vertices of degree 3, as shown in Fig. 4. By Lemma 8, since we can move an inserted vertex of degree 3 to any outer edge by diagonal flips, any two outer-triangulations with the notation $G + \Sigma_m$ can be transformed into each other by diagonal flips (i.e., independent of the choice of the edges subdivided by the m vertices of degree 3).

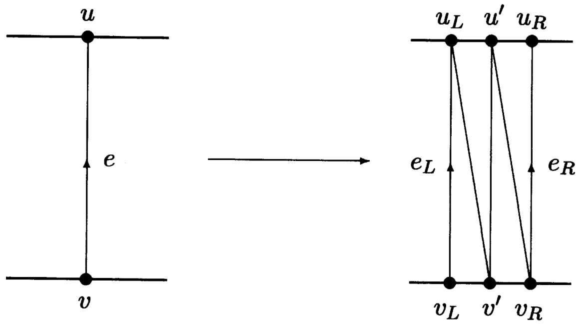

Fig. 5. Strip extension along e.

Lemma 9. *Let G and T be two outer-triangulations on a closed surface F^2. If G is contractible to T, then G can be transformed into $T+\Sigma_m$, up to isotopy, by a sequence of diagonal flips, where $m=|V(G)|-|V(T)|$.*

4. From outer-pseudo-triangulations to outer-triangulations

In this section, we show how to use the equivalence of outer-pseudo-triangulations described in Section 2.

Let P be an outer-pseudo-triangulation on a closed surface F^2 with boundary C. Let $e=uv$ be an inner edge of P. We consider the following operation for P. We now regard the outer face of P as a hole of F^2 (i.e., we regard C as the boundary of the surface). We denote the punctured surface with boundary C by $\tilde{F}^2$. Now cut $\tilde{F}^2$ along $e=uv$ from v to u, and denote the right- and left-side images of e by e_{R} and e_{L}, respectively (then the resulting surface has the boundary $C\cup e_{\mathrm{R}}\cup e_{\mathrm{L}}$, which might be disconnected). Suppose that e_{R} (resp., e_{L}) starts from a vertex v_{R} (resp., v_{L}) and terminates in a vertex u_{R} (resp., u_{L}). Join v_{R} and v_{L} by a path $v_{\mathrm{R}}v'v_{\mathrm{L}}$ of length 2, join u_{R} and u_{L} by a path $u_{\mathrm{R}}u'u_{\mathrm{L}}$ of length 2, and add an edge $v'u'$. Regarding each of the 4-cycles $v_{\mathrm{R}}u_{\mathrm{R}}u'v'$ and $v'u'u_{\mathrm{L}}v_{\mathrm{L}}$ as a quadrilateral region, we finally add two diagonals $u_{\mathrm{L}}v'$ and $u'v_{\mathrm{R}}$ (as shown in Fig. 5), or $u'v_{\mathrm{L}}$ and $u_{\mathrm{R}}v'$. In particular, if e is a self-incident edge, then by Lemma 3, we may assume that $u_{\mathrm{L}}=v_{\mathrm{R}}$ and $v_{\mathrm{L}}\neq u_{\mathrm{R}}$. In this case, we add diagonals $v_{\mathrm{L}}u'$ and $v'u_{\mathrm{R}}$, not to make multiple edges between v' and u_{L} and between v' and u_{L}. Here, $u_{\mathrm{L}}=v_{\mathrm{R}}$ and $v_{\mathrm{L}}=u_{\mathrm{R}}$ do not happen simultaneously, and hence we can add these diagonals without breaking the simpleness of graphs.

Clearly, the resulting embedding is also an outer-pseudo-triangulation on F^2. We call this operation the *strip extension* along e.

We call the union of the two quadrilateral regions the *strip*, denoted $\tilde{e}$, corresponding to e, and call each of e_{R} and e_{L} the *brim* of the strip.

Let P be an outer-pseudo-triangulation on a closed surface F^2. Apply the strip extensions for all inner edges of P. We call the resulting outer-pseudo-triangulation the *brick graph* of P, and denote it by $B(P)$.

Lemma 10. *Given an outer-pseudo-triangulation P on a closed surface F^2, the brick graph $B(P)$ is an outer-triangulation (i.e., simple).*

Proof. By construction, it is clear that any edge in a strip except brims is neither a loop nor multiple edges. So, consider a brim in a strip $\tilde{e}$ coming from an edge $e=uv$ of G and first suppose that it is a loop in $B(G)$. Then, e is also a loop at a vertex $u=v$ in G. If the two ends of e are not consecutive in the rotation around $u=v$, then the strip extension along edges between the two ends of e split $u=v$ into two or more distinct vertices and neither e_{L} nor e_{R} can be a loop in $B(G)$, a contradiction. On the other hand, if there is no edge between the two ends of e around $u=v$, the loop e must be 1-sided; otherwise, it would bound a monogonal face. In this case, the two edges e_{L} and e_{R} form together a path of length 2 and both of them are not a loop in G, a contradiction again. Since even in this case, we can add diagonals not to make multiple edges as described in the definition of the strip extension, $B(G)$ has no loop.

Now suppose that $B(G)$ includes a pair of multiple edges, which are brims coming from two edges e_1 and e_2 of G. Then e_1 and e_2 also form a pair of multiple edges between u and v in G. As well as a loop, strip extension cannot transform them into a pair of multiple edges in $B(G)$ unless their ends form consecutive pairs in the rotations around u and v and unless e_1 and e_2 form a 2-sided 2-cycle. However, in the exceptional case, the 2-cycle $e_1 \cup e_2$ would bound a diagonal face, a contradiction. Thus, $B(G)$ has no multiple edges. □

Lemma 11. *Let G be an outer-triangulation on a closed surface F^2. Then, the brick graph $B(G)$ is contractible to G.*

Proof. Let e be an inner edge of G. The outer-triangulation obtained from G by a strip extension along e is clearly contractible to G. Thus, the lemma follows. □

Let G and G' be two outer-triangulations on the same closed surface F^2. When G and G' are equivalent to each other, up to isotopy, keeping the simpleness of graphs, then we simply denote $G \sim G'$. Combining Lemmas 9 and 11, we have the following:

Lemma 12. *Let F^2 be a closed surface with Euler characteristic $\chi(F^2)$, and let G be an outer-triangulation on F^2. Then,*

$$B(G) \sim G + \Sigma_{4(|V(G)|-3\chi(F^2)+3)}.$$

Proof. By Euler's formula, we have $|E(G)|=2|V(G)|-3\chi(F^2)+3$. Thus, the number of inner edges of G is equal to

$$|E(G)|-|V(G)|=|V(G)|-3\chi(F^2)+3.$$

Since each strip extension increases the number of vertices by four, we have

$$|V(B(G))|-|V(G)|=4(|V(G)|-3\chi(F^2)+3).$$

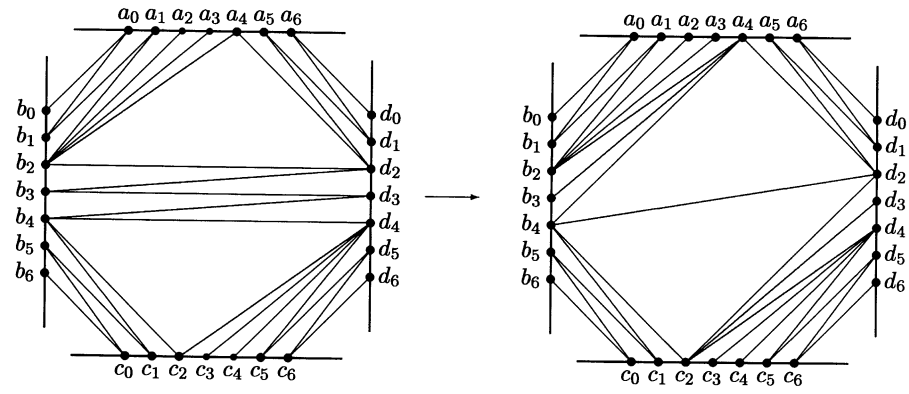

Fig. 6. A sequence of diagonal flips from $B(P)+\Sigma_4$.

Since $B(G)$ is contractible to G, we have $B(G)\sim G+\Sigma_{4(|V(G)|-3\chi(F^2)+3)}$, by Lemmas 9 and 11. □

Lemma 13. *Let P and P' be two outer-pseudo-triangulations on a closed surface F^2 such that P' can be obtained from P by flipping one edge. Then, two outer-triangulations $B(P)+\Sigma_4$ and $B(P')+\Sigma_4$ are equivalent.*

Proof. Let $Q=abcd$ be a quadrilateral in P formed by two triangular faces abd and bcd. Suppose that the edge bd is flipped to obtain P'. Since strip extensions to construct $B(P)$ and $B(P')$ are performed along the inner edges in P and P', the proof naturally falls into four cases, depending on the number of inner edges on the cycle $abcd$, but we shall show the lemma only when every edge in Q is an inner edge because the remaining cases may be handled in the same way.

See Fig. 6. We label the vertices of $B(P)$ corresponding to a, b, c and d as in the figure. The left-hand figure represents the local structure of $B(P)+\Sigma_4$ corresponding to the face $abcd$ in P. By Lemma 8, we may suppose that a_2, a_3 and c_3, c_4 are the inserted vertices of degree 3. By Theorem 4, we do not apply any diagonal flip of a self-incident edge, and hence we may suppose that the faces abd and bcd are different in P. Thus, the vertices $a_1,\dots,a_5$, $b_1,\dots,b_5$, $c_1,\dots,c_5$, $d_1,\dots,d_5$ are all distinct. Thus, the sequence of diagonal flips transforming the left-hand figure into the right-hand figure keeps the simpleness of graphs. From the right-hand figure, applying diagonal flips in the region bounded by $a_2a_3a_4b_4b_3b_2$ and the region bounded by $d_2d_3d_4c_4c_3c_2$, we can put edges a_2b_i and b_4a_i for $i=2,3,4$ and edges d_2c_i and c_4d_i for $i=2,3,4$. The resulting graph can easily be transformed into $B(P')+\Sigma_4$, similarly to the sequence from the left-hand to the right-hand in Fig. 6. □

Lemma 14. *Let F^2 be a closed surface with Euler characteristic $\chi(F^2)$, and let G and G' be any two outer-triangulations on F^2 with the same number of* *vertices. Then,*

$$G + \Sigma_{4(|V(G)|-3\chi(F^2)+4)} \sim G' + \Sigma_{4(|V(G')|-3\chi(F^2)+4)}.$$

Proof. By Theorem 4, if we neglect the simpleness of graphs, G and G' are equivalent up to isotopy, that is, there exists a sequence $G = T_0, T_1, \ldots, T_\ell = G'$ such that

(i) T_0 and T_l are outer-triangulations (i.e., simple),
(ii) for $i = 1, \ldots, \ell - 1$, T_i is an outer-pseudo-triangulation, and
(iii) for $j = 0, \ldots, \ell - 1$, T_{j+1} is obtained from T_j by one diagonal flip.

Now take the brick graphs of $T_0, \ldots, T_\ell$ with four extra inserted vertices of degree 3 added, that is, $B(T_0) + \Sigma_4, \ldots, B(T_\ell) + \Sigma_4$. Then, by Lemma 10, since $B(T_i)$ is simple, so is $B(T_i) + \Sigma_4$ for $i = 0, \ldots, \ell$. Moreover, by Lemma 13, $B(T_j) + \Sigma_4 \sim B(T_{j+1}) + \Sigma_4$ for $j = 0, \ldots, \ell - 1$. On the other hand, by Lemma 12, $B(T_0) \sim T_0 + \Sigma_{4(|V(T_0)|-3\chi(F^2)+3)}$ and $B(T_\ell) \sim T_\ell + \Sigma_{4(|V(T_\ell)|-3\chi(F^2)+3)}$, and hence we have $B(T_0) + \Sigma_4 \sim T_0 + \Sigma_{4(|V(T_0)|-3\chi(F^2)+4)}$ and $B(T_\ell) + \Sigma_4 \sim T_\ell + \Sigma_{4(|V(T_\ell)|-3\chi(F^2)+4)}$. Thus, we have

$$G + \Sigma_{4(|V(G)|-3\chi(F^2)+4)} \sim B(T_0) + \Sigma_4 \sim \cdots \sim B(T_\ell) + \Sigma_4 \sim G' + \Sigma_{4(|V(G')|-3\chi(F^2)+4)}.$$

Therefore, the lemma follows. □

5. Proof of the main theorem

Now we have prepared all to prove Theorem 2.

Proof of Theorem 2. By Lemma 6, F^2 admits only finitely many irreducible outer-triangulations, up to homeomorphism. Let $\{\tilde{I}_1, \ldots, \tilde{I}_p\}$ be the set of irreducible outer-triangulations on F^2, up to homeomorphism. Now, let I_j be an outer-triangulation represented by $\tilde{I}_j$, fixed up to isotopy, for $j = 1, \ldots, p$. Note that though any outer-triangulation G is contractible to one of $I_1, \ldots, I_p$, up to homeomorphism, G might be contractible to none of them, up to isotopy.

Without loss of generality, we may suppose that $|V(I_1)| \geqslant |V(I_2)| \geqslant \cdots \geqslant |V(I_p)|$. Let $m_i = |V(I_1)| - |V(I_j)|$ for $j = 1, \ldots, p$, and put $I'_j = I_j + \Sigma_{m_j}$ for $j = 1, \ldots, p$. By Lemma 14, since $|V(I'_1)| = \cdots = |V(I'_p)|$, we have

$$I'_1 + \Sigma_{4(|V(I'_1)|-3\chi(F^2)+4)} \sim I'_2 + \Sigma_{4(|V(I'_2)|-3\chi(F^2)+4)} \sim \cdots \sim I'_p + \Sigma_{4(|V(I'_p)|-3\chi(F^2)+4)}.$$

Moreover, by Lemma 14 again, for any $j \in \{1, \ldots, p\}$ and any homeomorphism $h: F^2 \to F^2$, we also have

$$h(I_j) + \Sigma_{4(|V(I_j)|-3\chi(F^2)+4)} \sim I_j + \Sigma_{4(|V(I_j)|-3\chi(F^2)+4)}.$$

Now put

$$N(F^2) = |V(I_1')| + 4(|V(I_1')| - 3\chi(F^2) + 4) = 5|V(I_1')| - 12\chi(F^2) + 16.$$

Then, putting $k_i = N(F^2) - |V(I_i)|$ for $i = 1, \ldots, p$,

$$I_1 + \Sigma_{k_1} \sim I_2 + \Sigma_{k_2} \sim \cdots \sim I_p + \Sigma_{k_p}.$$

Let G be an outer-triangulation on F^2 with $|V(G)| \geqslant N(F^2)$. Since G is contractible to an irreducible outer-triangulation, we may suppose that G is contractible to a homeomorphic image of I_t, denoted by $h(I_t)$. Hence, by Lemma 9, we have $G \sim h(I_t) + \Sigma_m$, where $m = |V(G)| - |V(I_t)|$. Since $|V(G)| \geqslant N(F^2)$, we have $m' = m - k_t \geqslant 0$. Thus, we have, by Lemmas 8 and 9,

$$\begin{aligned} G &\sim h(I_t) + \Sigma_m \\ &\sim I_t + \Sigma_m \\ &\sim I_t + \Sigma_{k_t} + \Sigma_{m'} \\ &\sim I_1 + \Sigma_{k_1} + \Sigma_{m'} \\ &\sim I_1 + \Sigma_{k_1+m'}. \end{aligned}$$

Similarly, if G' is an outer-triangulation on F^2 with $|V(G)| = |V(G')| \geqslant N(F^2)$, then we have $G' \sim I_1 + \Sigma_{k_1+m'}$. Thus, $G \sim G'$. □

The following corollary is an immediate consequence of this.

Corollary 15. $N(F^2) \leqslant 1371 - 867\chi(F^2)$.

Proof. In the proof of Theorem 2, we have $N(F^2) = 5|V(I_1')| - 12\chi(F^2) + 16$. The number $|V(I_1')| = |V(I_1)|$ can be bounded by $|V(I_1)| \leqslant 171(2 - \chi(F^2)) - 71$. (This inequality was obtained in the proof of Lemma 6.) Thus,

$$N(F^2) \leqslant 5(171(2 - \chi(F^2)) - 71) - 12\chi(F^2) + 16 = 1371 - 867\chi(F^2). \qquad \square$$

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