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Homological models for semidirect products of finitely generated Abelian groups

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Abstract Let G be a semidirect product of finitely generated Abelian groups. We provide a method for constructing an explicit contraction (special homotopy equivalence) from the reduced bar construction of the group ring of G, $\overline{B}(\mathbb{Z}[G])$, to a much smaller DGA-module hG. Such a contraction is called a homological model for G and is used as the input datum in the methods described in Álvarez et al. (J Symb Comput 44:558–570, 2009; 2012) for calculating a generating set for representative 2-cocycles and n-cocycles over G, respectively. These computations have led to the finding of new cocyclic Hadamard matrices (Álvarez et al. in 2006).

Keywords Semidirect product of groups · Homological model · Contraction · Homological perturbation theory

Mathematics Subject Classification (2000) 20J05 · 20J06 · 20J05 · 20J06

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1 Motivation of the problem: introduction

Hadamard matrices have a long history in combinatorics and arise in numerous applications, among others, in electrical engineering (circuit design) and statistics (experimental designs). Horadam's book [22] is an excellent reference for the use of these matrices in signal and data processing. Hadamard matrices have been actively studied for over 140 years and still remain a very challenging issue. Problems involving Hadamard matrices usually sound very easy, but they are notoriously difficult to solve. For instance, it is well-known that a Hadamard matrix must have order 1, 2 or a multiple of 4, but the *Hadamard conjecture* about whether there exists a Hadamard matrix of order $4t$ for every natural number t has remained open for over a century. A related problem is constructing all Hadamard matrices of a particular order is as difficult mainly because the search space expands exponentially with the order of the matrix. Horadam and de Launey [11,12] found an interesting application of 2-cocycles to tackle the problem of constructing Hadamard matrices.

Given a multiplicative group $G = \{g_1 = 1, g_2, \ldots, g_{4t}\}$, not necessarily Abelian. Functions $\psi\colon G \times G \to \langle -1 \rangle \cong \mathbf{Z}_2$ which satisfy

$$\psi(g_i, g_j)\psi(g_i g_j, g_k) = \psi(g_j, g_k)\psi(g_i, g_j g_k), \quad \forall g_i, g_j, g_k \in G \tag{1}$$

are called (binary) 2-cocycles (over G) [31]. A 2-cocycle is a 2-coboundary $\partial\phi$ if it is derived from a set mapping $\phi\colon G \to \langle -1 \rangle$ by $\partial\phi(a, b) = \phi(a)\phi(b)\phi(ab)^{-1}$. The set of 2-cocycles forms an Abelian group $Z(G)$ under pointwise multiplication, and the 2-coboundaries form a subgroup $B(G)$. It is a well-known fact that $Z(G)/B(G) \cong H^2(G; \mathbb{Z}_2)$. Thus, a basis $\mathcal{B}$ for 2-cocycles over G consists of some elementary 2-coboundaries ∂_i and some representative 2-cocycles in cohomology.

A 2-cocycle ψ is naturally displayed as *a cocyclic matrix* M_ψ; that is, the entry in the (i, j)th position of the cocyclic matrix is $\psi(g_i, g_j)$, for all $1 \leq i, j \leq n$.

The main advantages of the cocyclic approach concerning the construction of Hadamard matrices may be summarized in the following facts:

- The additional internal structure in a matrix which represents a 2-cocycle (a cocyclic matrix) is sufficient to provide a substantial cut-down in computational complexity of the problem of testing if it is Hadamard.
- The search space is reduced to the set of cocyclic matrices over a given group G. That is, 2^s matrices, provided that a basis for 2-cocycles over G consists of s generators.

Cocyclic construction is revealed to be the most uniform construction technique for Hadamard matrices yet known. Furthermore, a stronger version of the Hadamard conjecture, has been posed in [22], the *cocyclic Hadamard conjecture*: this asserts that there exists a cocyclic Hadamard matrix at every possible order. These facts have produced and increased interest in calculating a generating set for representative 2-cocycles (and n-cocycles, in general).

In [22, Sect. 6.3], three methods have been proposed in order to compute a generating set for representative 2-cocycles. The first method is the foundational work on the subject [12,13], and is applied over Abelian groups. The second one (see [17]) applies over groups G for which the word problem is solvable, and uses the inflation and transgression maps. Both methods rely on the Universal Coefficient Theorem

$$H^2(G, \mathbb{Z}_2) \cong Ext(G/[G,G], \mathbb{Z}_2) \oplus Hom(H_2(G), \mathbb{Z}_2).$$

The third approach to this question, which we term the *homological reduction method*, is described in [4]. Provided a *homological model* hG for G is known (that is, a differential graded module of finite type which shares the homology groups with G), it explicitly describes an algorithm for constructing a basis for 2-cocycles over G in a straightforward manner. In fact, the goodness of this approach is supported by the efficiency in which both $H_1(G) \simeq G/[G,G]$ and $H_2(G)$ are computed from the homological model hG. In [5], the cohomological analogous to this method is described and applied for computing n-cocycles in general. It might be a potential source of examples for cocyclic matrices of higher dimensions, which may not be supplied by the other methods.

In this paper, we provide a method for constructing a homological model for a semidirect product of finitely generated Abelian groups. Theoretically, this method provides explicit formulas in any degree. Although, from the practical perspective it is only appropriate for numerical calculations in low degrees.

We organize the paper as follows. In Sect. 2 we try to explain our approach to the computation of homology of groups. Section 3 is devoted to describing a homological model for a semidirect product of finite generated Abelian groups. For the sake of clarity it begins by introducing some notations and results on Simplicial Topology and Homological Algebra. In Sect. 4, some comments about several related topics are given. First, we indicated a $\mathbb{Z}[K \rtimes_\chi H]$-resolution. Later on, a homological model for iterated semidirect products of finite generated Abelian groups is determined. We included some comments about the simplification of the formulas that our method provides. Finally, the homology of some groups are computed and the matrices involved in the method are shown.

2 On the computation of the homology of groups

The (co)homology theory of groups arose form both topological and algebraic sources (see [7] for details). The starting point for the topological aspect of the theory was the work of Hurewicz [27] on *aspherical spaces* (that is to say, a space whose only non-null homotopy groups is the first, fundamental one). Given a group G and a contractible topological space with a free action of G, then an aspherical space can be obtained by means of the space of orbits of the action endowed with a convenient quotient topology. The homology of this aspherical space is, by definition, the homology of G, and it does not depend on the choosing of the contractible space or of the action. Each aspherical space (unique up to homotopy type) is a particular *Eilenberg-Mac Lane space* for G, and is denoted by $K(G, 1)$.

This topological approach presented a serious drawback because the contractible spaces to be constructed are frequently of either infinite type or too big which apparently closes the possibility of a computational treatment. However, Eilenberg-Mac

Lane in [15] computed the homology of finitely generated Abelian groups under this approach. By the mid-1940's a purely algebraic definition of group homology and cohomology was stated (see [30]). Indeed, the low-dimensional cohomology groups were seen to coincide with groups which had been introduced much earlier in connection with various algebraic problems. This algebraic approach is based on the definition of *resolution* (replacing the group under study with an acyclic object of a suitable category of modules) and it was chosen for being more adequate in practical computations. For instance, the package HAP [23] of the computer algebra system GAP [38] contains an impressive number of algorithms dealing with resolutions.

Due mainly to the progress in Homological Perturbation Theory [20,21] and working in the setting of Simplicial Topology [32], the topological approach has been revised and can be considered as a valid alternative from a computational point of view. For instance, Kenzo [10] is a Common Lisp program devoted to Symbolic Computation in Algebraic Topology (carryed out by means of simplicial sets and using techniques of Algebraic Topology), it makes use of Sergeraert's effective homology method (see [36]) to determine homology groups of complicated spaces and homology of groups [34,35].

Our method fits in the topological approach for computing the homology of groups. Given a group G, we compute the homology groups of G by means of the combinatorial description of $K(G,1)$ in Simplicial Topology, that is, $K(G,1) := \overline{W}(G)$. The enormous size of this space makes it difficult to obtain real calculations, even when G is finite, and therefore is necessary construct an explicit chain homotopy equivalence (a contraction)

$$C(\overline{W}(G)) \Rightarrow hG \tag{2}$$

where $C(\overline{W}(G))$ is the normalized chain complex canonically associated with $\overline{W}(G)$ and hG is a free DG-module of finite type, in general with a non-null differential, whose homology groups $H_*(hG)$ can be determined by an elementary algorithm. In addition, from the homotopy equivalence one can deduce the isomorphism $H_*(\overline{W}(G)) := H_*(C(\overline{W}(G)) \cong H_*(hG)$, which allows the computation of the homology groups of G. If G is an ordinary discrete group, then $C(\overline{W}(G))$ amount to the reduced bar construction $\overline{B}(\mathbb{Z}[G])$. Thus, (2) is rewritten as

$$\overline{B}(\mathbb{Z}[G]) \overset{\varphi}{\cong} C(\overline{W}(G)) \;\Rightarrow\; hG.$$

Such contraction is called a *homological model* for G.

Constructing a homological model for the semidirect product $K \rtimes_\chi H$ requires three steps. Let H be an (either simplicial or ordinary discrete) group and K be an (either simplicial or ordinary discrete) H-group. The first one consists of establishing a simplicial isomorphism (Theorem 1) between the simplicial set $\overline{W}(K \rtimes_\chi H)$, the $\overline{W}$-construction functor applied to the semidirect product $K \rtimes_\chi H$, and the twisted cartesian product $\overline{W}(K) \times_\tau \overline{W}(H)$ relative to the universal twisting function $\tau : \overline{W}(H) \to H$ and H-action on $\overline{W}(K)$.

Secondly, the twisted Eilenberg–Zilber Theorem yields a contraction from the normalized chain complex of $\overline{W}(K)\times_\tau \overline{W}(H)$ (we will denote by $C(\overline{W}(K)\times_\tau \overline{W}(H))$) to a twisted tensor product $C(\overline{W}(K))\otimes_t C(\overline{W}(H))$. Explicit formulas for a contraction of this type are given in [33].

Henceforth, we will assume that K and H are ordinary discrete groups. In this particular situation, $C(\overline{W}(K))$, $C(\overline{W}(H))$ and $C(\overline{W}(K\rtimes_\chi H))$ amount to the reduced bar constructions $\overline{B}(\mathbb{Z}[K])$, $\overline{B}(\mathbb{Z}[H])$ and $\overline{B}(\mathbb{Z}[K\rtimes_\chi H])$, respectively. So far, we have

$$\begin{array}{ccccc} \overline{B}(\mathbb{Z}[K\rtimes_\chi H]) & \overset{\varphi}{\cong} & C(\overline{W}(K\rtimes_\chi H)) & \overset{\psi}{\cong} & C(\overline{W}(K)\times_\tau \overline{W}(H)) \\ & & & & \Downarrow th.3 \\ & & \overline{B}(\mathbb{Z}[K])\otimes_t \overline{B}(\mathbb{Z}[H]) & \overset{\varphi^{-1}}{\cong} & C(\overline{W}(K))\otimes_t C(\overline{W}(H)). \end{array}$$

Finally, we constructed a contraction from $\overline{B}(\mathbb{Z}[K])\otimes_t \overline{B}(\mathbb{Z}[H])$ to a significantly smaller free DG-module of finite type, hKH. This last object is a certain twisted tensor product $hK\tilde{\otimes}hH$ of small DG-modules hK and hH onto which $\overline{B}(\mathbb{Z}[K])$ and $\overline{B}(\mathbb{Z}[H])$ contract respectively. In the case that K and H are finitely generated Abelian groups, such explicit contractions to hK and hH exist [15]. The key point is to guarantee the convergence of the related perturbation process (Theorem 4). The method works for any groups K and H under the hypothesis that explicit contractions to hK and hH exist, and the related perturbation process converges.

From this homological model,

$$\overline{B}(\mathbb{Z}[K\rtimes_\chi H]) \Rightarrow hKH,$$

it is easy to derive at once a small free resolution of the ground ring over $\mathbb{Z}[K\rtimes_\chi H]$ (Theorem 5). This amounts to putting a $\mathbb{Z}[K\rtimes_\chi H]$-linear differential on $\mathbb{Z}[K\rtimes_\chi H]\otimes hKH$ such that an acyclic chain complex results. In addition, a contracting homotopy on this resolution can be constructed by a formula involving the contracting homotopy on $B(\mathbb{Z}[K\rtimes_\chi H])$ (the standard bar resolution on $\mathbb{Z}[K\rtimes_\chi H]$). Let us point out that a free resolution without a contracting homotopy is a computationally limited object. It is a requirement for a method to be considered interesting. Therefore, in this way, we find a connection between the topological and algebraic approaches to the computation of homology of groups.

Furthermore, the method works over other semidirect products of groups (as well as iterated products of groups), even though the fibre groups K may not be a finitely generated Abelian group (see Remarks 3 and 5). An extended version of the method for iterated products of central extensions and semidirect products of finitely generated Abelian groups has been implemented in *Mathematica* by the authors (see [1,3]). Some calculations with this package have led to the finding new cocyclic Hadamard matrices [2,4].

3 Describing a homological model for $K\rtimes_\chi H$

In this section, we describe a homological model for a single semidirect product $K\rtimes_\chi H$ of (discrete) finitely generated Abelian groups K and H.

Firstly, we recall the definition of semidirect product of two groups H and K. Let χ be an action of H on K, i.e. $\chi : H \times K \to K$ with $\chi(h,k) = \alpha(h)(k)$ where $\alpha : H \to \mathrm{Aut}(K)$ is a homomorphism. The *semidirect product* of H and K with respect to χ, $K \rtimes_\chi H$ (or $K \rtimes_\alpha H$), is the set $K \times H$, endowed with the group law

$$(k,h)\cdot(k',h') = (k + \chi(h,k'), h + h').$$

We will write hk instead of $\chi(h,k)$ when no confusion can arise.

Example 1 The dihedral group

$$D_{2m} =< h, k : h^2 = 1, k^m = 1, hkh = k^{-1} >$$

is the semidirect product $\mathbb{Z}_m \rtimes_\chi \mathbb{Z}_2$, $m \geq 2$, for $\chi(0,k) = k$, $\chi(1,k) = -k$.

Step 1

In order to describe a homological model for $K \rtimes_\chi H$, we need to work in the framework of simplicial sets and use the techniques that the homological perturbation theory provides. We recall some basic concepts of Simplicial Topology and Homological Algebra. More details can be found in [32] and in [31] respectively.

A *simplicial group* G is a simplicial set $G = (G_n, \partial_i, s_i)$ where every G_n is a group and every face or degeneracy operator is compatible with the group structures. If G has only one 0-simplex, then G is called *reduced*.

The $\overline{W}$*-construction* (or the *classifying construction* $(\overline{W})$) for a simplicial group G, denotes by $\overline{W}(G)$, is a new simplicial set defined as follows:

$$\begin{aligned}
\overline{W}_0(G) &= \{[\,]\};\\
\overline{W}_n(G) &= G_{n-1}\times\cdots\times G_0, \quad n>0;\\
s_0[\,] &= [1];\\
\partial_i[g_0] &= [\,], \quad i = 0,1;\\
\partial_0[g_n,\dots,g_0] &= [g_{n-1},\dots,g_0],\\
\partial_{i+1}[g_n,\dots,g_0] &= [\partial_i g_n,\dots,\partial_1 g_{n-i+1}, g_{n-i-1}\partial_0 g_{n-i}, g_{n-i-2},\dots,g_0],\\
s_0[g_{n-1},\dots,g_0] &= [1, g_{n-1},\dots,g_0],\\
s_{i+1}[g_n,\dots,g_0] &= [s_i g_n,\dots,s_0 g_{n-i}, 1, g_{n-i-1},\dots,g_0];
\end{aligned}$$

where $[\,]$ denotes the unique element of $\overline{W}_0(G)$, 1 denotes the identity elements of G (at each simplicial degree) and $[g_{n-1},\dots,g_0]$ denotes a generic element of $\overline{W}_n(G)$, for $n>0$. $\overline{W}(G)$ is also called *a classifying space* for G.

If G is an ordinary discrete group then $\overline{W}(G) = \overline{W}({}^sG)$, for ${}^sG_m = G$, $\forall m \geq 0$, and all face and degeneracy operators are the identity maps. For clarity in the exposition, we denote ${}^sG$ simply by G itself in the sequel.

We need here the *reduced bar construction* $\overline{B}(A)$ of a DGA-algebra A. Recall that it is defined as the connected DGA-coalgebra, $\overline{B}(A) = T^c(s(\bar{A}))$, where $T^c(\,)$ is the tensor coalgebra, $s(\,)$ is the suspension functor and $\bar{A}$ is the augmentation ideal of A.

The element of $\overline{B}_0(A)$ corresponding to the identity element of Λ (ground ring) is denoted by [ ] and the element $s\bar{a}_1 \otimes \cdots \otimes s\bar{a}_n$ of $\overline{B}(A)$ is denoted by $[a_1|\cdots|a_n]$. The tensor and simplicial degrees of the element $[a_1|\cdots|a_n]$ are $|[a_1|\cdots|a_n]|_t = \sum |a_i|$ and $|[a_1|\cdots|a_n]|_s = n$, respectively; its total degree is the sum of its tensor and simplicial degree. The tensor and simplicial differential are defined by:

$$d_t([a_1|\cdots|a_n]) = -\sum_i (-1)^{e_{i-1}}[a_1|\cdots|d_A(a_i)|\cdots|a_n],$$

and

$$d_s([a_1|\cdots|a_n]) = \sum_i (-1)^{e_i}[a_1|\cdots|\mu_A(a_i \otimes a_{i+1})|\cdots|a_n]$$

where $e_i = i + |a_1| + \cdots + |a_i|$.

If the product of A is commutative, a product $*$ (called shuffle product) can be defined on $\overline{B}(A)$. For every discrete group G, $\overline{B}(\mathbb{Z}[G])$ amounts to $C(\overline{W}(G))$ by means of the following isomorphism

$$\varphi\colon \overline{B}(\mathbb{Z}[G]) \to C(\overline{W}(G)),$$
$$\varphi([g_0|\cdots|g_n]) = \begin{cases} (g_0,\ldots,g_n), & G \text{ is Abelian} \\ (-1)^{\lceil \frac{n+1}{2} \rceil + 1}(g_n,\ldots,g_0), & \text{Otherwise.} \end{cases}$$

Consider two simplicial sets F, B and a simplicial group G which operates on F from the left. A *twisted cartesian product* E with fibre F, base B and structural group G consists of a simplicial set $E_n = F_n \times B_n$ and

$$\begin{aligned} \partial_0(a,b) &= (\tau b \star \partial_0 a, \partial_0 b) \\ \partial_i(a,b) &= (\partial_i a, \partial_i b), \quad \text{for } i > 0 \\ s_i(a,b) &= (s_i a, s_i b), \quad \text{for } i \geq 0; \end{aligned}$$

as face and degeneracy operators. Here $\star\colon G \times F \to F$ is the action of G on F and τ is a *twisting function*, i.e., $\tau_n : B_n \to G_{n-1},\ n \geq 1$ satisfies

$$\begin{aligned} \partial_0 \tau(b) &= [\tau(\partial_0 b)]^{-1} \cdot \tau(\partial_1 b) \\ \partial_i \tau(b) &= \tau(\partial_{i+1} b), \quad \text{for } i > 0 \\ s_i \tau(b) &= \tau(s_{i+1} b), \quad \text{for } i \geq 0 \\ \tau(s_0 b) &= 1, \end{aligned}$$

where 1 denotes the identity element of the corresponding group G_n. We write $E = F \times_\tau B$.

Example 2 Let K and H be two simplicial groups where H operates on K from the left, then a TCP $\overline{W}(K) \times_\tau \overline{W}(H)$ with fibre $\overline{W}(K)$, base $\overline{W}(H)$ and structural group H can be defined via the action

$$\begin{aligned} \star : H \times \overline{W}(K) &\longrightarrow \overline{W}(K) \\ (h, [k_{n-1},\ldots,k_0]) &\longrightarrow [h \cdot k_{n-1},\ldots,h \cdot k_0]; \end{aligned}$$

and twisting function $\tau_n : \overline{W}_n(H) \longrightarrow H_{n-1}$,

$$\tau_n[h_{n-1}, \dots, h_0] = h_{n-1}.$$

Theorem 1 *In the conditions of the example above, there is an explicit simplicial isomorphism*

$$\psi : \overline{W}(K \rtimes_\chi H) \longrightarrow \overline{W}(K) \times_\tau \overline{W}(H).$$

Proof Define ψ and ψ^{-1} to be

$$\begin{aligned}
&\psi_n[(k_{n-1}, h_{n-1}), \dots, (k_0, h_0)]\\
&\quad = ([h_{n-1}^{-1} \cdot k_{n-1}, \dots, \partial_0^{i-1} h_{n-1}^{-1} \dots \partial_0 h_{n-i+1}^{-1}\, h_{n-i}^{-1}\\
&\qquad \cdot k_{n-i}, \dots, \partial_0^{n-1} h_{n-1}^{-1} \dots \partial_0 h_1^{-1}\, h_0^{-1} \cdot k_0], [h_{n-1}, \dots, h_0]);\\
&\psi_n^{-1}([k_{n-1}, \dots, k_0],\ [h_{n-1}, \dots, h_0])\\
&\quad = [(h_{n-1}k_{n-1}, h_{n-1}) \dots, (h_{n-i}\partial_0 h_{n-i+1} \dots \partial_0^{n-i+1} h_{n-1} \cdot k_{n-i}, h_{n-i}), \dots,\\
&\qquad (h_0 \partial_0 h_1 \dots \partial_0^{n-1} h_{n-1} \cdot k_0, h_0)].
\end{aligned}$$

Now the statement of the theorem follows by direct inspection. The proof is left to the reader. □

Step 2

Now, we make a precise definition of the objects studied in the homological perturbation theory and sketch a familiar example.

Let N and M be two DG-modules. Their differentials will be denoted respectively by d_N and d_M or simply by d when no confusion can arise. $d^{\otimes}$ denotes the trivial differential, $d_N \otimes 1 + 1 \otimes d_M$, on $N \otimes M$. A *contraction* (see [14,26]) is a data set $c : \{N, M, f, g, \phi\}$ where $f : N \to M$ and $g : M \to N$ are morphisms of DG-modules (called, respectively, *the projection* and *the inclusion*) and $\phi : N \to N$ is a morphism of graded modules of degree $+1$ (called *the homotopy operator*). These data are required to satisfy the rules: **(c1)** $fg = 1_M$, **(c2)** $\phi d_N + d_N \phi + gf = 1_N$ **(c3)** $\phi\phi = 0$, **(c4)** $\phi g = 0$ and **(c5)** $f\phi = 0$. These last three are called the side conditions [29]. In fact, these may always be assumed to hold, since the homotopy ϕ can be altered to satisfy these conditions [20]. These formulas imply that both chain complexes N and M have the same homology. We will also denote a contraction c by either $^{\phi:}\ N \underset{g}{\overset{f}{\rightleftharpoons}} M$ or $N \Rightarrow M$.

If we have two contractions (f_i, g_i, ϕ_i) from N_i to M_i, for $i = 1, 2$ then, the following contractions can be constructed (see [14]):

- The tensor product contraction $(f_2 \otimes f_1,\ g_1 \otimes g_2,\ \phi_1 \otimes g_2 f_2 + 1_{N_1} \otimes \phi_2)$ from $N_1 \otimes N_2$ to $M_1 \otimes M_2$.
- If $N_2 = M_1$, the composition contraction $(f_2 f_1,\ g_1 g_2,\ \phi_1 + g_1 \phi_2 f_1)$ from N_1 to M_2.

The Eilenberg–Zilber theorem [16] provides the most classic example of a contraction of chain complexes.

An Eilenberg–Zilber contraction is defined in [15] by the data set

$$^{SHI}{:}\, C(F \times B) \underset{EML}{\overset{AW}{\rightleftharpoons}} C(F) \otimes C(B)$$

where F and B are simplicial sets. Here $C(F)$ denotes the normalized chain complex associated to a simplicial set F with coefficients in $\mathbb{Z}$. The Alexander-Whitney operator $AW\colon C(F \times B) \to C(F) \otimes C(B)$, the Eilenberg–Zilber operator $EML\colon C(F) \otimes C(B) \to C(F \times B)$ and the Shih operator (of degree +1) $SHI\colon C(F \times B) \to C(F \times B)$ are defined by the following formulas:

$$AW(a_n \times b_n) = \sum_{i=0}^{n} \partial_{i+1} \cdots \partial_n a_n \otimes \partial_0 \cdots \partial_{i-1} b_n,$$

$$EML(a_p \otimes b_q) = \sum_{(\alpha,\beta)\in\{(p,q)-shuffles\}} (-1)^{sg(\alpha,\beta)} (s_{\beta_q} \cdots s_{\beta_1} a_p \times s_{\alpha_p} \cdots s_{\alpha_1} b_q),$$

$$SHI(a_n \times b_n) = \sum (-1)^{m+sg(\alpha,\beta)} (s_{\beta_q+m} \cdots s_{\beta_1+m} s_{m-1} \partial_{n-q+1} \cdots \partial_n a_n \times s_{\alpha_{p+1}+m} \cdots s_{\alpha_1+m} \partial_m \cdots \partial_{m+p-1} b_n);$$

the last sum is taken over the indices $0 \le q \le n-1$, $0 \le p \le n-q-1$ and $(\alpha, \beta) \in \{(p+1, q)\text{-shuffles}\}$ where $m = n - p - q$ and $sg(\alpha, \beta) = \sum_{i=1}^{p+1} (\alpha_i - (i-1))$. We define AW, EML and SHI to be the 1, 1 and 0 maps in degree 0, respectively.

Definition 1 The term *homological model for G* refers to a *contraction*

$^{\phi}{:}\, \overline{B}(\mathbb{Z}[G]) \underset{g}{\overset{f}{\rightleftharpoons}} hG$ from the *reduced bar construction* of the group ring of G (i.e. the reduced complex associated to the standard bar resolution [31]) to a differential graded module of finite type hG, so that

$$H_*(G) = H_*(\bar{B}(\mathbb{Z}[G])) \cong H_*(hG)$$

and the homology of hG may be effectively computed by means of Veblen's algorithm [39] (involving the Smith's normal forms of the matrices representing the differential operator).

Example 3 In this example, we show homological models for $\mathbb{Z}$ and $\mathbb{Z}_n$. They have been extracted from [15].

– A homological model for $\mathbb{Z}$.

$$^{\phi_{\mathbb{Z}}}{:}\, \overline{B}(\mathbb{Z}[\mathbb{Z}]) \underset{g_{\mathbb{Z}}}{\overset{f_{\mathbb{Z}}}{\rightleftharpoons}} E(u),$$

where $E(u)$ denotes the free DGA-algebra endowed with trivial differential and generators 1 (at degree 0) and u (at degree 1), so that $u \cdot u = 0$.
The explicit formulas for the morphisms are:
$f_{\mathbf{Z}}([n_1|\cdots|n_q]) = \begin{cases} n_1\, u, & \text{if } q = 1 \\ 0, & \text{if } q > 1 \end{cases}$, $g_{\mathbf{Z}}(u) = [1]$ and

$$\phi_{\mathbf{Z}}[n_1|\cdots|n_k] = \begin{cases} (-1)^k \sum_{i=1}^{n_k-1} [n_1|\cdots|n_{k-1}|i|1], & \text{if } n_k > 0, \\ 0, & \text{if } n_k = 0, \\ (-1)^{k+1} \sum_{i=1}^{|n_k|} [n_1|\cdots|n_{k-1}|-i|1], & \text{if } n_k < 0. \end{cases} \tag{3}$$

– A homological model for $\mathbb{Z}_n$.

$$\phi_{\mathbf{Z}_n} : \overline{B}(\mathbb{Z}[\mathbb{Z}_n]) \underset{g_{\mathbf{Z}_n}}{\overset{f_{\mathbf{Z}_n}}{\rightleftharpoons}} (E(u) \otimes \Gamma(v), d)$$

where $d(u) = 0$, $d(v) = n \cdot u$ and $\Gamma(v)$ denotes the free DGA-algebra endowed with trivial differential and generators $\gamma_k(v)$ (at degree $2k$, $k \geq 0$, $\gamma_0(v) = 1$), such that $\gamma_k(v)\gamma_h(v) = \frac{(k+h)!}{k!h!}\gamma_{k+h}(v)$.
The explicit formulas for the morphisms are:

$$\begin{aligned} f_{\mathbf{Z}_n}[x_1|y_1|\cdots|x_m|y_m] &= \left[\prod_{i=1}^m \delta_{x_i,y_i}\right] \gamma_m(v), \\ f_{\mathbf{Z}_n}[x_1|y_1|\cdots|x_m|y_m|z] &= \left[z \prod_{i=1}^m \delta_{x_i,y_i}\right] u\gamma_m(v), \end{aligned} \quad \text{for } \delta_{x_i,y_i} = \begin{cases} 0, & x_i + y_i < n, \\ 1, & x_i + y_i \geq n; \end{cases}$$

$$g_{\mathbf{Z}_n}(u) = [1], \quad g_{\mathbf{Z}_n}(\gamma_k(v)) = \sum_{x_i \in \mathbf{Z}_n} [1|x_1|\cdots|1|x_k],$$

$$g_{\mathbf{Z}_n}(u\gamma_k(v)) = \sum_{x_i \in \mathbf{Z}_n} [1|x_1|\cdots|1|x_k|1],$$

and

$$\phi_{\mathbf{Z}_n}([x_1|\cdots|x_k]) = -\varphi_{\mathbf{Z}_n}([x_1|\cdots|x_k]), \text{ for } \varphi_{\mathbf{Z}_n}[\,] = 0,\ \varphi_{\mathbf{Z}_n}[x] = \sum_{i=1}^{x-1}[1|i],$$

$$\varphi_{\mathbf{Z}_n}[x|y|\sigma] = \sum_{i=1}^{x-1}[1|i|y|\sigma] + \delta_{x,y} \sum_{k=1}^{n-1}[1|k|\varphi_{\mathbf{Z}_n}\sigma]. \tag{4}$$

Remark 1 It is well-known that if A is a finitely generated Abelian group then A can be written in the form $A = \mathbb{Z}^m \times \mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_n}$, where each l_i denotes a power of a prime. From the data above, a homological model for such an Abelian group

A may be constructed in a straightforward manner [15], by simply applying $n+m$ times the Eilenberg–Zilber theorem, and tensoring up the $n+m$ correspondent single homological models.

One of the cornerstones of the homological perturbation theory is the Basic Perturbation Lemma. It provides a beautiful way of unifying many disparate results in Algebraic Topology concerning chain homotopy equivalences, and it can be used to find new results as well.

Now, we recall the concept of a perturbation datum. Let N be a graded module and let $f : N \to N$ be a morphism of graded modules. The morphism f is *pointwise nilpotent* if for all $x \in N$ ($x \neq 0$), a positive integer n exists (in general, the number n depends on the element x) such that $f^n(x) = 0$. A *perturbation of a DG-module* N is a morphism of graded modules $\delta : N \to N$ of degree -1, such that $(d_N + \delta)^2 = 0$ and $\delta_1 = 0$, i.e. $d_N + \delta$ is a new differential on N. A *perturbation datum of the contraction* $c : \{N, M, f, g, \phi\}$ is a perturbation δ of the DGA-module N verifying that the composition $\phi\delta$ is pointwise nilpotent.

A *Transference Problem* consists of a contraction $c\colon \{M, N, f, g, \phi\}$ together with a perturbation δ of the DG-module N. The problem is to determine new morphisms $d_\delta,\ f_\delta,\ g_\delta$ and ϕ_δ such that $c_\delta : \{(N, d_N + \delta),\ (M, d_M + d_\delta),\ f_\delta,\ g_\delta,\ \phi_\delta\}$ is a contraction.

The Basic Perturbation Lemma ([8,20,21,33]) gives an explicit solution to the Transference Problem, assuming that δ is a perturbation datum of c.

Theorem 2 (BPL) *Let $c : \{N, M, f, g, \phi\}$ be a contraction and $\delta : N \to N$ a perturbation datum of c. Then, a new contraction*

$$c_\delta : \{(N, d_N + \delta),\ (M, d_M + d_\delta),\ \ f_\delta,\ g_\delta,\ \phi_\delta\}$$

is defined by the formulas: $d_\delta = f\delta\Sigma_c^\delta g$; $f_\delta = f(1 - \delta\Sigma_c^\delta\phi)$; $g_\delta = \Sigma_c^\delta g$; $\phi_\delta = \Sigma_c^\delta\phi$; where

$$\Sigma_c^\delta = \sum_{i\geq 0}(-1)^i\,(\phi\delta)^i = 1 - \phi\delta + \phi\delta\phi\delta - \cdots + (-1)^i(\phi\delta)^i + \cdots .$$

Let us note that $\Sigma_c^\delta(x)$ is a finite sum for each $x \in N$, because of the pointwise nilpotency of the composition $\phi\delta$. Moreover, it is obvious that the morphism d_δ is a perturbation of the DG-module (M, d_M).

The twisted Eilenberg–Zilber theorem can be seen as an important example of the usefulness of this lemma (see [37]). It solves the Transference Problem for twisted cartesian products.

Theorem 3 (Twisted Eilenberg–Zilber Theorem) [19,37]

Let $F \times_\tau B$ be the TCP with fibre F, base B and structural group G. Then, the morphism

$$\delta(a, b) = (\tau b \star \partial_0 a, \partial_0 b) - (\partial_0 a, \partial_0 b), \quad (a, b) \in C_N(F \times B)$$

is a perturbation datum of the contraction,

$$^{SHI:}C(F \times B) \underset{EML}{\overset{AW}{\rightleftharpoons}} C(F) \otimes C(B).$$

From these data a new contraction (called the twisted Eilenberg–Zilber contraction) is obtained by applying BPL:

$$^{SHI_\delta:}C(F \times_\tau B) \underset{EML_\delta}{\overset{AW_\delta}{\rightleftharpoons}} C(F) \otimes_t C(B)$$

where the bigger chain complex is associated to $F \times_\tau B$, and the smaller one consists of a twisted tensor product along the twisting cochain t, for $t = p \circ d_\delta \circ \rho$

$$C(B) \xrightarrow{\rho} C(G) \otimes C(B) \xrightarrow{d_\delta} C(G) \otimes C(B) \xrightarrow{p} C_N(G) \tag{5}$$

where

$$\rho(x) = 1_0 \otimes x, \quad 1_0 \text{ being the identity element of } G_0 \text{ and } p(y \otimes x) = \begin{cases} 0, & x \notin B_0 \\ y, & x \in B_0 \end{cases}$$

So that, $C(F) \otimes_t C(B)$ is a differential graded module whose underlying module structure is given by the ordinary tensor product $C(F) \otimes C(B)$ and whose differential is given by $d^{\otimes} + t\cap$, where $d^{\otimes} = d \otimes 1 + 1 \otimes d$ and $t\cap$ is given by:

$$t\cap = (\mu_{C(F)} \otimes 1)(1 \otimes t \otimes 1)(1 \otimes \Delta_{C(B)}), \tag{6}$$

where $\mu_{C(F)}$ is the module action induced by the the action $\star$: $G \times F \to F$. Hence,

$$d_\delta = t\cap.$$

Applying the above theorem to $\overline{W}(K) \times_\tau \overline{W}(H)$, the TCP defined in Example 2, it follows

$$^{SHI_\delta:}C(\overline{W}(K) \times_\tau \overline{W}(H)) \underset{EML_\delta}{\overset{AW_\delta}{\rightleftharpoons}} C(\overline{W}(K)) \otimes_t C(\overline{W}(H)). \tag{7}$$

Furthermore, if K and H are ordinary discrete groups we will give an explicit formula for the twisting cochain t and for the morphism $t\cap$ (see Lemmas 1 and 2).

To sum up, given the semidirect product $K \rtimes_\chi H$ where K and H are simplicial groups with H operating on K from the left, we have

1. $C(\overline{W}(K \rtimes_\chi H)) \overset{\psi}{\cong} C(\overline{W}(K) \times_\tau \overline{W}(H))$ (by Theorem 1).
2. $^{SHI_\delta:}C(\overline{W}(K) \times_\tau \overline{W}(H)) \underset{EML_\delta}{\overset{AW_\delta}{\rightleftharpoons}} C(\overline{W}(K)) \otimes_t C(\overline{W}(H))$ (by Theorem 3).

From now on, we will assume that K and H are ordinary discrete groups, unless otherwise stated.

Lemma 1 *An explicit formula for the twisting cochain* $t : C(\overline{W}(H)) \to C(H)$ *is given by*

$$t(h_{n-1}, \dots, h_0) = \begin{cases} h_0 - 1, & \text{if } n = 1, \\ 0, & \text{if } n \geq 2. \end{cases}$$

Proof Attending to Theorem 3 applied to the TCP $\overline{W}(K) \times_\tau \overline{W}(H))$ (see Example 2), the twisting cochain $t : C(\overline{W}(H)) \to C(H)$ is given by the composition $t = p\, d_\delta\, \rho$,

$$C(\overline{W}(H)) \xrightarrow{\rho} C(H) \otimes_t C(\overline{W}(H)) \xrightarrow{d_\delta} C(H) \otimes_t C(\overline{W}(H)) \xrightarrow{p} C(H),$$

where $\rho(h_{n-1}, \dots, h_0) = 1 \otimes (h_{n-1}, \dots, h_0)$, $p(h \otimes [\,]) = h$ (zero otherwise) and the morphism $d_\delta = AW\delta \sum_{i \geq 0} (-1)^i (SHI\delta)^i EML$ is the perturbation datum provided by BPL when

$$^{SHI:} C(H \times \overline{W}(H)) \underset{EML}{\overset{AW}{\rightleftarrows}} C(H) \otimes C(\overline{W}(H))$$

is perturbed by means of

$$\delta(h, (h_{n-1}, \dots, h_0)) = (h_{n-1} \cdot h, (h_{n-2}, \dots, h_0)) - (h, (h_{n-2}, \dots, h_0)).$$

It is readily checked that the composition $\delta\, EML\, \rho$ consists of

$$\begin{aligned} (h_{n-1}, \dots, h_0) &\xrightarrow{\rho} 1 \otimes (h_{n-1}, \dots, h_0) \\ &\xrightarrow{EML} (1, (h_{n-1}, \dots, h_0)) \\ &\xrightarrow{\delta} (h_{n-1}, (h_{n-2}, \dots, h_0)) - (1, (h_{n-2}, \dots, h_0)). \end{aligned}$$

Independent of the value of n, the application of SHI to the output above is always null. This is obvious for $n = 1$, since SHI is defined as the zero map acting on simplicial degree 0. For $n \geq 2$, the SHI map introduces some degeneracy operators s_j on the term in $\overline{W}(H)$, so that the final output in $C(H \times \overline{W}(H))$ is the image of the degeneracy operator (s_j, s_j), and hence zero (notice that H denotes here the simplicial version of the discrete group H, whose degeneracy and face operators are the identity map on H).

This way, the composition $p\, d_\delta\, \rho$ reduces to $p\, AW\delta EML\, i$.

All summands of

$$AW((h, (h_{n-2}, \dots, h_0)) = \sum_{i=0}^{n-1} \partial_{n-i}^{n-i} h \otimes \partial_0^i (h_{n-2}, \dots, h_0)$$

are zero but the one correspondent to $i = 0$, so that the element in $C(H)$ is located at simplicial degree 0 (and hence is not degenerated). Thus,
$AW\delta EML\, i(h_{n-1}, \ldots, h_0) = (h_{n-1}, (h_{n-2}, \ldots, h_0)) - (1, (h_{n-2}, \ldots, h_0))$.

Taking into account that the projection p is null acting on the elements of $C(\overline{W}(H))$ of simplicial degree greater than 0, we finally conclude that

$$t(h_{n-1}, \ldots, h_0) = p\, d_\delta\, i(h_{n-1}, \ldots, h_0) = \begin{cases} h_0 - 1, & n = 1, \\ 0, & n \geq 2. \end{cases}$$

□

Remark 2 If the basis group H of the semidirect product is located on the left-hand side, $H_\chi \ltimes K$, the precedent twisting cochain t must be changed in turn to the opposite $t' = -t$.

Lemma 2 *An explicit formula for the morphism*

$$t\cap\colon C(\overline{W}(K)) \otimes C(\overline{W}(H)) \to C(\overline{W}(K)) \otimes C(\overline{W}(H))$$

is given by

$$\begin{aligned} &t \cap ((k_{n-1}, \ldots, k_0) \otimes (h_{m-1}, \ldots, h_0)) \\ &\quad = (-1)^n ((h_{m-1}k_{n-1}, \ldots, h_{m-1}k_0) \otimes (h_{m-2}, \ldots, h_0) - (k_{n-1}, \ldots, k_0) \\ &\qquad \otimes (h_{m-2}, \ldots, h_0)) \end{aligned}$$

Proof It is a simple inspection. The formula for $t\cap$ is given in (6). □

Step 3

Our next goal will be to construct a contraction from $C(\overline{W}(K)) \otimes_t C(\overline{W}(H))$ to hKH (a DG-module of finite type). To this end, we assume knowing a homological model for K and H, respectively:

$$C(\overline{W}(K)) \overset{\varphi^{-1}}{\cong} {}^{\phi_K:}\overline{B}(\mathbb{Z}[K]) \underset{g_K}{\overset{f_K}{\rightleftharpoons}} hK \quad \text{and} \quad C(\overline{W}(H)) \overset{\varphi^{-1}}{\cong} {}^{\phi_H:}\overline{B}(\mathbb{Z}[H]) \underset{g_H}{\overset{f_H}{\rightleftharpoons}} hH.$$

With these homological models at hand we construct

$${}^{1\otimes\phi_H + \phi_K \otimes g_H f_H:} C(\overline{W}(K)) \otimes C(\overline{W}(H)) \underset{g_K \otimes g_H}{\overset{f_K \otimes f_H}{\rightleftharpoons}} hK \otimes hH.$$

If the morphism $t\cap$ (see Lemma 2) is a perturbation datum of the contraction above, then the BPL yields the desired contraction.

Now, we have all the necessary elements to state the following result.

Theorem 4 *Let K and H be finitely generated Abelian groups, and let $K \rtimes_\chi H$ be the semidirect product of H and K with respect to the action χ. Then, the morphism $t\cap$ (Lemma 2) is a perturbation datum of*

$$^{1\otimes\phi_H+\phi_K\otimes g_H f_H}: C(\overline{W}(K)) \otimes C(\overline{W}(H)) \underset{g_K\otimes g_H}{\overset{f_K\otimes f_H}{\rightleftharpoons}} hK \otimes hH, \tag{8}$$

and hence a homological model for $K \rtimes_\chi H$ is completely determined.

Proof Obviously, $t\cap$ is a perturbation of the complex $C(\overline{W}(K)) \otimes C(\overline{W}(H))$, so if we prove that $(1 \otimes \phi_H + \phi_K \otimes g_H f_H)t\cap$ is pointwise nilpotent then $t\cap$ will be a perturbation of the contraction (8).

To this end, we look for a filtration $\{D_q\}_{q\geq 0}$ on $C(\overline{W}(K)) \otimes C(\overline{W}(H))$, such that $t\cap$ reduces the filtration degree, as $(1 \otimes \phi_H + \phi_K \otimes g_H f_H)$ preserves the filtration degree. Consequently, the composition $(1\otimes\phi_H+\phi_K\otimes g_H f_H)t\cap$ reduces the filtration degree, and is shown to be pointwise nilpotent.

Assume that $H = \mathbb{Z}^m \times \mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_n}$. We define $F_q(C(W(H))$ to be the sub-DG-module generated by those tuples $(x^1_{t-1} \times \cdots \times x^{m+n}_{t-1}, \ldots, x^1_0 \times \cdots \times x^{m+n}_0)$ such that $\sum_{i,j} |x^j_i| \leq q$. We define the filtration $\{D_q\}_{q\geq 0}$ so that

$$D_q = C(\overline{W}(K)) \otimes F_q(C(\overline{W}(H)).$$

Taking into account formulas (3), (4) and Remark 1, it is readily checked that the homotopy operator ϕ_H and the composition $g_H f_H$ preserve the filtration degree. Furthermore, using the formula giving in Lemma 2 and by a simple inspection, we can state that $t\cap$ decreases the filtration degree, at least in one degree. So, $\{D_q\}_{q\geq 0}$ is the desired filtration.

Thus, BPL gives rise to the contraction

$$^{(1\otimes\phi_H+\phi_K\otimes g_H f_H)_{t\cap}}: C(\overline{W}(K)) \otimes_t C(\overline{W}(H)) \underset{(g_K\otimes g_H)_{t\cap}}{\overset{(f_K\otimes f_H)_{t\cap}}{\rightleftharpoons}} (hK \otimes hH, 1\otimes d + d\otimes 1 + d_{t\cap}).$$

For the sake of simplicity, we note $\phi_t = (\phi_K \otimes g_H f_H + 1 \otimes \phi_H)_{t\cap}$, $f_t = (f_K \otimes f_H)_{t\cap}$, $g_t = (g_K \otimes g_H)_{t\cap}$ and $hKH = (hK \otimes hH, 1 \otimes d + d \otimes 1 + d_{t\cap})$. □

Remark 3 Notice that the proof of the theorem above works on any semidirect product $K \rtimes_\chi H$, for H a finitely generated Abelian group, and for K a group with a known homolological model, but not necessarily Abelian. We rely on this fact to extend the above theorem to iterated semidirect products in the next section.

To sum up, under the hypothesis of Theorem 4 we can link the next complexes

$$\begin{array}{ccccc}
\overline{B}(\mathbb{Z}[K\rtimes_\chi H]) & \overset{\varphi}{\cong} & C(\overline{W}(K\rtimes_\chi H)) & \overset{\psi}{\cong} & C(\overline{W}(K)\times_\tau \overline{W}(H)) \\
 & & & & \Downarrow th.3 \\
hKH & \overset{th.4}{\Leftarrow} & \overline{B}(\mathbb{Z}[K])\otimes_t \overline{B}(\mathbb{Z}[H]) & \overset{\varphi^{-1}}{\cong} & C(\overline{W}(K))\otimes_t C(\overline{W}(H))
\end{array}$$

Composing the contractions above, we get a homological model for $K \rtimes_\chi H$,

$$\phi \circlearrowright \overline{B}(\mathbb{Z}[K \rtimes_\chi H]) \underset{g}{\overset{f}{\rightleftharpoons}} hKH \tag{9}$$

where

$$\begin{aligned} f &= f_t \circ AW_\delta \circ \psi \circ \varphi, \\ g &= \varphi^{-1} \circ \psi^{-1} \circ EML_\delta \circ g_t, \\ \phi &= \varphi^{-1} \circ \psi^{-1} \circ (SHI_\delta + EML_\delta \circ \phi_t \circ AW_\delta) \circ \psi \circ \varphi. \end{aligned}$$

Let us observe that these formulas are not recursive.

Example 4 In this example, we give a homological model only up to degree 3 for the dihedral group of $2n$ elements, $D_{2n} = \mathbb{Z}_n \rtimes_\chi \mathbb{Z}_2$, $\chi : \mathbb{Z}_2 \times \mathbb{Z}_n \to \mathbb{Z}_n$ such that $\chi(1, x) = -x$ and $\chi(0, x) = x$. In the sequel, we use the following notation. We define the set map $\lambda^{2n} : \mathbb{Z} \to \mathbb{Z}_2$, so that $\lambda^{2n}(j) = \lambda^{2n}_j = 1$ if $j \geq 2n$ and 0 otherwise. The notation $[x]_m$ refers to x mod m.

$$\phi_{D_{2n}} \circlearrowright \overline{B}(\mathbb{Z}[D_{2n}]) \underset{g_{D_{2n}}}{\overset{f_{D_{2n}}}{\rightleftharpoons}} (E(u) \otimes E(u') \otimes \Gamma(v) \otimes \Gamma(v'), d),$$

where the differential on elements of degrees less than or equal to 4, non null, is:

$$\begin{aligned} d(v) &= 2n\, u, \quad d(uu') = (2 - 2n)\, u, \quad d(v') = 2\, u'; \\ d(vu') &= 2n\, uu' + (2n - 2)\, v, \quad d(uv') = -2n\, uu' - (2n - 2)\, v; \\ d(\gamma_2(v)) &= 2n\, uv, \quad d(uvu') = (-1 + (2n - 1)^2)\, uv; \\ d(vv') &= 2n\, uv' + 2n\, vu' - n(2n - 1)(2n - 2)\, uv, \qquad d(\gamma_2(v') = 2\, u'v'; \\ d(uu'v') &= (2 - 2n)\, uv' + (2 - 2n)\, vu' + \frac{(2t - 1)(2t - 2)^2}{2}\, uv. \end{aligned}$$

The formula for the projection $f_{D_{2n}}$ on elements of degrees less than or equal to 3 is:

$$\begin{aligned} & f_{D_{2n}}[(g, h)] = h\, u' + [(-1)^h g]_{2n}\, u. \\ & f_{D_{2n}}[(g, h)|(b, a)] = a \cdot h\, v' + b \cdot h\, uu' + \, (\lambda^{2n}_{[(-1)^{a+ah} b]_{2n} + [(-1)^{a+h+ah} g]_{2n}} \\ & \qquad\qquad + \sum_{i=1}^{b-1} \lambda^{2n}_{[(-1)^h 1]_{2n} + [(-1)^h i]_{2n}})\, v \\ & f_{D_{2n}}[(g, h)|(b, a)|(j, i)] = -(i \cdot a \cdot h)\, uv' - (j \cdot a \cdot h)\, uv' \\ & \quad -(h \cdot \lambda^{2n}_{[(-1)^a j]_{2n} + b} + a \cdot h \cdot (j - 1))\, vu' \\ & \quad +(-g \lambda^{2n}_{[(-1)^{a+h} j]_{2n} + [(-1)^h b]_{2n}} + h([(-1)^a j]_{2n} - 1)[(-1)^h b]_{2n} \\ & \quad + \sum_{l=1}^{j-1} a \cdot h \cdot (2n - 2)[(-1)^{a+h} l]_{2n})\, uv. \end{aligned}$$

The formula for the injection $g_{D_{2n}}$ on elements of degrees less than or equal to 3 is:

$$g_{D_{2n}}(u) = [(1,0)], \quad g_{D_{2n}}(u') = [(0,1)], \quad g_{D_{2n}}(v) = -\sum_{i=1}^{2n-1}[(i,0)|(1,0)];$$

$$g_{D_{2n}}(uu') = -[(0,1)|(1,0)] + [(2n-1,1)|(0,1)] + \sum_{i=1}^{2n-2}[(i,0)|(1,0)];$$

$$g_{D_{2n}}(v') = -[(0,1)|(0,1)], \quad g_{D_{2n}}(uv) = -\sum_{i=1}^{2n-1}[(1,0|(i,0)|(1,0)];$$

$$\begin{aligned} g_{D_{2n}}(uv') =& -[(0,1)|(0,1)|(1,0)] + [(0,1)|(2n-1,0)|(0,1)] - [(1,0)|(0,1)|(0,1)] \\ &+ \sum_{i=2}^{2n-2}([(0,1)|(i,0)|(1,0)] - [(-i,0)|(0,1)|(1,0)] + [(-i,0)|(2n-1,0)|(0,1)]) \\ &+ \sum_{i=2}^{2n-1}\sum_{j=1}^{2n-2}[(i,0)|(j,0)|(1,0)]; \end{aligned}$$

$$\begin{aligned} g_{D_{2n}}(vu') =& \sum_{i=1}^{2n-1}(-[(-i,0)|(2n-1,0)|(0,1)] + [(-i,0)|(0,1)|(1,0)] \\ &-[(0,1)|(i,0)|(1,0)]) \\ &+ \sum_{i=1}^{2n-1}\sum_{j=1}^{2n-2}[(i,0)|(j,0)|(1,0)]; \end{aligned}$$

$$g_{D_{2n}}(u'v') = -[(0,1)|(0,1)|(0,1)].$$

Further degrees are computed in a similar way, but they become more and more complicated.

Using only the projection f and the differential d described in the example above, a generating set of representative 2-cocycles and 3-cocycles over D_{4t} are given in [4,5], respectively. These computations have led to the finding of new cocyclic Hadamard matrices [4, Table 1].

4 Related questions

We include here some comments about several related topics. The first section is devoted to indicate briefly how a resolution of $\mathbb{Z}$ over $\mathbb{Z}[K \rtimes_\chi H]$ arises from a homological model of $K \rtimes_\chi H$. In the following sections, we will see that the method described in this paper is suitable for iterated semidirect products and simplicial semi-direct products. Finally, we will give some simplifications of the formulas that our method provides and some examples.

4.1 A resolution of integers over the group ring of $K \rtimes_\chi H$

The homology of a group G is usually determined from a resolution of the integers over the group ring of G (see [7]). Resolutions for semidirect products of groups have been given in [6,9] among others. The homological perturbation theory has been applied to compute resolutions for a wide range of groups (e.g. finitely generated two-step nilpotent groups [24], metacyclic groups [25], finite p-groups [18]). Using homological perturbation theory, we show that a resolution R of $\mathbb{Z}$ over $\mathbb{Z}[K \rtimes_\chi H]$ (which splits off of the bar resolution) arises from a homological model for $K \rtimes_\chi H$. Furthermore, a contracting homotopy on R can be constructed by a formula involving the contracting homotopy on $B(\mathbb{Z}[G])$. From a practical point of view, this method is only appropriate for numerical calculations in low degrees.

Definition 2 Lambe [28] A resolution X over $\mathbb{Z}[G]$ *splits off of the bar construction* if there is a contraction from $B(\mathbb{Z}[G])$ (the bar resolution over $\mathbb{Z}[G]$) to X.

Theorem 5 *Suppose that $K \rtimes_\chi H$ is a semidirect product of finitely generated Abelian groups. There exists a resolution $(\mathbb{Z}[K \rtimes_\chi H] \otimes hKH, d)$ which splits off of the bar resolution $B(\mathbb{Z}[K \times_\chi H])$.*

Proof To Construct a resolution of the integers over $\mathbb{Z}[K \rtimes_\chi H]$ boils down to putting a $\mathbb{Z}[K \rtimes_\chi H]$-linear differential on $\mathbb{Z}[K \rtimes_\chi H] \otimes hKH$ such that an acyclic DG-module results. To this end, we follow these steps:

1. The tensor product of (9) and the trivial contraction provides

$$^{1\otimes\phi:}\mathbb{Z}[K \rtimes_\chi H] \otimes \overline{B}(\mathbb{Z}[K \rtimes_\chi H]) \underset{1\otimes g}{\overset{1\otimes f}{\rightleftharpoons}} (\mathbb{Z}[K \rtimes_\chi H] \otimes hKH,\ d^{\otimes}). \qquad (10)$$

2. Perturb the contraction above with $\theta\cap = d - d'$ where d is the differential on the bar resolution, $B(\mathbb{Z}[K \rtimes_\chi H]) = \mathbb{Z}[K \rtimes_\chi H] \otimes_\theta \overline{B}(\mathbb{Z}[K \rtimes_\chi H])$, and d' is the trivial differential on $\mathbb{Z}[K \rtimes_\chi H] \otimes \overline{B}(\mathbb{Z}[K \rtimes_\chi H])$. Obtaining:

$$^{(1\otimes\phi)_{\theta\cap}:}\mathbb{Z}[K \rtimes_\chi H] \otimes_\theta \overline{B}(\mathbb{Z}[K \rtimes_\chi H]) \underset{(1\otimes g)_{\theta\cap}}{\overset{(1\otimes f)_{\theta\cap}}{\rightleftharpoons}} (\mathbb{Z}[K \rtimes_\chi H] \otimes hKH, d^{\otimes} + d_{\theta\cap}).$$

Obviously, $\epsilon : (\mathbb{Z}[K \rtimes_\chi H] \otimes hKH,\ d^{\otimes} + d_{\theta\cap}) \to \mathbb{Z}$ is the desired resolution and $d_{\theta\cap}$ is given explicitly by BPL.

Hence, we have to prove that the universal twisting cochain θ is a perturbation datum of (10). We organize the proof in three steps.

1. The contraction

$$^{0:}\mathbb{Z}[K \rtimes_\chi H] \otimes C(\overline{W}(K \rtimes_\chi H)) \underset{1\otimes\psi^{-1}}{\overset{1\otimes\psi}{\rightleftharpoons}} \mathbb{Z}[K \rtimes_\chi H] \otimes C(\overline{W}(K) \times_\tau \overline{W}(H))$$

may be perturbed by means of the perturbation datum $\theta\cap$

$$\theta \cap ((k, h) \otimes ((k_1, h_1), \ldots, (k_n, h_n))) = (k, h) \cdot (k_1, h_1) \otimes ((k_2, h_2), \ldots, (k_n, h_n)) - (k, h) \otimes ((k_2, h_2), \ldots, (k_n, h_n))$$

induced by the universal twisting cochain $\theta : \overline{B}(\mathbb{Z}[K \times_\chi H]) \to \mathbb{Z}[K \times_\chi H]$,

$$\theta([(k_1, h_1)|\cdots|(k_n, h_n)]) = \begin{cases} (k_1, h_1) - (e_K, e_H), & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

In fact, this step defines an isomorphism, since the homotopy operator is the zero map.
The perturbed differential $d_{\theta\cap}$ consists of

$$\begin{aligned} &d_{\theta\cap}((k, h) \otimes (\{k_{n-1}, \ldots, k_0\}, \{h_{n-1}, \ldots, h_0\})) \\ &\quad = ((k, h) \cdot (h_{n-1}k_{n-1}, h_{n-1}) - (k, h)) \otimes (\{h_{n-1}k_{n-2}, \ldots, h_{n-1}k_0\}, \\ &\qquad \{h_{n-2}, \ldots, h_0\}). \end{aligned}$$

2. We now prove that $d_{\theta\cap}$ induces a finite perturbation process from

$$\begin{aligned} &(1 \otimes AW_\delta,\ 1 \otimes EML_\delta,\ 1 \otimes SHI_\delta): \\ &\mathbb{Z}[K \rtimes_\chi H] \otimes C(\overline{W}(K) \times_\tau \overline{W}(H)) \Rightarrow \mathbb{Z}[K \rtimes_\chi H] \otimes C(\overline{W}(K)) \otimes_t C(\overline{W}(H)) \end{aligned}$$

to

$$\begin{aligned} &((1 \otimes AW_\delta)_{d_{\theta\cap}},\ (1 \otimes EML_\delta)_{d_{\theta\cap}},\ (1 \otimes SHI_\delta)_{d_{\theta\cap}}): \\ &\mathbb{Z}[K \rtimes_\chi H] \otimes_\theta C(\overline{W}(K) \times_\tau \overline{W}(H)) \Rightarrow \mathbb{Z}[K \rtimes_\chi H] \tilde{\otimes} C(\overline{W}(K)) \otimes_t C(\overline{W}(H)). \end{aligned}$$

Certainly, the map $(1 \otimes SHI_\delta)d_{\theta\cap}$ is pointwise nilpotent, as the filtration

$$\begin{aligned} F_q = \{(k, h) \otimes (\{k_{n-1}, \ldots, k_0\}, \{h_{n-1}, \ldots, h_0\}) : \\ \#(\{i : k_i = 0 \text{ or } h_i = 0\}) \geq n - q\} \end{aligned}$$

shows. It is readily checked that $d_{\theta\cap}$ increases the filtration degree at most by 1 unit, since k_{n-1} and h_{n-1} cannot be simultaneously zero (we are working with normalized chain complexes). Taking into account that

$$SHI_\delta = \sum_{i \geq 0} (-1)^i [SHI((\delta\partial_0, \partial_0) - (\partial_0, \partial_0))]^i SHI,$$

it is evident that SHI_δ diminishes the filtration degree at least by 2 units, accordingly to the formulas for SHI (the filtration degree decreases by 2) and $SHI((\tau\partial_0, \partial_0) - (\partial_0, \partial_0))$ (the filtration degree decreases by 1).
An explicit formula for $\rho = d_{d_{\theta\cap}}$ is

$$\begin{aligned} &\rho((k, h) \otimes \{k_{p-1}, \ldots, k_0\} \otimes \{h_{q-1}, \ldots, h_0\}) \\ &\quad = ((k, hh_{q-1}) - (k, h)) \otimes \{h_{q-1}k_{p-1}, \ldots, h_{q-1}k_0\} \otimes \{h_{q-2}, \ldots, h_0\} \\ &\qquad + ((k + hk_{p-1}, h) - (k, h)) \otimes \{k_{p-2}, \ldots, k_0\} \otimes \{h_{q-1}, \ldots, h_0\}. \end{aligned}$$

3. Finally, the perturbation of the contraction

$$^{1\otimes\phi_t:}\mathbb{Z}[K \rtimes_\chi H] \otimes C(\overline{W}(K)) \otimes_t C(\overline{W}(H)) \underset{1\otimes g_t}{\overset{1\otimes f_t}{\rightleftharpoons}} \mathbb{Z}[K \times_\chi H] \otimes hKH$$

by means of ρ converges, since $(1 \otimes \phi_t)\rho$ is pointwise nilpotent, as it may be concluded from the filtration

$$F_q = \{(k, h) \otimes \{k_{p-1}, \ldots, k_0\} \otimes \{h_{q-1}, \ldots, h_0\} : \sum_{i=0}^{p-1} |k_i| + \sum_{j=0}^{q-1} |h_j| \leq q\}.$$

□

Remark 4 The morphism $(1 \otimes f_t)_{\theta\cap} \circ s \circ (1 \otimes g_t)_{\theta\cap}$ is a contracting homotopy on the resolution above where $s\colon B(\mathbb{Z}[G]) \to B(\mathbb{Z}[G])$ with $s(g \otimes [g_1|\cdots|g_n]) = [g|g_1|\cdots|g_n]$ is the contracting homotopy on the bar resolution.

4.2 Iterated semidirect products of finitely generated Abelian groups

The definition of semidirect product of two groups G_1 and G_2 with respect to the homomorphism $\alpha\colon G_1 \to \mathrm{Aut}(G_2)$ denoted by $G_2 \rtimes_\alpha G_1$ can of course be iterated. Assume we are given groups $G_1, \ldots, G_l$ and, for each $1 < q \leq l$, homomorphisms

$$\alpha_q : G_{q-1} \to \mathrm{Aut}((\ldots(G_l \rtimes_{\alpha_l} G_{l-1}) \rtimes \ldots) \rtimes_{\alpha_{q+1}} G_q).$$

Then, we define the *iterated semidirect product* of $G_1, \ldots, G_l$ with respect to α_q to be the group

$$G = ((\ldots(G_l \rtimes_{\alpha_l} G_{l-1}) \rtimes \ldots) \rtimes_{\alpha_3} G_2) \rtimes_{\alpha_2} G_1).$$

In this section we extend the preceding work to the case of iterated semidirect products of finitely generated Abelian groups.

Theorem 6 *Let G be an iterated semidirect product of finitely generated Abelian groups. There exists a homological model* $^{\varphi:}\overline{B}(\mathbb{Z}[G]) \underset{g}{\overset{f}{\rightleftharpoons}} hG$ *for G.*

Proof The filtrations used in the proof of Theorem 4 extend directly to this situation. □

Remark 5 Notice that the proof of the theorem above fits with iterated semidirect products of groups with G_i finitely generated Abelian groups for $1 \leq i \leq l-1$, and with group G_l not necessarily Abelian. This is the case of the iterated products of central extensions and semidirect products of finitely generated Abelian groups considered in [1,3].

Theorem 7 *Suppose that G is an iterated semidirect product of finitely generated Abelian groups. There exists a resolution $(\mathbb{Z}[G] \otimes hG, d)$ which splits off of the bar resolution $B(\mathbb{Z}[G])$.*

Proof Once again, the filtrations used in the proof of Theorem 5 extend in a straightforward way to this situation. □

4.3 On homological models for simplicial semidirect products of groups

In the setting of the simplicial groups we have an analogous result to Theorem 4 under certain hypothesis. More concretely, let us assume that K and H are two simplicial groups where H operates on K from the left. Then we have the following chain of contractions:

$$C(\overline{W}(K \rtimes_\chi H)) \overset{\psi}{\cong} C(\overline{W}(K) \times_\tau \overline{W}(H)) \overset{th.3}{\Rightarrow} C(\overline{W}(K)) \otimes_t C(\overline{W}(H))$$

Furthermore, if two finite DG-modules hK and hH exist such that $C(\overline{W}(K)$ and $C(\overline{W}(H)$ contract to hK and hH, respectively, and the twisting cochain t (see 5) vanishes on simplicial degree 1 in $C(\overline{W}(H))$. Then the morphism $t\cap$ (see 6) is a perturbation datum of the contraction

$$C(\overline{W}(K) \otimes C(\overline{W}(H) \Rightarrow hK \otimes hH \qquad (11)$$

(see [29, lemma 3.4.]). Hence, we can state the following theorem:

Theorem 8 *Under the circumstances displayed above. There exists a homological model for the semidirect product $K \rtimes_\chi G$ of simplicial groups H and K.*

Proof This homological model is the composition of the following chain of contractions:

$$\begin{array}{c} C(\overline{W}(K \rtimes_\chi H)) \overset{\psi}{\cong} C(\overline{W}(K) \times_\tau \overline{W}(H)) \overset{th.3}{\Rightarrow} C(\overline{W}(K)) \otimes_t C(\overline{W}(H)) \\ \Downarrow \\ (hK \otimes hH, d^{\otimes} + d_{t\cap}) \end{array}$$

The BPL yields the last contraction in the diagram above where the input data are the contraction (11) and the perturbation $t\cap$. □

Remark 6 If H is reduced, then the twisting cochain t (see (5)) vanishes on simplicial degree 1 in $C(\overline{W}(H))$, as the following theorem states.

Theorem 9 [32] *Let $F \times_\tau B$ be a TCP with structural group G, and let e_0 denote the unit of G_0. If $\tau(b) = e_0, \quad \forall b \in B_1$, then $t(b) = 0, \quad \forall b \in B_1$ where t denotes the cochain (5).*

4.4 Some simplifications on the morphisms involved in the perturbation process

In spite of the fact that a perturbation process is involved, the formulas for the morphisms EML_δ, AW_δ and SHI_δ (see (7)) in our method may be substantially reduced.

Proposition 1 *Consider the contraction*

$$^{SHI_\delta}\!:C(\overline{W}(K)\times_\tau \overline{W}(H)) \underset{EML_\delta}{\overset{AW_\delta}{\rightleftharpoons}} C(\overline{W}(K))\otimes_t C(\overline{W}(H)).$$

Then $SHI_\delta = SHI$, $EML_\delta = EML$ *and* $AW_\delta = AW - AW\delta SHI$.

Proof As we noted before (cf. Sect. 3), the perturbation datum associated to the perturbation process above, $\delta : C(\overline{W}(K)\times\overline{W}(H)) \to C(\overline{W}(K)\times\overline{W}(H))$, consists of

$$\begin{aligned}&\delta((k_{n-1},\dots,k_0),(h_{n-1},\dots,h_0))\\&\quad=((h_{n-1}k_{n-2},\dots,h_{n-1}k_0),(h_{n-2},\dots,h_0))-((k_{n-2},\dots,k_0),(h_{n-2},\dots,h_0)).\end{aligned}$$

Defining $\overline{\kappa} : C(\overline{W}(K)\times\overline{W}(H)) \to C(\overline{W}(K)\times\overline{W}(H))$ is given by

$$\overline{\kappa}((k_{n-1},\dots,k_0),(h_{n-1},\dots,h_0)) = ((h_{n-1}k_{n-1},\dots,h_{n-1}k_0),(h_{n-1},\dots,k_0)),$$

it is easily checked that $\delta = \partial_0\overline{\kappa} - \partial_0$, for $\partial_0 = (\partial_0,\partial_0)$ being the degeneracy operator in $C(\overline{W}(K)\times\overline{W}(H))$.

It may be seen by inspection that an explicit formula for SHI consists of

$$\begin{aligned}&SHI((k_{n-1},\dots,k_0),(h_{n-1},\dots,h_0))\\&\quad=\sum_{q=0}^{n-1}\sum_{p=0}^{n-q-1}\pm((k_{n-1},\dots,k_{p+q+1},0),(h_{n-1},\dots,h_{p+q+1},h_{p+q}\cdots h_q))\,|\\&\qquad((k_{p+q},\dots,k_q),(1,\dots,1))*((0,\dots,0),(h_{q-1},\dots,h_0)),\end{aligned}$$

where $*$ denotes the shuffle product and | is used for juxtaposition.

Hence,

$$\begin{aligned}&SHI\overline{\kappa}((k_{n-1},\dots,k_0),(h_{n-1},\dots,h_0))\\&\quad=\overline{\kappa}SHI((k_{n-1},\dots,k_0),(h_{n-1},\dots,h_0))\\&\qquad-\sum_{q=0}^{n-1}[(0,h_{n-1}\cdots h_q)|((h_{n-1}\cdots h_qk_{n-1},\dots,h_{n-1}\cdots h_qk_q),(1,\dots,1))\\&\qquad*((0,\dots,0),(h_{q-1},\dots,h_0))\\&\qquad+(0,h_{n-1}\cdots h_q)|((h_{n-1}k_{n-1},\dots,h_{n-1}k_q),(1,\dots,1))\\&\qquad*((0,\dots,0),(h_{q-1},\dots,h_0))].\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\partial_0\overline{\kappa}SHI((k_{n-1},\ldots,k_0),(h_{n-1},\ldots,h_0))\\
&\quad=\partial_0 SHI\overline{\kappa}((k_{n-1},\ldots,k_0),(h_{n-1},\ldots,h_0))\\
&\qquad+\sum_{q=0}^{n-1}[((h_{n-1}\cdots h_q k_{n-1},\ldots,h_{n-1}\cdots h_q k_q),(1,\ldots,1))\\
&\qquad*((0,\ldots,0),(h_{q-1},\ldots,h_0))\\
&\qquad-((h_{n-1}k_{n-1},\ldots,h_{n-1}k_q),(1,\ldots,1))*((0,\ldots,0),(h_{q-1},\ldots,h_0))]. \quad (12)
\end{aligned}$$

As a preliminary to the next step, it is necessary to note the following identities: (Due to $\partial_0 SHI=-SHI\partial_0+EML\,AW$ (extracted from [15]) and the side conditions of the Eilenberg–Zilber contraction. We have)

$$SHI\partial_0 SHI=-SHI\,SHI\partial_0+SHI\,EML\,AW=0.$$

and (Taking into account (12), $EML\simeq *$ and the side conditions of the Eilenberg-Zilber contraction again)

$$SHI\partial_0\overline{\kappa}SHI=SHI\partial_0 SHI\overline{\kappa}+SHI\,EML(\,,\,)-SHI\,EML(\,,\,)=0.$$

In these circumstances, we have:

$$SHI\delta SHI=SHI\partial_0\overline{\kappa}SHI-SHI\partial_0 SHI=0.$$

Thus,

$$EML_\delta=\sum_{i\geq 0}(-1)^i(SHI\delta)^i EML=EML+SHI\delta EML.$$

Moreover, it is easy to check that

$$\delta EML=\partial_0\overline{\kappa}EML-\partial_0 EML=EML(\,,\,)-EML(\,,\,),$$

then $SHI\delta EML=0$ and $EML_\delta=EML$.

Analogously, $SHI_\delta=\sum_{i\geq 0}(-1)^i(SHI\delta)^i SHI=SHI$.

Finally, $AW_\delta=\sum_{i\geq 0}(-1)^i AW(\delta SHI)^i=AW-AW\delta SHI$, so that

$$\begin{aligned}
&AW_\delta((k_{n-1},\ldots,k_0),(h_{n-1},\ldots,h_0))\\
&\quad=(2AW-AW\overline{\kappa})((k_{n-1},\ldots,k_0),(h_{n-1},\ldots,h_0))\\
&\qquad-\sum_{q=0}^{n-1}[((h_{n-1}\cdots h_q k_{n-1},\ldots,h_{n-1}\cdots h_q k_q,0,\ldots,0),(1,\ldots,1,h_{q-1},\ldots,h_0))\\
&\qquad-((h_{n-1}k_{n-1},\ldots,h_{n-1}k_q,0,\ldots,0),(1,\ldots,1,h_{q-1},\ldots,h_0))].
\end{aligned}$$

□

4.5 Examples

All the executions and examples of this section have been worked out with aid of the *Mathematica 4.0* notebook [3] described in [1].

We now include some calculations for dihedral groups and an iterated product of a central extension by a semidirect product of finite abelian groups by means of their homological models. These groups have provided a large amount of cocyclic Hadamard matrices in [2,4].

In the sequel, for brevity, we only show the groups $H_i(G)$ and the matrices M_i, for some values of i. The matrix M_i represents the differential operator d_i. We include M_i in order to have an idea about its dimension and sparsity.

Finite dihedral groups

$D_{4t} = \mathbb{Z}_2 \,{}_\chi\!\rtimes \mathbb{Z}_{2t}$, $\chi : \mathbb{Z}_2 \times \mathbb{Z}_{2t} \to \mathbb{Z}_{2t}$ such that $\chi(1, x) = -x$ and $\chi(0, x) = x$. Notice that D_4 is abelian, but D_{4t} is not abelian, for $t > 1$. We next compute $H_i(D_{20})$ for $0 \leq i \leq 5$.

i	$M_i(d_i)$	$H_i(D_{20})$
0	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\mathbb{Z}$
1	$\begin{pmatrix} 10 & 0 \\ -8 & 0 \\ 0 & 2 \end{pmatrix}$	$\mathbb{Z}_2^2$
2	$\begin{pmatrix} 0 & 0 & 0 \\ 8 & 10 & 0 \\ -8 & -10 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}_2$
3	$\begin{pmatrix} 10 & 0 & 0 & 0 \\ -80 & 0 & 0 & 0 \\ -360 & 10 & 10 & 0 \\ 288 & -8 & -8 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_{10}$
4	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 80 & 10 & 0 & 0 & 0 \\ -656 & -82 & 0 & 0 & 0 \\ -2880 & -360 & 8 & 10 & 0 \\ 2304 & 288 & -8 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}_2^2$
5	$\begin{pmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ -728 & 0 & 0 & 0 & 0 & 0 \\ -29520 & 82 & 10 & 0 & 0 & 0 \\ 238464 & -656 & -80 & 0 & 0 & 0 \\ 733440 & -2880 & -360 & 10 & 10 & 0 \\ -523776 & 2304 & 288 & -8 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$	$\mathbb{Z}_2^4$

An iterated product of finite groups

$G_t = (\mathbb{Z}_t \rtimes_f \mathbb{Z}_2) \ltimes_\chi \mathbb{Z}_2$, for χ being the dihedral action $\chi(a,b) = \begin{cases} -b \text{ if } a = 1 \\ b \text{ if } a = 0 \end{cases}$ and $f : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_t$ being the 2-cocycle $f(g_i, g_j) = \begin{cases} \lceil \frac{t}{2} \rceil + 1 \text{ if } g_i = g_j = 1 \\ 0 \qquad\qquad \text{otherwise} \end{cases}$

Notice that $\mathbb{Z}_t \rtimes_f \mathbb{Z}_2$ is abelian (since f is symmetric), but G_t is not abelian for $t \neq 2$ (because of the dihedral action). Furthermore $G_t \simeq D_{4t}$ for odd t, since f is a 2-coboundary in these circumstances: $f = f_\alpha$, for $\alpha : \mathbb{Z}_2 \to \mathbb{Z}_t$ such that $\alpha(0) = 0$, $\alpha(1) = \dfrac{t^2+3}{4}\, mod\, t$. Analogously, the extension is also trivial for $t \equiv 2\, mod\, 4$, since $f = f_\alpha$, for $\alpha(0) = 0$, $\alpha(1) = \lfloor \frac{t}{4} \rfloor + 1$, so that $G_t \simeq (\mathbb{Z}_t \times \mathbb{Z}_2) \ltimes_\chi \mathbb{Z}_2$.

We next compute $H_i(G_4)$ for $0 \leq i \leq 4$.

i	$M_i(d_i)$	$H_i(G_4)$
0	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$	$\mathbb{Z}$
1	$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\mathbb{Z}_2^2$
2	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ -2 & 0 & -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}_2$
3	$\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -2 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ -12 & 0 & 4 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\ -7 & -2 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_8$

i	$M_i(d_i)$	$H_i(G_4)$
4	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -20 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 3 & 1 & 2 & 2 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -12 & -3 & 0 & -4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -24 & 0 & -12 & 0 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & -7 & -3 & -2 & 0 & 0 & 0 & -4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & -6 & -7 & 0 & -2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 6 & 0 & 0 & -2 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & -3 & -4 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 8 & -2 & 3 & -1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ -2 & -1 & -2 & 0 & -1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\mathbb{Z}_2^2$

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References

1. Álvarez, V., Armario, J.A., Frau, M.D., Real, P.: A Mathematica notebook for computing the homology of iterated products of groups. In: Iglesias, A., Takayama N. (eds.) ICMS-2 Proceedings, LNCS, vol. 4151, pp. 47–57. Springer, Berlin (2006)
2. Álvarez, V., Armario, J.A., Frau, M.D., Real, P.: Calculating cocyclic Hadamard matrices in Mathematica: exhaustive and heuristic searches. In: Iglesias, A., Takayama N. (eds.) ICMS-2 Proceedings. LNCS, vol. 4151, pp. 419–422. Springer, Berlin (2006)
3. Álvarez, V., Armario, J.A., Frau, M.D., Real, P.: http://library.wolfram.com/infocenter/MathSource/6516/
4. Álvarez, V., Armario, J.A., Frau, M.D., Real, P.: The homological reduction method for computing cocyclic matrices. J. Symb. Comput. **44**, 558–570 (2009)
5. Álvarez, V., Armario, J.A., Frau, M.D., Real, P.: The cohomological reduction method for computing n-dimensional cocycles Hadamard matrices. Preprint, p. 15 (2012). arXiv:1201.4026vl
6. Brady, T.: Free resolutions for semi-direct products. Tohoku Math. J. **45**, 535–537 (1993)
7. Brown, K.S.: Cohomology of Groups. Graduate Texts in Math, vol. 8. Springer, New York (1982)
8. Brown, R.: The twisted Eilenberg-Zilber theorem. In: Oderisi, G. (eds.) Celebrazioni Achimedee del secolo XX, Simposio di topologia, pp. 33–37. (Messina, 1964) (1965)
9. Cohen, D.C., Suciu, A.I.: Homology of iterated semidirect products of free groups. J. Pure Appl. Algebra **126**, 87–120 (1998)
10. Dousson, X., Rubio, J., Sergeraert, F., Siret, Y.: The Kenzo Program, Institut Fourier, Grenoble, 1999. http://www-fourier.ujf-grenoble.fr/~sergerar/Kenzo

11. de Launey, W., Horadam, K.J.: A weak difference set construction for higher-dimensional designs. Des. Codes Cryptogr. **3**, 75–87 (1993)
12. de Launey, W., Horadam, K.J.: Cocyclic development of designs. J. Algebraic Combin. **2**(3), 267–290 (1993). Erratum: J. Algebraic Combin. **129**(1), (1994)
13. de Launey, W., Horadam, K.J.: Generation of cocyclic Hadamard matrices. In: Proceedings of Computational Algebra and Number Theory (Sydney, 1992), vol. 325, Math. Appl., pp. 279–290. Kluwer, Dordrecht (1995)
14. Eilenberg, S., Mac Lane, S.: On the groups $H(\pi, n)$ I. Ann. Math. **58**, 55–106 (1953)
15. Eilenberg, S., Mac Lane, S.: On the groups $H(\pi, n)$ II. Ann. Math. **66**, 49–139 (1954)
16. Eilenberg, S., Zilber, J.A.: On the products of complexes. Am. J. Math. **75**, 200–204 (1953)
17. Flannery, D.L.: Calculation of cocyclic matrices. J. Pure Appl. Algebra **112**, 181–190 (1996)
18. Grabmeier, J., Lambe, L.A.: Computing resolutions over finite p-groups. In: Betten, A., Kohnert, A., Lave, R., Wassermann, A. (eds.) Proceedings ALCOMA'99, LNCSE, pp. 157–195. Springer, Heidelberg (2001)
19. Gugenheim, V.K.A.M.: On a the chain complex of a fibration. Illinois J. Math. **3**, 398–414 (1972)
20. Gugenheim, V.K.A.M., Lambe, L.A.: Perturbation theory in differential homological algebra I. Illinois J. Math. **33**, 556–582 (1989)
21. Gugenheim, V.K.A.M., Lambe, L.A., Stasheff, J.D.: Perturbation theory in differential homological algebra II. Illinois J. Math. **35**(3), 357–373 (1991)
22. Horadam, K.J.: Hadamard Matrices and Their Applications. Princeton University Press, Princeton (2007)
23. Homological Algebra programming. A package for the GAP computational algebra system. http://www.gap-system.org/Packages/hap.html
24. Huebschmann, J.: Cohomology of nilpotent groups of class 2. J. Algebra **126**, 400–450 (1989)
25. Huebschmann, J.: Cohomology of metacyclic groups. Trans. Am. Math. Soc. **328**(1), 1–72 (1991)
26. Huebschmann, J., Kadeishvili, T.: Small models for chain algebras. Math. Z. **207**, 245–280 (1991)
27. Hurewicz, W.: Beitrge zur Topologie der Deformationen IV. Asphrische Rume. Nerderl. Akad. Wetensch. Proc. **39**, 215–224 (1936)
28. Lambe, L.A.: Resolutions which split off of the bar construction. J. Pure Appl. Algebra **84**, 311–329 (1993)
29. Lambe, L.A., Stasheff, J.D.: Applications of perturbation theory to iterated fibrations. Manuscripta Math. **58**, 367–376 (1987)
30. Mac Lane, S.: Origins of the cohomology groups. Enseign. Math. **24**(2), 1–29 (1978)
31. Mac Lane, S.: Homology. Classics in Mathematics. Springer, Berlin (1995). Reprint of the 1975 edition
32. May, J.P.: Simplicial objects in Algebraic Topology. Chicago Lectures in Mathematics. The University of Chicago Press, Chicago (1992). Reprint of the 1967 edition
33. Real, P.: Homological perturbation theory and associativity. Homology Homotopy Appl. **2**, 51–88 (2000)
34. Romero, A., Ellis, G., Rubio, J.: Interoperating between computer algebra systems: computing homology of groups wiht Kenzo and GAP. In: Proceeding ISSAC 2099, pp. 303–310, ACM, New York (2009)
35. Romero, A., Rubio, J.: Computing the homology of groups: the geometric way, p. 23 (2011) Preprint. arXiv:1107.3396vl
36. Sergeraert, F.: The computability problem in algebraic topology. Adv. Math. **1104**, 1–29 (1994)
37. Shih, W.: Homologie des espaces fibrés. Inst. Hautes Etudes Sci. **13**, 293–312 (1962)
38. The GAP group. GAP—Groups, Algorithms, and Programming. http://www.gap-system.org
39. Veblen, O.: Analysis Situs. Amer. Math. Soc. Coll. Publ. II, vol. 5, New York (1931)