Existence of global weak solution for a 2D viscous bi-layer Shallow Water model

Jean De Dieu Zabsonré*, Gladys Narbona-Reina†

February 19, 2008

Abstract

We consider a non-linear viscous bi-layer shallow water model with capillarity effects and extra friction terms in a two-dimensional space. This system is issued from a derivation of a three-dimensional Navier-Stokes equations with water-depth depending on friction coefficients. We prove an existence result for global weak solution in a periodic domain $\Omega = \mathbb{T}^2$.

Keywords. Shallow water, bi-layer, viscous models, energetic consistency, global weak existence.
AMS classification: 35Q30.

1 Introduction

The Shallow-Water flows cover a very large number of geophysical and engineering applications as ocean circulation, coastal areas, rivers, lakes, avalanches, ... But, in many situations one layer of Shallow-Water cannot be

*LAMA, UMR5127CNRS, Université de Savoie, 73376 Le Bourget du lac 73376 Le Bourget du Lac cedex, France. Université de Ouagadougou, UFR SEA 03 BP 7021 Ouagadougou 03 Burkina Faso. email: wend-woaga.zabsonre@univ-savoie.fr, jzabsonre@univ-ouaga.bf

†Dpto. Matemática Aplicada I, E. T. S. Arquitectura, Universidad de Sevilla, Avda Reina Mercedes N.2, 41012 Sevilla, España. email: gnarbona@us.es

used to model the system. The simplest example is the flow in the Strait of Gibraltar. It is necessary in this case to consider two layers of water. Indeed, the conservation of the volume of water and salinity in the basin indicates the presence of two opposite flows: the surface Atlantic water and the deeper, denser Mediterranean water flowing into the Atlantic. Thus, it is necessary to consider at least two layers model if we want to simulate the flow in this region. We assume that for this phenomena one can make an appropriate Shallow-Water approximation. For this purpose we can find many derivations of bi-layer and multi-layers Shallow-Water models. In [1], AUDUSSE derived a multi-layer Shallow-Water model to extend the case of one layer established by GERBEAU and PERTHAME in [17]. In this work, using the hydrostatic pressure and the kinematic boundary conditions, he derived momentum equations of the form:

$$\partial_t \int_{H_{\alpha-1}}^{H_\alpha} u dz + \partial_x \int_{H_{\alpha-1}}^{H_\alpha} u^2 dz + g h_\alpha \partial_x h = \frac{\nu_0}{\epsilon} \partial_z u(H_\alpha(t,x)) - \frac{\nu_0}{\epsilon} \partial_z u(H_{\alpha-1}(t,x))$$

and use at the leading order a finite difference method with respect to the vertical variable when the equation is an interface equation to deduce the friction term:

$$\mu \partial_z u(H_\alpha) = \mu \frac{U_{\alpha+1} - U_\alpha}{h_{\alpha+1} + h_\alpha}.$$

In [25], PEYBERNES deduce a bi-layer viscous Shallow-Water model which take into account the friction at the interface. But instead of asymptotic analysis several assumptions of simplifications are used in the boundary conditions to deduce the final system. Also, the energy of the system is obtained under restrictive hypothesis on the data.

On the other hand, we propose in this paper a new viscous bi-layer Shallow-Water model with different constant densities. Following the work performed in [17] for one layer in one dimensional case and in [20] for one layer but in the two dimensional case, here the considered model is a simplified system of a general obtained in [16]. In [20], a viscous one layer of two dimensional Shallow-Water system is derived by MARCHE. The originality in this work is the introduction of surface-tension term through the capillary effects at the free surface and quadratic friction term at the bottom. Such surface-tension and quadratic friction terms have been useful to establish the existence of global weak solutions in [2]. Our model also take into account friction term on the bottom and capillary term on the interface and on the free surface. Another work related to the derivation of 2D Shallow-Water model has been done by FERRARI and Saleri in [15]. In particular the au-

thors include the atmospheric pressure in the derivation. For the sake of brevity, we have not included in this work the deduction of our new viscous bi-layers model, see [16] for detail.

We prove the existence of global weak solution for the considered system. The analysis developed here is based on the techniques used by Bresch, Desjardins and Lin in [2] and [6]. In these works, they obtain the existence of global weak solution for a 2D Shallow Water system and a Korteweg system with a diffusion term of type ν div $(hD(u))$. They prove that the considered systems is energetically consistent without any restriction on the data. The key point of this proof is based in a estimate of a new entropy (in mathematical sense), called "mathematical BD entropy", which gives a bound of the term $\nabla\sqrt{h}$. This inequality is extended later to a more general Navier-Stokes equation with an algebraic relation between the shear and the bulk viscosities coefficients. But the authors used quadratic frictions terms and capillary effects to get the stability of the system in [2]. More recently, another proof based also on the "BD entropy" estimate of the stability for the Navier-Stokes equations for barotropic compressible fluids is developed in [21] by Mellet and Vasseur. Notice that this analysis includes the case of Shallow-Water without any regularizing term. Their analysis is based on the estimate of ρu^2 which is enough to get the compactness result. In fact this estimate replace that of $h^{1/3}u$ in [2] obtained by using a drag term of the form $r|h||u|u$. But it is not actually possible to construct a suitable approximate sequences of weak solutions with this method.

In [14] and [25], the authors prove the existence of global weak solution of a bi-layer Shallow-Water model without any friction term but with a diffusion term of the form $\nu\Delta u$. This analysis uses the method developed by Orenga in [24] and the system is energetically consistent only for small enough initial data. Others works concerning the existence of global weak solution of a bi-layer Shallow-Water using the preceding method can also find in [11] and [23].

In this work we consider in a periodic domain Ω, a system composed by two layers of immiscible fluids with different and constant densities (ρ_1 and ρ_2, resp.) and viscosities (ν_1 and ν_2, resp.).

From now on, index 1 refers to the deeper layer and index 2 to the upper layer of the flow. So, h_i, u_i for $i = 1, 2$ denote the thickness and the velocity field of each layer. We define h to be $h = h_1 + h_2$. We assume that the friction coefficient at the bottom c_0 and the coefficients α_1, α_2 representing respectively the interface and free surface tensions coefficients are positive.

The model proposed here reads as:

$$\partial_t h_1 + \text{ div } (h_1 v_1) = 0; \tag{1}$$

$$\rho_1 \partial_t (h_1 v_1) + \rho_1 \text{ div } (h_1 v_1 \otimes v_1) - 2\nu_1 \text{ div } (h_1 D(v_1))$$

$$+\rho_1 g h_1 \nabla h_1 + \rho_2 g h_1 \nabla h_2 - \left(1 + \frac{c_0 \beta(h_1) h_1}{6\nu_1}\right) fric(v_1, v_2) + c_0 \beta(h_1) v_1$$

$$-\alpha_1 h_1 \nabla(\Delta h_1) - \alpha_2 h_1 \nabla(\Delta h_2) = 0; \tag{2}$$

$$\partial_t h_2 + \text{ div } (h_2 v_2) = 0; \tag{3}$$

$$\rho_2 \partial_t (h_2 v_2) + \rho_2 \text{ div } (h_2 v_2 \otimes v_2) - 2\nu_2 \text{ div } (h_2 D(v_2))$$

$$+\rho_2 g h_2 \nabla h_2 + \rho_2 g h_2 \nabla h_1 + fric(v_1, v_2) - \alpha_2 h_2 \nabla(\Delta h) = 0 \tag{4}$$

with initial conditions:

$$h_{i|t=0} = h_{i_0} \geq 0, \quad h_i v_{i|t=0} = m_{i_0}, \tag{5}$$

for which we assume the following regularity:

$$\begin{array}{l} h_{i_0} \in L^2(\Omega), \qquad \nabla h_{i_0} \in (L^2(\Omega))^2, \quad \nabla\sqrt{h_{i_0}} \in (L^2(\Omega))^2 \\ \dfrac{|m_{i_0}|^2}{h_{i_0}} \in L^1(\Omega), \quad \log_-(h_{i_0}) \in L^1(\Omega). \end{array} \tag{6}$$

The function β depending on h_1 is one of the drags coefficients given by:

$$\beta(h_1) = \left(1 + \frac{c_0}{3\nu_1} h_1\right)^{-1}. \tag{7}$$

We denote by $D(v)$ the strain tensor, defined by $D(v) = \frac{\nabla v + \nabla^t v}{2}$, and by $A(v)$, the vorticity tensor such that $A(v) = \frac{\nabla v - \nabla^t v}{2}$.
The friction term between the two layers, denoted by $fric(v_1, v_2)$ is proportional to $v_1 - v_2$ and is taken as follows:

$$fric(v_1, v_2) = -c_1 B(h_1, h_2)(v_1 - v_2), \tag{8}$$

where

$$B(h_1, h_2) = \frac{h_1 h_2}{\frac{\nu_1}{\nu_2} h_1 + \frac{\nu_2}{\nu_1} h_2} \tag{9}$$

is the other drag coefficient (friction term at the interface). c_1 is taken constant and strictly positive. The drag coefficient B is also used in [18]. It seems that it is not possible to control the interface friction term of the form $c_1|v_1 - v_2|(v_1 - v_2)$. But the friction coefficient B makes possible the control of friction term (8). Note that CHUESHOV and all in [12] study a system of 3D Navier-Stokes equations in a two-layer thin domain with an interface condition

$$(\nu_i \partial_3 u_j^i - k(u_j^1 - u_j^2))|_{x_3=0} = 0 \qquad i, j = 1, 2.$$

This condition is the same of type as the condition appearing in the Primitive Equations of the Coupled Atmosphere and Ocean which describes the atmosphere-ocean interaction. They prove the existence of strong solution corresponding to a large set of initial data and forcing terms.

An other important particular case of our system is that when the viscosity coefficient ν_1 and ν_2 tend to zero, $\left(1 + \frac{c_0 \beta(h_1) h_1}{6\nu_1}\right)$ tends to $3/2$ but not to 0 and B tends to 0; so the limit system with respect to ν_1 is not closed to those obtained at the leading order as for one layer case.

We assume the following hypothesis on the data:

$$\rho_1 > \rho_2, \quad \nu_1 < \nu_2, \quad \alpha_1 > \alpha_2, \tag{10}$$

and the "mathematical relationship" between viscosity and tension coefficients given by:

$$\frac{\nu_1}{\nu_2} > \frac{\alpha_2}{\alpha_1} \frac{\rho_1}{\rho_2}. \tag{11}$$

This paper is organized as follows: In Section 2 we define the notion of a weak solution and we give our main existence result. Section 3 is devoted to the classical physical energy and the mathematical BD entropy. We prove the existence theorem in Section 4 and finally, in Section 5, we give the proof of the classic energy and BD entropy inequalities stated in Section 3.

2 Existence of weak solution

In this section we state the results of existence of weak solution for the system (1)-(4). Previously we introduce in what sense this weak solution is defined.

We introduce the initial energy associated to the system (1)-(4):

$$\begin{aligned}\mathcal{E}_0 = &\frac{1}{2}\rho_1\int_\Omega h_{1_0}|v_{1_0}|^2 + \frac{1}{2}\rho_2\int_\Omega h_{2_0}|v_{2_0}|^2 + \frac{1}{2}g(\rho_1-\rho_2)\int_\Omega |h_{1_0}|^2 \\ &+\frac{1}{2}\rho_2 g\int_\Omega |h_{1_0}+h_{2_0}|^2 + \frac{1}{2}(\alpha_1-\alpha_2)\int_\Omega |\nabla h_{1_0}|^2 \\ &+\frac{1}{2}\alpha_2\int_\Omega |\nabla(h_{1_0}+h_{2_0})|^2;\end{aligned} \tag{12}$$

and the expression:

$$\mathcal{F}_0 = \frac{1}{2}\int_\Omega |\nabla\sqrt{h_{1_0}}|^2 + \frac{1}{2}\int_\Omega |\nabla\sqrt{h_{2_0}}|^2. \tag{13}$$

And we assume both of them are bounded.

Definition 2.1 *We shall say that (h_1, h_2, v_1, v_2) is a weak solution of* (1)-(4) *if* (1) *and* (3) *hold in $(\mathcal{D}'(0,T)\times\Omega)^2$; $h_{1_{|t=0}} = h_1^0 \geq 0$ and $h_{2_{|t=0}} = h_2^0$ in $\mathcal{D}'(\Omega)$; the following assumptions are satisfied:*

$$\begin{array}{l} h_i \in L^\infty(0,T;L^2(\Omega)); \\ \nabla h_i \text{ and } \sqrt{h_i}v_i \in L^\infty(0,T;(L^2(\Omega))^2); \\ v_1 \in L^2(0,T;(L^2(\Omega))^2); \\ \sqrt{h_i}D(v_i) \in L^2(0,T;(L^2(\Omega))^4); \\ \sqrt{B(h_1,h_2)}(v_1-v_2) \in L^2(0,T;(L^2(\Omega))^2); \\ \nabla\sqrt{h_{i_n}} \in L^2(0,T;L^2(\Omega)^2); \\ \Delta h_{i_n} \in L^2(0,T;L^2(\Omega)). \end{array} \tag{14}$$

And for any $\varphi \in \mathcal{C}^\infty((0,T)\times\Omega)^2$ with $\varphi(T,\cdot) = 0$, (φ with compact support), we have:

$$-\rho_1 h_1^0 v_1^0 h_1^0\varphi(0,\cdot) - \int_0^T\int_\Omega \rho_1 h_1^2 v_1\partial_t\varphi + \rho_1\int_0^T\int_\Omega h_1^2(v_1\cdot\varphi)\ div\ v_1$$

$$-\rho_1\int_0^T\int_\Omega (h_1v_1\otimes h_1v_1) : D(\varphi) + 2\nu_1\int_0^T\int_\Omega h_1^2(D(v_1):D(\varphi))$$

$$
\begin{aligned}
&+2\nu_1 \int_0^T \int_\Omega h_1(D(v_1) : (\nabla h_1 \otimes \varphi)) + c_0 \int_0^T \int_\Omega \beta(h_1) v_1 h_1 \varphi \\
&-c_1 \int_0^T \int_\Omega \left(1 + \frac{c_0 \beta(h_1) h_1}{6\nu_1}\right) B(h_1, h_2)(v_2 - v_1) h_1 \varphi \\
&+\frac{1}{2}\rho_1 g \int_0^T \int_\Omega h_1^3 \ div\ \varphi + \frac{1}{2}\rho_1 g \int_0^T \int_\Omega h_1^2 (\varphi \cdot \nabla h_1) \\
&+\rho_2 g \int_0^T \int_\Omega h_1^2 (\varphi \cdot \nabla h_2) + \alpha_1 \int_0^T \int_\Omega h_1^2 \Delta h_1 \ div\ \varphi \\
&+2\alpha_1 \int_0^T \int_\Omega h_1 \Delta h_1 (\varphi \cdot \nabla h_1) + \alpha_2 \int_0^T \int_\Omega h_1^2 \Delta h_2 \ div\ \varphi \\
&+\alpha_2 \int_0^T \int_\Omega h_1 (\varphi \cdot \nabla h_1) \Delta h_2 = 0
\end{aligned} \tag{15}
$$

and

$$
\begin{aligned}
&-\rho_2 h_2^0 v_2^0 h_2^0 \varphi(0,\cdot) - \int_0^T \int_\Omega \rho_2 h_2^2 v_2 \partial_t \varphi + \rho_2 \int_0^T \int_\Omega h_2^2 (v_2 \cdot \varphi) \ div\ v_2 \\
&-\rho_2 \int_0^T \int_\Omega (h_2 v_2 \otimes h_2 v_2) : D(\varphi) + 2\nu_2 \int_0^T \int_\Omega h_2^2 (D(v_2) : D(\varphi)) \\
&+2\nu_2 \int_0^T \int_\Omega h_2 (D(v_2) : (\nabla h_2 \otimes \varphi)) \\
&+c_1 \int_0^T \int_\Omega B(h_1, h_2)(v_2 - v_1) h_2 \varphi \\
&+\frac{1}{2}\rho_2 g \int_0^T \int_\Omega h_2^3 \ div\ \varphi + \frac{1}{2}\rho_2 g \int_0^T \int_\Omega h_2^2 (\varphi \cdot \nabla h_2) \\
&+\rho_2 g \int_0^T \int_\Omega h_2^2 (\varphi \cdot \nabla h_1) + \alpha_2 \int_0^T \int_\Omega h_2^2 \Delta (h_1 + h_2) \ div\ \varphi \\
&+2\alpha_2 \int_0^T \int_\Omega h_2 \Delta (h_1 + h_2)(\varphi \cdot \nabla h_2) = 0.
\end{aligned} \tag{16}
$$

Remark 2.2 *This definition of weak solution with test functions depending on the solutions itself was first introduced in [13] by* Desjardins *and*

ESTEBAN *when dealing with motion of rigid or elastic bodies evolving in viscous compressible or incompressible fluids. In [6], the authors use the same definition. It will allow us to get the compacity when limit height vanishes.*

We will prove the following theorem:

Theorem 2.3 *There exists a global weak solution* (h_1, h_2, v_1, v_2) *of* (1)-(4) *satisfying entropy inequalities* (17) *and* (19).

3 Energy inequalities

We give in this section, the classical energy estimate and the mathematical BD entropy. These two inequalities will allow us to prove the main theorem.

Lemma 3.1 *Let* (h_1, h_2, v_1, v_2) *be a solution of the system* (1)-(4). *Then, the following inequality holds:*

$$\begin{aligned}
&\frac{1}{2}\rho_1\frac{d}{dt}\int_\Omega h_1|v_1|^2 + \frac{1}{2}\rho_2\frac{d}{dt}\int_\Omega h_2|v_2|^2 + 2\nu_1\int_\Omega h_1(D(v_1):D(v_1)) \\
&\quad +2\nu_2\int_\Omega h_2(D(v_2):D(v_2)) + \frac{1}{2}g(\rho_1-\rho_2)\frac{d}{dt}\int_\Omega |h_1|^2 + c_0\int_\Omega |v_1|^2 \\
&\quad +\frac{1}{2}\rho_2 g\frac{d}{dt}\int_\Omega |h_1+h_2|^2 + \frac{1}{2}(\alpha_1-\alpha_2)\frac{d}{dt}\int_\Omega |\nabla h_1|^2 \qquad (17)\\
&\quad +\frac{\alpha_2}{2}\frac{d}{dt}\int_\Omega |\nabla(h_1+h_2)|^2 + \frac{1}{2}c_1\int_\Omega B(h_1,h_2)|v_1-v_2|^2 \\
&\qquad\qquad \leq \left(\frac{c_0^2}{3\nu_1} + c_1\frac{\nu_1}{\nu_2}\right)\int_\Omega h_1|v_1|^2.
\end{aligned}$$

Remark 3.2 *1. Notice that the two terms in the right can be controled using Gronwall's lemma.*

2. From this energy estimate (17), *we deduce the following bounds:*

$$\sqrt{h_1}\,v_1 \in L^\infty(0,T;(L^2(\Omega))^2); \quad \sqrt{h_2}\,v_2 \in L^\infty(0,T;(L^2(\Omega))^2);$$

$$h_1 \in L^\infty(0,T;L^2(\Omega)); \qquad \sqrt{h_1}\,D(v_1) \in L^2(0,T;(L^2(\Omega))^4);$$

$$
\begin{aligned}
&h_2 \in L^\infty(0,T;L^2(\Omega)); \qquad \sqrt{h_2}\, D(v_2) \in L^2(0,T;(L^2(\Omega))^4);\\
&\nabla h_1 \in L^\infty(0,T;(L^2(\Omega))^2); \qquad \nabla h_2 \in L^\infty(0,T;(L^2(\Omega))^2);\\
&v_1 \in L^2(0,T;(L^2(\Omega))^2);\\
&\sqrt{B(h_1,h_2)}(v_1-v_2) \in L^2(0,T;(L^2(\Omega))^2).
\end{aligned}
\tag{18}
$$

But it is well-known that these estimates are not enough to pass to the limit and get the stability of the system.

So we are going to obtain further estimates from the BD entropy that we state in the following lemma, (see [6]).

Lemma 3.3 *If we assume that (h_1, h_2, v_1, v_2) is a smooth solution of system (1)-(4), then*

$$
\begin{aligned}
&\frac{1}{2}\rho_2\frac{d}{dt}\int_\Omega h_1|\rho_1 v_1 + 2\nu_1\nabla\log h_1|^2 + \frac{1}{2}\rho_1\frac{d}{dt}\int_\Omega h_2|\rho_2 v_2 + 2\nu_2\nabla\log h_2|^2\\
&\quad+\rho_1\rho_2\left(\frac{1}{2}g(\rho_1-\rho_2)\frac{d}{dt}\int_\Omega|h_1|^2 + \frac{1}{2}\rho_2 g\frac{d}{dt}\int_\Omega|h_1+h_2|^2 + c_0\int_\Omega|v_1|^2\right)\\
&\quad+2\nu_2\rho_1\rho_2\int_\Omega h_2(A(v_2):A(v_2)) + 2\nu_1\rho_1\rho_2\int_\Omega h_1(A(v_1):A(v_1))\\
&\quad+\frac{1}{2}c_1\rho_1\rho_2\int_\Omega B(h_1,h_2)|v_1-v_2|^2 + \frac{1}{2}\rho_1\rho_2(\alpha_1-\alpha_2)\frac{d}{dt}\int_\Omega|\nabla h_1|^2\\
&\quad+\frac{1}{2}\alpha_2\rho_1\rho_2\frac{d}{dt}\int_\Omega|\nabla(h_1+h_2)|^2 + 2\nu_1\rho_1\rho_2 g\int_\Omega|\nabla h_1|^2\\
&\quad+2\nu_2\rho_1\rho_2 g\int_\Omega|\nabla h_2|^2 + 2\nu_1\alpha_1\rho_2\int_\Omega|\Delta h_1|^2 + 2\nu_2\alpha_2\rho_1\int_\Omega|\Delta h_2|^2\\
&\quad-2\nu_1 c_0\rho_2\frac{d}{dt}\int_\Omega\log\left(\frac{h_1}{3\nu_1+c_0 h_1}\right)\\
&\quad+2\nu_1 c_0\rho_2\int_\Omega\beta'(h_1)v_1\nabla h_1 + 2\rho_2 g(\rho_2\nu_1+\rho_1\nu_2)\int_\Omega\nabla h_1\nabla h_2
\end{aligned}
\tag{19}
$$

$$+2\alpha_2(\rho_2\nu_1+\rho_1\nu_2)\int_\Omega \Delta h_1\Delta h_2$$

$$-2\nu_2 c_1\rho_1\int_\Omega \frac{B(h_1,h_2)}{h_2}((v_1-v_2)\cdot\nabla h_2)$$
$$+2\nu_1 c_1\rho_2\int_\Omega \left(1+\frac{c_0\beta(h_1)h_1}{6\nu_1}\right)\frac{B(h_1,h_2)}{h_1}((v_1-v_2)\cdot\nabla h_1)$$

$$\leq \rho_1\rho_2\left(\frac{c_0^2}{3\nu_1}\int_\Omega h_1|v_1|^2+c_1\frac{\nu_1}{\nu_2}\int_\Omega h_1|v_1|^2\right).$$

Remark 3.4 *We would like to point out the boundness of the 'non usual' terms appearing above.*

1. *The term including* $\log\left(\frac{h_1}{3+c_0\nu_1^{-1}h_1}\right)$ *is bounded. In fact, we write it as:*

$$\log\left(\frac{h_1}{3+c_0\nu_1^{-1}h_1}\right)=\log h_1-\log(3+c_0\nu_1^{-1}h_1).$$

Since $3+c_0\nu_1^{-1}h_1>1$ *the second term is bounded. If we denote* $\log_+ h_1=\log(\max\{h_1,1\})$ *and* $\log_- h_1=\log(\min\{h_1,1\})$*, and using the regularity assumed for initial conditions, it is sufficient to control* $\log_+ h_1$*. But* $0\leq\log_+ h_1\leq h_1$*, so we can bound this term because* $h_1\in L^\infty(0,T;L^2(\Omega))$*.*

2. *In the energy equality* (19)*, it remains to control the four last terms on l.h.s.*

   - *Pressure:*
     We use all pressure terms to write them together as follows: We only take the sum

$$\nu_2\rho_1 g\int_\Omega|\nabla h_2|^2+2(\nu_1\rho_2+\nu_2\rho_1)g\int_\Omega\nabla h_1\nabla h_2$$

since the remainder being positive. We have:

$$\nu_2\rho_1 g\int_\Omega|\nabla h_2|^2+2(\nu_1\rho_2+\nu_2\rho_1)g\int_\Omega\nabla h_1\nabla h_2$$

$$\leq(\nu_1\rho_2+\nu_2\rho_1)g\int_\Omega|\nabla h_2|^2$$

$$+2(\nu_1\rho_2+\nu_2\rho_1)g\int_\Omega \nabla h_1\nabla h_2+(\nu_1\rho_2+\nu_2\rho_1)g\int_\Omega|\nabla h_1|^2 \quad (20)$$

$$=(\nu_1\rho_2+\nu_2\rho_1)g\int_\Omega|\nabla(h_1+h_2)|^2$$

and these two terms on the right member can be controled by Gronwall's lemma.

- *Tension:*

$$\begin{aligned}&2\nu_1\alpha_1\rho_2\int_\Omega|\Delta h_1|^2+2\nu_2\alpha_2\rho_1\int_\Omega|\Delta h_2|^2+2\alpha_2(\rho_2\nu_1+\rho_1\nu_2)\int_\Omega\Delta h_1\Delta h_2\\ &\quad=\alpha_2(\nu_2\rho_1+\nu_1\rho_2)\int_\Omega|\Delta(h_1+h_2)|^2+\alpha_2(\nu_2\rho_1-\nu_1\rho_2)\int_\Omega|\Delta h_2|^2\\ &\qquad+(\alpha_1\nu_1\rho_2-\alpha_2\nu_2\rho_1)\int_\Omega|\Delta h_1|^2+(\alpha_1\nu_1\rho_2-\alpha_2\nu_1\rho_2)\int_\Omega|\Delta h_1|^2.\end{aligned}$$

Thanks to hypothesis (10) and (11), each term appearing in the right is positive.

- *Friction terms: First we have*

$$\begin{aligned}I_1&=2\nu_2c_1\rho_1\int_\Omega\frac{B(h_1,h_2)}{h_2}(v_2-v_1)\nabla h_2\\ &=4\nu_2c_1\rho_1\int_\Omega\sqrt{h_2}\sqrt{\frac{B(h_1,h_2)}{h_2}}(v_2-v_1)\sqrt{\frac{B(h_1,h_2)}{h_2}}\nabla\sqrt{h_2}.\end{aligned}$$

Then, Young's inequality allows us to conclude that

$$I_1\le 2\nu_2c_1\rho_1\int_\Omega B(h_1,h_2)|v_1-v_2|^2+2c_1\rho_1\frac{\nu_2^2}{\nu_1}\int_\Omega|\nabla\sqrt{h_2}|^2.$$

Next in the same way, one can write

$$\begin{aligned}I_2&=2\nu_1c_1\rho_2\int_\Omega\left(1+\frac{c_0\beta(h_1)h_1}{6\nu_1}\right)\frac{B(h_1,h_2)}{h_1}((v_1-v_2)\cdot\nabla h_1)\\ &\le 3\nu_1c_1\rho_2\int_\Omega\frac{B(h_1,h_2)}{h_1}|(v_1-v_2)\cdot\nabla h_1|\\ &\le 3\nu_1c_1\rho_2\int_\Omega\sqrt{h_1}\sqrt{\frac{B(h_1,h_2)}{h_1}}|v_1-v_2|\sqrt{\frac{B(h_1,h_2)}{h_1}}|\nabla h_1|\end{aligned}$$

$$
\begin{aligned}
&\leq 6\nu_1 c_1\rho_2\int_\Omega \sqrt{h_1}\sqrt{\frac{B(h_1,h_2)}{h_1}}|v_1-v_2|\sqrt{\frac{B(h_1,h_2)}{h_1}}\nabla\sqrt{h_1}\\
&\leq 3\nu_1 c_1\rho_2\int_\Omega B(h_1,h_2)|v_1-v_2|^2+3\nu_1 c_1\rho_2\int_\Omega \frac{B(h_1,h_2)}{h_1}|\nabla\sqrt{h_1}|^2\\
&\leq 3\nu_1 c_1\rho_2\int_\Omega B(h_1,h_2)|v_1-v_2|^2+3c_1\rho_2\frac{\nu_1^2}{\nu_2}\int_\Omega|\nabla\sqrt{h_1}|^2.
\end{aligned}
$$

It is easy to note that both terms

$$
3c_1\rho_2\frac{\nu_1^2}{\nu_2}\int_\Omega|\nabla\sqrt{h_1}|^2 \text{ and } c_1\rho_1\frac{\nu_2^2}{\nu_1}\int_\Omega|\nabla\sqrt{h_2}|^2
$$

can be absorbed by Gronwall's lemma.

3. *These results and the BD entropy allow us to find the estimates:*

$$
\begin{aligned}
&\nabla\sqrt{h_1}\in L^2(0,T;(L^2(\Omega))^2);\quad \nabla\sqrt{h_2}\in L^2(0,T;(L^2(\Omega))^2);\\
&\Delta h_1\in L^2(0,T;L^2(\Omega));\quad \Delta h_2\in L^2(0,T;L^2(\Omega));\\
&\nabla h_1\in L^\infty(0,T;(L^2(\Omega))^2);\quad \nabla h_2\in L^\infty(0,T;(L^2(\Omega))^2).
\end{aligned}
\tag{21}
$$

4 Proof of the Theorem 2.3

To perform the proof, first we justify the existence of an approximate solution satisfying the energy inequalities of Section 3. Secondly we pass to the limit in the fluid transport equation and finally in the momentum equation.

We assume that a sequence of approximate solution $(h_{1_n}, h_{2_n}, v_{1_n}, v_{2_n})$ has been constructed and has suitable regularity to justify the formal energy estimates. In the case of one-layer, in [4] and in [5], such approximate solution is constructed. The method used by the authors can be applied in our case to get an approximate solution. We need only to prove the stability of the system.

Thus, using the classical energy estimate and the mathematical BD entropy, we obtain the following uniform bounds:

$$
\|h_{1_n}\|_{L^\infty(0,T;L^2(\Omega))}\leq C;\quad \|h_{2_n}\|_{L^\infty(0,T;L^2(\Omega))}\leq C;
$$

$$
\|\sqrt{h_{1_n}}\,v_{1_n}\|_{L^\infty(0,T;(L^2(\Omega))^2)}\leq C;
$$

$$
\begin{aligned}
&\|\sqrt{h_{2_n}}\, v_{2_n}\|_{L^\infty(0,T;(L^2(\Omega))^2)} \le C;\\
&\|\sqrt{h_{1_n}} D(v_{1_n})\|_{L^2(0,T;(L^2(\Omega))^4)} \le C;\\
&\|\sqrt{h_{2_n}} D(v_{2_n})\|_{L^2(0,T;(L^2(\Omega))^4)} \le C;\\
&\|\nabla h_{1_n}\|_{L^\infty(0,T;(L^2(\Omega))^2)} \le C;\\
&\|\nabla h_{2_n}\|_{L^\infty(0,T;(L^2(\Omega))^2)} \le C;\\
&\|v_{1_n}\|_{L^2(0,T;(L^2(\Omega))^2)} \le C;\\
&\|\sqrt{B(h_{1_n}, h_{2_n})}(v_{1_n} - v_{2_n})\|_{L^2(0,T;(L^2(\Omega))^2)} \le C
\end{aligned}
\tag{22}
$$

and

$$
\begin{aligned}
&\|\nabla\sqrt{h_{1_n}}\|_{L^2(0,T;(L^2(\Omega))^2)} \le C;\\
&\|\nabla\sqrt{h_{2_n}}\|_{L^2(0,T;(L^2(\Omega))^2)} \le C;\\
&\|\Delta h_{1_n}\|_{L^2(0,T;L^2(\Omega))} \le C;\\
&\|\Delta h_{2_n}\|_{L^2(0,T;L^2(\Omega))} \le C.
\end{aligned}
\tag{23}
$$

Convergence in the fluid transport equations.

For $i = 1, 2$, we have h_{i_n} bounded in $L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$. Moreover $\partial_t h_{i_n} = -\operatorname{div}(h_{i_n} v_{i_n})$ is bounded in $L^2(0,T;H^{-1}(\Omega))$.
Thanks to the Sobolev's imbedding (cf. [8]), we have $\forall s \in (0,1)$

$$
\begin{array}{rcl}
H^1(\Omega) & \hookrightarrow & H^s(\Omega)\\
H^s(\Omega) & \subset\subset & H^{-1}(\Omega)
\end{array}
$$

and

$$
\begin{array}{rcl}
H^2(\Omega) & \hookrightarrow & H^{1+s}(\Omega)\\
H^{1+s}(\Omega) & \subset\subset & H^{-1}(\Omega).
\end{array}
$$

Then, due to [26], up to the extraction of a sequence, there exists $h_i \in L^2(0,T;H^2(\Omega))$, $(i=1,2)$ such that ∇h_i and $\nabla\sqrt{h_i}$ belongs to $L^\infty(0,T;L^2(\Omega)^2)$ and

$$h_{i_n} \longrightarrow h_i \quad \text{in} \quad L^p(0,T;H^{1+1/p}(\Omega)) \cap \mathcal{C}([0,T];H^{1/p}(\Omega)), \quad \forall p \in (2,+\infty) \text{ and } p \in (2,\infty). \tag{24}$$

Next, since $\sqrt{h_{i_n}}\, v_{i_n}$ is bounded in $L^\infty(0,T;(L^2(\Omega))^2)$, we deduce that it converges weakly in $L^2(0,T;(L^2(\Omega))^2)$ up to a subsequence to some limit $z_i \in L^2(0,T;(L^2(\Omega))^2)$.

Let us define v_i to be:

$$v_i = \begin{cases} \dfrac{z_i}{\sqrt{h_i}} & \text{if } h_i > 0; \\ 0 & \text{if } h_i = 0. \end{cases} \tag{25}$$

To prove the convergence of $h_{i_n} v_{i_n}$, we write it as $h_{i_n} v_{i_n} = \sqrt{h_{i_n}}\, v_{i_n} \sqrt{h_{i_n}}$. Notice that

$$\sqrt{h_{i_n}}\, v_{i_n} \rightharpoonup z_i \quad \text{in } L^2(0,T;(L^2(\Omega))^2), \tag{26}$$

and so, it suffices to prove the strong convergence for $\sqrt{h_{i_n}}$ in $L^2(0,T;L^2(\Omega))$. This proof is given as follows:

Thanks to (22) and (23),

$$\begin{aligned} \|\nabla\sqrt{h_{i_n}}\|_{L^2(0,T;(L^2(\Omega))^2)} &\leq C \\ \text{and} \quad \|\sqrt{h_{i_n}}\|_{L^\infty(0,T;(L^4(\Omega))^2)} &\leq C, \end{aligned} \tag{27}$$

so, we can write that $\sqrt{h_{i_n}}$ is bounded in $L^\infty(0,T;H^1(\Omega))$ and $\partial_t\sqrt{h_{i_n}} = \frac{1}{2}\sqrt{h_{i_n}}$ div $v_{i_n} - \frac{1}{2}$ div $(\sqrt{h_{i_n}}\, v_{i_n})$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. Consequently, up to a subsequence, $\sqrt{h_{i_n}}$ converges strongly to some $\sqrt{h_i}$ in $L^2(0,T;L^2(\Omega))$. From (26) we deduce that

$$h_{i_n}\, v_{i_n} \rightharpoonup \sqrt{h_i} z_i = h_i\, v_i \quad \text{in } L^2(0,T;L^2(\Omega)^2). \tag{28}$$

We then prove the convergence in the mass equations that means we have

$$\partial_t h_i + \text{ div } (h_i\, v_i) = 0 \text{ in } (\mathcal{D}'(0,T)\times\Omega) \tag{29}$$

$$\text{and} \qquad h_{i_{|t=0}} = h_{i_0} \text{ in } (\mathcal{D}'(\Omega)). \tag{30}$$

Convergence in the momentum equation

We prove it in two steps:
Step 1: Compactness of $h_{i_n} v_{i_n}$.
We first give important two lemmas that will be useful in this part.

Lemma 4.1 *$(h_{i_n}^\gamma v_{i_n})$ converges strongly to $(h_i^\gamma v_i)$ in $L^2(0,T;L^2(\Omega)^3)$ up to a subsequence for all $\gamma > 1/2$*

Lemma 4.2 *Let $f \in L^\infty(0,T;L^2(\Omega))$ such that $f \geq 0$ a.e. on $(0,T)\times\Omega$, $\nabla f \in L^\infty(0,T;L^2(\Omega)^3)$ and $\Delta f \in L^2(0,T;L^2(\Omega))$. Let us also consider a vector field h such that $\sqrt{f}h$ and $\sqrt{f}\nabla h$ respectively belong to $L^\infty(0,T;L^2(\Omega)^3)$ and $L^2((0,T)\times\Omega)^{3\times 3}$.*
Then $\nabla(f^{3/2}h) \in L^2((0,T);L^{3/2}(\Omega)^3)$ and there exists $C>0$ independent of f and h such that

$$\begin{array}{l}\|\nabla(f^{3/2}h)\|_{L^2(0,T;(L^{3/2}(\Omega))^3)} \leq \|\nabla f\|_{L^2(0,T;(L^6(\Omega))^3)}\|\sqrt{f}h)\|_{L^\infty(0,T;(L^2(\Omega))^3)} \\ \qquad C\|\sqrt{f}\nabla h)\|_{L^2(((0,T)\times(\Omega))^3)}(\|f\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla f\|_{L^\infty(0,T;L^2(\Omega))}).\end{array} \tag{31}$$

For the proofs of these lemmas we refer the reader to [6] and [5]. We only make here the remark that the proof of the first lemma uses the second one.

- First, we remark that h_{i_n} and v_{i_n} verify the conditions of this lemma. So we deduce that $\nabla(h_{i_n}^{3/2}v_{i_n})$ is uniformly bounded in $L^2(0,T;(L^{3/2}(\Omega))^4)$. Moreover, since we work in dimension 2, Sobolev's embedding implies that $h_{i_n}^{3/2}v_{i_n}$ is uniformly bounded in $L^2(0,T;(L^6(\Omega))^2)$.

- Secondly, we estimate $\partial_t(h_{i_n}v_{i_n})$. More precisely, we will prove that $\partial_t(h_{i_n}v_{i_n})$ is uniformly bounded in $L^2(0,T;H^{-s}(\Omega)^2)$. We only prove the estimate $\partial_t(h_{2_n}v_{2_n})$; it can be adapted to estimate $\partial_t(h_{1_n}v_{1_n})$.

  Using (4), we deduce the value of $\rho_2\partial_t(h_{2_n}v_{2_n})$. So we have to find bounds for every term which compose it. We have:

  - $h_{2_n}v_{2_n}\otimes v_{2_n} = h_{2_n}^{1/2}v_{2_n}\otimes h_{2_n}^{1/2}v_{2_n} \in L^\infty(0,T;(L^1(\Omega))^4)$.
  - $h_{2_n}D(v_{2_n}) = h_{2_n}^{1/2}h_{2_n}^{1/2}D(v_{2_n}) \in L^2(0,T;(L^{2p/(2+p)}(\Omega))^2)$.
  - $h_{2_n}\nabla h_{2_n}$ and $h_{2_n}\nabla h_{1_n}$ are bounded in $L^\infty(0,T;(L^{p/2}(\Omega))^2)$.
  - Since Δh is bounded in $L^2(0,T;(L^2(\Omega))^2)$, $\nabla\Delta h$ is in $L^2(0,T;H^{-1}(\Omega))^2$. Combining this result with $h_{2_n} \in L^\infty(0,T;(H_0^2(\Omega))^2)$ we deduce that $h_{2_n}\nabla\Delta h$ is in $L^2(0,T;(W^{-1,1}(\Omega))^2)$.

– $\sqrt{B}$ is in $L^\infty(0,T;L^\infty((\Omega)))$ and $\sqrt{B}(v_{2_n}-v_{1_n})$ is in $L^2(0,T;(L^2((\Omega)))^2)$. Then $fric(v_{1_n}-v_{2_n}) \in L^2(0,T;(L^2(\Omega))^2)$.

We then conclude that $\partial_t(h_{2_n}v_{2_n})$ is uniformly bounded in $L^2(0,T;H^{-s}(\Omega)^2)$ with s large enough.
Conclusion: We get the strong convergence of $h_{2_n}v_{2_n}$ to h_2v_2 in $L^2(0,T;(L^q(\Omega))^2;\ q>1$.

In the same way, using (2), we obtain that $h_{1_n}v_{1_n}$ converges strongly to h_1v_1 in $L^2(0,T;(L^q(\Omega))^2);\ q>1$.

Step 2: Passage to the limit.

Remark 4.3 *We can now pass to the limit in the convection terms* $-\rho_i\int_0^T\int_\Omega(h_{i_n}v_{i_n}\otimes h_{i_n}v_{i_n}) : D(\varphi)$ *and in* $\rho_i\int_0^T\int_\Omega h_{i_n}^2v_{i_n}\partial_t\varphi$ *since we have the strong convergences of* $h_{i_n}v_{i_n}$ *and* h_{i_n}. *Also we can pass to the limit in* $-\rho_i(h_{i_n}^0)^2v_{i_n}^0\varphi(0,\cdot)$. *It remains to see the other terms which appear in (15) and (16):*

- Using the strong convergence of h_{i_n} to h_i in $\mathcal{C}([0,T];H^s(\Omega))$ for all $s\in(0,1)$, we deduce the strong convergence of $h_{i_n}^2$ to h_i^2 in $L^2(0,T;L^2(\Omega))$.
- The pressure terms. $h_{i_n}^3$ is bounded in $L^\infty(0,T;L^{p/3}(\Omega))$. Moreover $h_{i_n}^2\nabla h_{i_n}$ and $h_{i_n}^2\nabla h_{j_n}$ $(i\neq j)$ are bounded in $L^\infty(0,T;(L^{p/3}(\Omega))^2)$. Hence, they weakly converge to $h_i^2\nabla h_i$ and to $h_i^2\nabla h_j$ and so we can pass to the limit in the six pressure terms.
- The friction terms.
  For the first one, we must prove the convergence of $\beta(h_{1_n})v_{1_n}h_{1_n}$.
  We have:

$$\begin{aligned}\beta(h_{1_n})-\beta(h_1) &= \frac{3\nu_1}{3\nu_1+c_0h_{1_n}}-\frac{3\nu_1}{3\nu_1+c_0h_1}\\ &= \frac{3\nu_1c_0(h_1-h_{1_n})}{(3\nu_1+c_0h_{1_n})(3\nu_1+c_0h_1)}. \qquad (32)\end{aligned}$$

  So, (32) gives

$$|\beta(h_{1_n})-\beta(h_1)|\leq\frac{c_0}{3\nu_1}|h_1-h_{1_n}| \qquad (33)$$

and this leads to

$$\beta(h_{1_n}) \longrightarrow \beta(h_1) \text{ in } L^p(0,T;L^p(\Omega)) \,\forall p \in (2,+\infty). \tag{34}$$

Since $h_{1_n}v_{1_n}$ converges strongly to h_1v_1 in $L^2(0,T;L^2(\Omega))$. This gives weak convergence of $\beta(h_{1_n})v_{1_n}h_{1_n}$ to $\beta(h_1)v_1h_1$ in $L^{2p/2+p}(0,T;L^{2p/2+p}(\Omega))$.

Next, we prove the convergence of $B(h_{1_n},h_{2_n})$. By a simple calculation,

$$\begin{aligned} B(h_{1_n},h_{2_n}) - B(h_1,h_2) &= \frac{h_{1_n}h_{2_n}}{\frac{\nu_1}{\nu_2}h_{1_n}+\frac{\nu_2}{\nu_1}h_{2_n}} - \frac{h_1h_2}{\frac{\nu_1}{\nu_2}h_1+\frac{\nu_2}{\nu_1}h_2} \\ &= \frac{\frac{\nu_1}{\nu_2}h_{1_n}h_1(h_{2_n}-h_2)+\frac{\nu_2}{\nu_1}h_{2_n}h_2(h_{1_n}-h_1)}{(\frac{\nu_1}{\nu_2}h_{1_n}+\frac{\nu_2}{\nu_1}h_{2_n})(\frac{\nu_1}{\nu_2}h_1+\frac{\nu_2}{\nu_1}h_2)}. \end{aligned}$$

So we obtain immediately that,

$$|B(h_{1_n},h_{2_n}) - B(h_1,h_2)| \leq \frac{\nu_2}{\nu_1}|h_{2_n}-h_2| + \frac{\nu_2}{\nu_1}|h_{1_n}-h_1| \tag{35}$$

which gives the strong convergence of: $B(h_{1_n},h_{2_n})$ to $B(h_1,h_2)$ in $L^p(0,T;L^p(\Omega))$.

The convergence of the friction term in layer 2 is achieved due to the strong convergence of $h_{2_n}v_{2_n}$ and h_{2_n} and the weak convergence of v_{1_n}. In fact,

$$B(h_{1_n},h_{2_n})(v_{1_n}-v_{2_n})h_{2_n} = -B(h_{1_n},h_{2_n})h_{2_n}v_{2_n} + B(h_{1_n},h_{2_n})v_{1_n}h_{2_n}.$$

Then $B(h_{1_n},h_{2_n})(v_{1_n}-v_{2_n})h_{2_n}$ converges weakly to $B(h_1,h_2)(v_1-v_2)h_2$ in $L^{2p/2+p}(0,T;(L^{2p/2+p}(\Omega))^2)$. It remains to establish the convergence of the second friction terms for the first layer. For the coefficient, $1+\frac{c_o\beta(h_{1_n})h_{1_n}}{6\nu_1}$ converges strongly to $1+\frac{c_o\beta(h_1)h_1}{6\nu_1}$ in $L^p(0,T;L^p(\Omega))$ and therefore, thanks to the strong convergences of $B(h_{1_n},h_{2_n})$ and $h_{1_n}v_{1_n}$,
$(1+\frac{c_o\beta(h_{1_n})h_{1_n}}{6\nu_1})B(h_{1_n},h_{2_n})h_{1_n}v_{1_n}$ converges weakly to
$(1+\frac{c_o\beta(h_1)h_1}{6\nu_1})B(h_1,h_2)h_1v_1$ in $L^{2p/p+3}(0,T;L^{2p/p+3}(\Omega))$.

Next, for the term including v_{2_n}, multiplying and dividing by $\sqrt{h_{2_n}}$, we write

$$B(h_{1_n}, h_{2_n})v_{2_n}h_{1_n} = \frac{B(h_{1_n}, h_{2_n})}{\sqrt{h_{2_n}}}\sqrt{h_{2_n}}v_{2_n}h_{1_n}. \tag{36}$$

Thus, it suffices to prove the strong convergence of $\frac{B(h_{1_n},h_{2_n})}{\sqrt{h_{2_n}}}$. To do this, we study the difference $\left|\frac{B(h_{1_n}, h_{2_n})}{\sqrt{h_{2_n}}} - \frac{B(h_1, h_2)}{\sqrt{h_2}}\right|$. We have:

$$\begin{aligned}
\left|\frac{B(h_{1_n}, h_{2_n})}{\sqrt{h_{2_n}}} - \frac{B(h_1, h_2)}{\sqrt{h_2}}\right| &= \left|\frac{h_{1_n}\sqrt{h_{2_n}}}{\frac{\nu_1}{\nu_2}h_{1_n} + \frac{\nu_2}{\nu_1}h_{2_n}} - \frac{h_1\sqrt{h_2}}{\frac{\nu_1}{\nu_2}h_1 + \frac{\nu_2}{\nu_1}h_2}\right| \\
&= \left|\frac{\frac{\nu_1}{\nu_2}h_{1_n}h_1(\sqrt{h_{2_n}} - \sqrt{h_2}) + \frac{\nu_2}{\nu_1}(h_{1_n}h_2\sqrt{h_{2_n}} - h_1h_{2_n}\sqrt{h_2})}{(\frac{\nu_1}{\nu_2}h_{1_n} + \frac{\nu_2}{\nu_1}h_{2_n})(\frac{\nu_1}{\nu_2}h_1 + \frac{\nu_2}{\nu_1}h_2)}\right| \qquad (37) \\
&\leq \frac{\nu_2}{\nu_1}\left|\sqrt{h_{2_n}} - \sqrt{h_2}\right| + \frac{\nu_2}{\nu_1}h_{1_n}h_2|\sqrt{h_{2_n}} - \sqrt{h_2}| \\
&\quad + \frac{\nu_2}{\nu_1}\left(h_2^{3/2}|h_{1_n} - h_1| + h_1\sqrt{h_2}|h_2 - h_{2_n}|\right).
\end{aligned}$$

The above last inequality gives the strong convergence of $\frac{B(h_{1_n},h_{2_n})}{\sqrt{h_{2_n}}}$ to $\frac{B(h_1,h_2)}{\sqrt{h_2}}$ in $L^\infty(0,T;L^{1+s}(\Omega))$ where $s \in (0,1)$, s small enough. Combining this result with the weakly convergence of the product $\sqrt{h_{2_n}}v_{2_n}h_{1_n}$ in $L^{2p/p+2}(0,T;(L^{2p/p+2}(\Omega))^2)$, we deduce the weak convergence of $B(h_{1_n}, h_{2_n})v_{2_n}h_{1_n}$ in $L^{2p/p+2}(0,T;(L^t(\Omega))^2)$, t being stricly greater than 1 and defined by:

$$\frac{1}{t} = \frac{p+2}{2p} + \frac{1}{1+s}.$$

- The surface-tension terms. Formally, they appear in the following form:

$$h_{i_n}^2\Delta(h_{i_n} + h_{j_n}) + h_{i_n}\nabla h_{i_n}\Delta(h_{j_n} + h_{i_n}),\ (i \neq j).$$

Due to the strong convergence of $h_{i_n}^2$ in $L^{p/2}(0,T;L^{p/2}(\Omega))$ and the weak convergence of $\Delta(h_{i_n}+h_{j_n})$ in $L^2(0,T;L^2(\Omega))$, the first converges weakly in $L^{2p/p+4}(0,T;L^{2p/p+4}(\Omega))$. Next, since h_{i_n} converges strongly to h_i in $L^p(0,T;H^{1+1/p}(\Omega))$, ∇h_{i_n} converges strongly in $L^p(0,T;L^p(\Omega))$ and then we get the weak convergence of the second in $L^{2p/p+4}(0,T;L^{2p/p+4}(\Omega))$.

- Finally it remains to do the proof of the convergence of the six diffusion terms namely

$$\int_0^T\int_\Omega h_{i_n}^2(v_1\cdot\varphi)\ \text{div}\ v_{i_n},\quad \int_0^T\int_\Omega h_{i_n}(D(v_{i_n}):(\nabla h_{i_n}\otimes\varphi)),$$
$$\int_0^T\int_\Omega h_{i_n}^2(D(v_{i_n}):D(\varphi)).$$

For this aim we follow the lines performed in [6].

We define a function $\delta\in\mathcal{C}^\infty(\mathbb{R})$, such that $0\le\delta(\cdot)\le 1$ and

$$\delta(s)=\begin{cases}1 & s\ge 2;\\ 0 & s\le 1.\end{cases}\tag{38}$$

For a given positive number τ we denote $\delta_\tau(s)=\delta(s/\tau)$.
Using the function δ_τ defined above we write $h_{i_n}^2v_{i_n}\ \text{div}\ v_{i_n}$ as follows:

$$h_{i_n}^2v_{i_n}\ \text{div}\ v_{i_n}=(1-\delta_\tau(h_{i_n}))h_{i_n}^2v_{i_n}\ \text{div}\ v_{i_n}+\delta_\tau(h_{i_n})h_{i_n}^2v_{i_n}\ \text{div}\ v_{i_n}.$$

So now, we study each of the terms of the sum separately.
Note that we can write under the same form the others terms:

$$h_{i_n}^2D(v_{i_n})\ \text{and}\ h_{i_n}(D(v_{i_n}):D(v_{i_n})).$$

First we estimate the part including $1-\delta_\tau(h_{i_n})$. We can write that for all $\tau>0$,

$$\begin{aligned}&\|(1-\delta_\tau(h_{i_n}))h_{i_n}^2v_{i_n}\ \text{div}\ v_{i_n}\|_{L^1(0,T;L^1(\Omega)^2)}\\ &\quad\le\|\sqrt{h_{i_n}}\ \text{div}\ v_{i_n}\|_{L^2(0,T;L^2(\Omega))}\|h_{i_n}^{3/2}v_{i_n}\|_{L^2(0,T;L^6(\Omega)^2)}\\ &\qquad\times\|(1-\delta_\tau(h_{i_n}))\|_{L^\infty(0,T;L^3(\Omega))}\le C\tau.\end{aligned}\tag{39}$$

Similarly we can estimate the others diffusion terms as follows:

$$\begin{aligned}&\|(1-\delta_\tau(h_{i_n}))h_{i_n}^2 D(v_{i_n})\|_{L^1(0,T;L^1(\Omega)^4)}\\ &\leq \|\sqrt{h_{i_n}}D(v_{i_n})\|_{L^2(0,T;L^2(\Omega)^4)}\|(1-\delta_\tau(h_{i_n}))h_{i_n}^{3/2}\|_{L^2(0,T;L^2(\Omega))}\\ &\qquad \leq C\tau^{3/2},\end{aligned} \tag{40}$$

and

$$\begin{aligned}&\|(1-\delta_\tau(h_{i_n}))h_{i_n} D(v_{i_n})_{ij}\partial_j h_{i_n}\|_{L^1(0,T;L^1(\Omega)^4)}\\ &\leq \|\sqrt{h_{i_n}}D(v_{i_n})\|_{L^2(0,T;L^2(\Omega)^4)}\|(1-\delta_\tau(h_{i_n}))\sqrt{h_{i_n}}\|_{L^\infty(0,T;L^\infty(\Omega))}\\ &\qquad \times\|\nabla h_{i_n}\|_{L^2(0,T;L^2(\Omega))} \leq C\sqrt{\tau}.\end{aligned} \tag{41}$$

Therefore all of them converge to 0 when τ tends to 0. So it remains to study the sequences:

$$\begin{aligned}&\delta_\tau(h_{i_n})h_{i_n}^2 v_{i_n}\ \text{div}\ v_{i_n}, \quad \delta_\tau(h_{i_n})h_{i_n}^2 D(v_{i_n}),\\ &\delta_\tau(h_{i_n})h_{i_n} D(v_{i_n})_{ij}\partial_j h_{i_n},\end{aligned}$$

for a given positive τ.

Notice that $\delta_\tau(h_{i_n})\sqrt{h_{i_n}}D(v_{i_n})$ converges weakly to some ξ_τ in $L^2(0,T;L^2(\Omega)^4)$. We want to prove that

$$\xi_\tau = \delta_\tau(h_i)\sqrt{h_i}D(v_i). \tag{42}$$

We write

$$\begin{aligned}\delta_\tau(h_{i_n})\sqrt{h_{i_n}}D(v_{i_n}) &= D(\delta_\tau(h_{i_n})\sqrt{h_{i_n}}v_{i_n})\\ &\quad -\sqrt{h_{i_n}}v_{i_n}\underline{\otimes}\nabla h_{i_n}\left(\delta'_\tau(h_{i_n})+\frac{\delta_\tau(h_{i_n})}{2h_{i_n}}\right)\end{aligned} \tag{43}$$

Next, using the strong convergence of $\delta_\tau(h_{i_n})$ and $\delta'_\tau(h_{i_n})+\frac{\delta_\tau(h_{i_n})}{2h_{i_n}}$ in $\mathcal{C}([0,T];L^p(\Omega))$, $\forall p<+\infty$, the strong convergence of ∇h_{i_n} in $L^2(0,T;(L^2(\Omega))^2)$ and the weak convergence of $\sqrt{h_{i_n}}v_{i_n}$ to $\sqrt{h_i}v_i$ in $L^2(0,T;(L^2(\Omega))^2)$, we get the following identity in $\mathcal{D}'((0,T)\times\Omega)^4$:

$$\xi_\tau = D(\delta_\tau(h_i)\sqrt{h_i}v_i) - \sqrt{h_i}v_i\underline{\otimes}\nabla h_i\left(\delta'_\tau(h_i)+\frac{\delta_\tau(h_i)}{2h_i}\right), \tag{44}$$

and therefore, (42).
So we have

$$\delta_\tau(h_{i_n})\sqrt{h_{i_n}}D(v_{i_n}) \longrightarrow \delta_\tau(h_i)\sqrt{h_i}D(v_i) \quad \text{in } \mathcal{D}'((0,T)\times\Omega)^4 \quad (45)$$

We write the remaining three terms as:

$$\begin{aligned}
\delta_\tau(h_{i_n})h_{i_n}^2 D(v_{i_n}) &= \delta_\tau(h_{i_n})\sqrt{h_{i_n}}D(v_{i_n})h_{i_n}^{3/2}\,\chi_{\{h_{i_n}\geq\alpha\}} \\
&\quad +\delta_\tau(h_{i_n})\sqrt{h_{i_n}}D(v_{i_n})h_{i_n}^{3/2}\,\chi_{\{h_{i_n}<\alpha\}}, \\
\delta_\tau(h_{i_n})h_{i_n}^2 v_{i_n}\ \text{div}\ v_{i_n} &= \delta_\tau(h_{i_n})h_{i_n}^2 v_{i_n}\ \text{div}\ v_{i_n}\,\chi_{\{h_{i_n}\geq\alpha\}} \\
&\quad +\delta_\tau(h_{i_n})h_{i_n}^2 v_{i_n}\ \text{div}\ v_{i_n}\,\chi_{\{h_{i_n}<\alpha\}}
\end{aligned}$$

and

$$\begin{aligned}
\delta_\tau(h_{i_n})h_{i_n}D(v_{i_n})_{ij}\partial_j h_{i_n} = \delta_\tau(h_{i_n})\sqrt{h_{i_n}}D(v_{i_n})_{ij}\sqrt{h_{i_n}}\partial_j h_{i_n}\,\chi_{\{h_{i_n}\geq\alpha\}} \\
+\delta_\tau(h_{i_n})\sqrt{h_{i_n}}D(v_{i_n})_{ij}\sqrt{h_{i_n}}\partial_j h_{i_n}\,\chi_{\{h_{i_n}<\alpha\}}.
\end{aligned}$$

Thanks to the definition of $\delta_\tau(\cdot)$ and using the strong convergence of $h_{i_n}^{3/2}\chi_{\{h_{i_n}\geq\alpha\}}$, $h_{i_n}^{3/2}v_{i_n}$ and $\sqrt{h_{i_n}}\nabla h_{i_n}\chi_{\{h_{i_n}\geq\alpha\}}$ in $L^2(0,T;(L^2(\Omega))^2)$ respectively to $h_i^{3/2}\chi_{\{h_i\geq\alpha\}}$, $h_i^{3/2}v_i$ and $\sqrt{h_i}\nabla h_i\chi_{\{h_i\geq\alpha\}}$, we can pass to the limit in the three terms.
Note that, due to (38)

$$\lim_{\tau\to 0}\delta_\tau(\cdot) = 1. \quad (46)$$

So, finally, let τ goes to zero to get the convergence of the diffusion terms in equations.

□

5 Proof of the energy inequalities

This section is devoted to prove the energies inequalities given by Lemmas 3.1 and 3.3.

5.1 Proof of the classic energy inequality (Lemma 3.1)

To find the system energy, we multiply the equations for the first layer (resp. second layer) by v_1 (resp. v_2) and integrate over Ω. First, we shall do it for equation (2). Let us simplify the first two terms and the diffusion term. We have

$$\rho_1 \int_\Omega (\partial_t(h_1 v_1) + \text{ div } (h_1 v_1 \otimes v_1))v_1 = \frac{1}{2}\rho_1 \frac{d}{dt}\left(\int_\Omega h_1 |v_1|^2\right) \tag{47}$$

and

$$-2\nu_1 \int_\Omega (\text{ div } (h_1 D(v_1))v_1 = 2\nu_1 \int_\Omega h_1 (D(v_1) : D(v_1)). \tag{48}$$

Thus, we obtain:

$$\begin{aligned} &\frac{1}{2}\rho_1 \frac{d}{dt}\left(\int_\Omega h_1 |v_1|^2\right) + \frac{1}{2} g\rho_1 \int_\Omega \nabla h_1^2 v_1 + g\rho_2 \int_\Omega h_1 \nabla h_2 v_1 \\ &+ \int_\Omega c_0 \beta(h_1)|v_1|^2 - \int_\Omega \left(1 + \frac{c_0 \beta(h_1) h_1}{6\nu_1}\right) fric(v_1, v_2)v_1 \\ &+2\nu_1 \int_\Omega h_1 (D(v_1) : D(v_1)) - \alpha_1 \int_\Omega h_1 \nabla(\Delta h_1) v_1 \\ &-\alpha_2 \int_\Omega h_1 \nabla(\Delta h_2) v_1 = 0. \end{aligned} \tag{49}$$

We do the same for the second layer, to get:

$$\begin{aligned} &\frac{1}{2}\rho_2 \frac{d}{dt}\left(\int_\Omega h_2 |v_2|^2\right) + \frac{1}{2} g\rho_2 \int_\Omega \nabla h_2^2 v_2 + g\rho_2 \int_\Omega h_2 \nabla h_1 v_2 \\ &+2\nu_2 \int_\Omega h_2 (D(v_2) : D(v_2)) + \int_\Omega fric(v_1, v_2)v_2 \\ &-\alpha_2 \int_\Omega h_2 \nabla(\Delta h) v_2 = 0. \end{aligned} \tag{50}$$

We sum up the above two equations, and we study the pressure, friction and tension terms, that we denote respectively by PT, FT and TT given by:

$$\begin{aligned} PT &= \frac{1}{2} g\rho_1 \int_\Omega \nabla h_1^2 v_1 + g\rho_2 \int_\Omega h_1 \nabla h_2 v_1 \\ &\quad + \frac{1}{2} g\rho_2 \int_\Omega \nabla h_2^2 v_2 + g\rho_2 \int_\Omega h_2 \nabla h_1 v_2; \end{aligned}$$

$$\begin{aligned} FT = & - \int_\Omega fric(v_1,v_2)v_1 - \int_\Omega \frac{c_0\beta(h_1)h_1}{6\nu_1} fric(v_1,v_2)v_1 \\ & + \int_\Omega c_0\beta(h_1)|v_1|^2 + \int_\Omega fric(v_1,v_2)v_2 \end{aligned}$$

and

$$TT = -\alpha_1 \int_\Omega h_1 \nabla(\Delta h_1)v_1 - \alpha_2 \int_\Omega h_1 \nabla(\Delta h_2)v_1 - \alpha_2 \int_\Omega h_2 \nabla(\Delta h)v_2.$$

Pressure Terms: Integrating by parts and using equations (1) and (3), one can write

$$\begin{aligned} PT &= -g\rho_1 \int_\Omega h_1 \text{ div } (h_1 v_1) - g\rho_2 \int_\Omega h_2 \text{ div } (h_2 v_2) \\ &\quad + g\rho_2 \int_\Omega (h_2 \partial_t h_1 + h_1 \partial_t h_2) \\ &= g\rho_1 \int_\Omega h_1 \partial_t h_1 + g\rho_2 \int_\Omega h_2 \partial_t h_2 + g\rho_2 \int_\Omega \partial_t (h_2 h_1). \end{aligned}$$

Now we add and subtract $\frac{1}{2} g\rho_2 \int_\Omega \partial_t h_1^2$ to obtain finally:

$$PT = \frac{1}{2} g(\rho_1 - \rho_2) \frac{d}{dt} \int_\Omega |h_1|^2 + \frac{1}{2} \rho_2 g \frac{d}{dt} \int_\Omega |h_1 + h_2|^2. \tag{51}$$

Friction Terms: Next, thanks to definition of $fric(v_1, v_2)$ given by (8), FT reads

$$\begin{aligned} FT &= \int_\Omega c_0\beta(h_1)|v_1|^2 + c_1 \int_\Omega B(h_1,h_2)|v_1 - v_2|^2 \\ &\quad + c_1 \int_\Omega \frac{c_0\beta(h_1)h_1}{6\nu_1} B(h_1,h_2)(v_1 - v_2)v_1. \end{aligned}$$

Also, due to the definition (7)

$$\frac{c_0\beta(h_1)h_1}{6\nu_1} = \frac{c_0 h_1}{6\nu_1} \frac{3\nu_1}{3\nu_1 + c_0 h_1} = \frac{1}{2} \frac{c_0 h_1}{3\nu_1 + c_0 h_1} \leq \frac{1}{2}.$$

This allows us to get:

$$\begin{aligned} & c_1 \int_\Omega \frac{c_0\beta(h_1)h_1}{6\nu_1} B(h_1,h_2)(v_2 - v_1)v_1 \\ & \leq \frac{1}{2} c_1 \int_\Omega B(h_1,h_2)|v_2 - v_1|^2 + \frac{1}{2} c_1 \int_\Omega B(h_1,h_2)|v_1|^2. \end{aligned} \tag{52}$$

Moreover, writing $c_0\beta(h_1) = c_0(1-(1-\beta(h_1)))$, we deduce that

$$\frac{1}{2}c_1\int_\Omega B(h_1,h_2)|v_1-v_2|^2 + c_0\int_\Omega |v_1|^2$$

$$\leq \frac{c_0^2}{3\nu_1}\int_\Omega h_1|v_1|^2 + \frac{1}{2}c_1\int_\Omega B(h_1,h_2)|v_1|^2$$

$$\leq \frac{c_0^2}{3\nu_1}\int_\Omega h_1|v_1|^2 + \frac{1}{2}c_1\frac{\nu_1}{\nu_2}\int_\Omega h_1|v_1|^2. \tag{53}$$

Tension Terms: Finally, let us transform the tension terms TT. If we use equations (1) and (3), we can write them as follows

$$\begin{aligned} TT &= \alpha_1\int_\Omega \Delta h_1 \text{ div } (h_1v_1) + \alpha_2\int_\Omega \Delta h_2 \text{ div } (h_1v_1) + \alpha_2\int_\Omega \Delta h \text{ div } (h_2v_2) \\ &= -\alpha_1\int_\Omega \Delta h_1\partial_t h_1 - \alpha_2\int_\Omega \Delta h_2\partial_t h_1 - \alpha_2\int_\Omega \Delta h\partial_t h_2. \end{aligned}$$

Using Leibnitz formula, we obtain

$$\begin{aligned} TT &= \frac{1}{2}\alpha_1\frac{d}{dt}\int_\Omega |\nabla h_1|^2 + \alpha_2\int_\Omega \nabla h_2\partial_t\nabla h_1 + \frac{1}{2}\alpha_2\frac{d}{dt}\int_\Omega |\nabla h_2|^2 \\ &\quad -\alpha_2\int_\Omega \Delta h_1\partial_t h_2 \\ &= \frac{1}{2}\alpha_1\frac{d}{dt}\int_\Omega |\nabla h_1|^2 + \alpha_2\int_\Omega \nabla h_2\partial_t\nabla h_1 + \frac{1}{2}\alpha_2\frac{d}{dt}\int_\Omega |\nabla h_2|^2 \\ &\quad +\alpha_2\int_\Omega \nabla h_1\partial_t(\nabla h_2) \\ &= \frac{1}{2}\alpha_1\frac{d}{dt}\int_\Omega |\nabla h_1|^2 + \frac{1}{2}\alpha_2\frac{d}{dt}\int_\Omega |\nabla h_2|^2 + \alpha_2\frac{d}{dt}\int_\Omega \nabla h_2\nabla h_1. \end{aligned}$$

So, finally, we write the energy inequality as follows:

$$\frac{1}{2}\rho_1\frac{d}{dt}\int_\Omega h_1|v_1|^2 + 2\nu_1\int_\Omega h_1(D(v_1):D(v_1)) + \frac{1}{2}\rho_2\frac{d}{dt}\int_\Omega h_2|v_2|^2$$

$$
\begin{aligned}
&+2\nu_2\int_\Omega h_2(D(v_2):D(v_2))+\frac{1}{2}g(\rho_1-\rho_2)\frac{d}{dt}\int_\Omega |h_1|^2\\
&+\frac{1}{2}\rho_2 g\frac{d}{dt}\int_\Omega |h|^2+\frac{1}{2}c_1\int_\Omega B(v_1,v_2)|v_1-v_2|^2+c_0\int_\Omega |v_1|^2\\
&+\frac{1}{2}\alpha_1\frac{d}{dt}\int_\Omega |\nabla h_1|^2+\frac{1}{2}\alpha_2\frac{d}{dt}\int_\Omega |\nabla h_2|^2+\alpha_2\frac{d}{dt}\int_\Omega \nabla h_1\nabla h_2\\
&+\frac{1}{2}\rho_2 g\frac{d}{dt}\int_\Omega |h|^2+\frac{1}{2}c_1\int_\Omega B(v_1,v_2)|v_1-v_2|^2+c_0\int_\Omega |v_1|^2\\
&+\frac{1}{2}\alpha_1\frac{d}{dt}\int_\Omega |\nabla h_1|^2+\frac{1}{2}\alpha_2\frac{d}{dt}\int_\Omega |\nabla h_2|^2+\alpha_2\frac{d}{dt}\int_\Omega \nabla h_1\nabla h_2\\
&\leq \frac{c_0^2}{3\nu_1}\int_\Omega h_1|v_1|^2+\frac{1}{2}c_1\frac{\nu_1}{\nu_2}\int_\Omega h_1|v_1|^2
\end{aligned}
\tag{54}
$$

□

5.2 Proof of the entropy inequality (Lemma 3.3)

We use the both transport equation and the renormalize technique to get:

$$
\begin{aligned}
\partial_t \nabla h_1 + \text{ div } (h_1\nabla^t v_1) + \text{ div } (v_1\otimes\nabla h_1) = 0\\
\partial_t \nabla h_2 + \text{ div } (h_2\nabla^t v_2) + \text{ div } (v_2\otimes\nabla h_2) = 0.
\end{aligned}
$$

Replacing ∇h_i by $h_i\nabla\log h_i$ and introducing the viscosity $2\nu_i$, they become

$$
\partial_t(2\nu_1 h_1\nabla\log h_1)+2\nu_1 \text{ div } (h_1\nabla^t v_1)+ \text{ div } (h_1v_1\otimes 2\nu_1\nabla\log h_1)=0
$$

and

$$
\partial_t(2\nu_2 h_2\nabla\log h_2)+2\nu_2 \text{ div } (h_2\nabla^t v_2)+ \text{ div } (h_2v_2\otimes 2\nu_2\nabla\log h_2)=0.
$$

Next, we add the momentum equation to obtain:

$$
\begin{aligned}
&\partial_t(h_1(\rho_1 v_1+2\nu_1\nabla\log h_1))+ \text{ div } (h_1v_1\otimes(\rho_1v_1+2\nu_1\nabla\log h_1))\\
&-2\nu_1 \text{ div } (h_1(D(v_1)-\nabla^t v_1))+\rho_1 g h_1\nabla h_1+\rho_2 g h_1\nabla h_2\\
&-\left(1+\frac{c_0\beta(h_1)h_1}{6\nu_1}\right)fric(v_1,v_2)+c_0\beta(h_1)v_1-\alpha_1 h_1\nabla(\Delta h_1)\\
&-\alpha_2 h_1\nabla(\Delta h_2)=0
\end{aligned}
\tag{55}
$$

and

$$\partial_t(h_2(\rho_2 v_2 + 2\nu_2 \nabla \log h_2)) + \text{ div } (h_2 v_2 \otimes (\rho_2 v_2 + 2\nu_2 \nabla \log h_2))$$

$$-2\nu_2 \text{ div } (h_2(D(v_2) - \nabla^t v_2)) + \rho_2 g h_2 \nabla h_2 + \rho_2 g h_2 \nabla h_1 \tag{56}$$

$$+fric(v_1, v_2) - \alpha_2 h_2 \nabla(\Delta(h_1 + h_2)) = 0.$$

We multiply every equation i by $\rho_j\,(\rho_i v_i + 2\nu_i \nabla \log h_i)$, with $i \neq j$, take the integrate over Ω and sum the two equalities. We transform now every term.

- From the first two ones:

$$\rho_j \int_\Omega \partial_t(h_i(\rho_i v_i + 2\nu_i \nabla \log h_i))(\rho_i v_i + 2\nu_i \nabla \log h_i)$$

$$+\rho_j \int_\Omega \text{ div } (h_i v_i \otimes (\rho_i v_i + 2\nu_i \nabla \log h_i))(\rho_i v_i + 2\nu_i \nabla \log h_i)$$

$$= \frac{1}{2}\rho_j \frac{d}{dt} \int_\Omega h_i |\rho_i v_i + 2\nu_i \nabla \log h_i|^2.$$

- Using the definition of the deformation tensor and vorticity tensor we obtain:

$$2\nu_i \rho_j \int_\Omega \text{ div } (h_i(D(v_i) - \nabla^t v_i))(\rho_i v_i + 2\nu_i \nabla \log h_i)$$

$$= -2\nu_i \rho_i \rho_j \int_\Omega h_i (A(v_i) : A(v_i)).$$

Next, we only study all terms which are not appear in the classical energy.

- The pressure terms become:

$$2\nu_1 \rho_1 \rho_2 g \int_\Omega h_1 \nabla h_1 \nabla \log h_1 + 2\nu_1 \rho_2^2 g \int_\Omega h_1 \nabla h_2 \nabla \log h_1$$

$$+2\nu_2 \rho_1 \rho_2 g \int_\Omega h_2 \nabla h_2 \nabla \log h_2 + 2\nu_2 \rho_1 \rho_2 g \int_\Omega h_2 \nabla h_1 \nabla \log h_2$$

$$= 2\nu_1 \rho_1 \rho_2 g \int_\Omega |\nabla h_1|^2 + 2\nu_2 \rho_1 \rho_2 g \int_\Omega |\nabla h_2|^2$$

$$+2\rho_2 g(\rho_2 \nu_1 + \rho_1 \nu_2) \int_\Omega \nabla h_1 \nabla h_2.$$

- Now, we change the tension term as follow:

$$-2\nu_1\alpha_1\rho_2\int_\Omega h_1\nabla(\Delta h_1)\nabla\log h_1 - 2\nu_1\alpha_2\rho_2\int_\Omega h_1\nabla(\Delta h_2)\nabla\log h_1$$

$$-2\nu_2\alpha_2\rho_1\int_\Omega \nabla(\Delta(h_1+h_2))\nabla\log h_2$$

$$= 2\nu_1\alpha_1\rho_2\int_\Omega|\Delta h_1|^2 + 2\nu_2\alpha_2\rho_1\int_\Omega|\Delta h_2|^2 + 2\alpha_2(\rho_2\nu_1+\rho_1\nu_2)\int_\Omega\Delta h_1\Delta h_2.$$

- Friction at bottom:

$$\begin{aligned} c_0\rho_2\int_\Omega\beta(h_1)v_1\nabla\log h_1 &= \rho_2\int_\Omega\frac{3\nu_1c_0}{3\nu_1+c_0h_1}v_1\nabla\log h_1 \\ &= -\rho_2\int_\Omega\frac{3\nu_1c_0}{3\nu_1+c_0h_1}\left(\frac{\partial_t h_1}{h_1}+\ \text{div}\ v_1\right). \end{aligned}$$

So, define a function f such that $f(h_1) = c_0\log\left(\dfrac{h_1}{3+c_0\nu_1^{-1}h_1}\right)$, we then have
$f'(h_1) = \dfrac{3\nu_1c_0}{3\nu_1+c_0h_1}\dfrac{1}{h_1} = c_0\beta(h_1)\dfrac{1}{h_1}$ and therefore

$$\rho_2c_0\int_\Omega\beta(h_1)v_1\nabla\log h_1 = -\rho_2\frac{d}{dt}\int_\Omega f(h_1) + \rho_2c_0\int_\Omega\beta'(h_1)v_1\nabla h_1. \quad (57)$$

- And finally, the interface friction terms are:

$$2\nu_1c_1\rho_2\int_\Omega\left(1+\frac{c_0\beta(h_1)h_1}{6\nu_1}\right)\frac{B(h_1,h_2)}{h_1}((v_1-v_2)\cdot\nabla h_1)$$

$$-2\nu_2c_1\rho_1\int_\Omega\frac{B(h_1,h_2)}{h_2}((v_1-v_2)\cdot\nabla h_2).$$

So we find the inequality (19).

□

Acknowledgments.

The first author is partially supported by "Réseau EDP-MC, ISP and SARIMA". The research of G. Narbona-Reina to carry on this work was partially supported by the Spanish Government Research project MTM2006-01275.

The authors wish to thank Pr. Didier Bresch for fruitful discussions.

References

[1] Audusse E. *A multilayer Saint-Venant model*. Disc. Cont. Dyn. System. Series B, 5(2): 189-214, 2005.

[2] Bresch D. and Desjardins B. *Existence of weak solutions for 2d viscous shallow water equations and convergence to the quasi.geostrophic model*. Comm. Math. Phys., 238 (1,2): 211-223, 2003.

[3] Bresch D. and Desjardins B. *Some diffusive capillary models of Korteweg type*. C.R. Acad. Sci. Paris, Section Mécanique, 332(11): 881-886, 2004.

[4] Bresch D. and Desjardins B. *On the construction of approximate solutions for the 2d viscous shallow water model and for compressible Navier-Stokes models*. J. Math. Pure Appl., 86: 362-368, 2006.

[5] Bresch D., Desjardins B., Ghidaglia J-M, Grenier E. *On global weak solutions to a generic two-fluid model.* (Soumis).

[6] Bresch D., Desjardins B. and Lin C.K. *On some compressible fliud models: Korteweg, lubrication and shallow water systems*. Comm. Partial Diff. Equations, 28(3,4): 843-868, 2003.

[7] Bresch D., Desjardins B. and Gérard-Varet D. *On compressible Navier-Stokes equations with density dependent viscosities in bounded domain*. J. Math. Pures Appl., 87: 227-235, 2007.

[8] Brézis, H.: *Análisis funcional. Teoría y aplicaciones*, Alianza Editorial, Madrid, (1984).

[9] Castro M.J., García Rodríguez J.A., González-Vida J.M. and Parés C. *A parallel 2d finite volume scheme for solving systems of balance laws with non-conservative products: Application to shallow flows*. Comput. Meth. Appl. Mech. Engry., 195: 2788-2815, 2006.

[10] Castro M.J., García Rodríguez J.A., Macías J. and Parés C. *A two-layer finite volume model for flows through channels with irregular geometry: computation of maximal exchange solutions, applications to strait of gibraltar*. Comm. Nonlin. Sci. Num. Simul., 9: 241-249, 2004.

[11] Chatelon F.J., Muñoz-Ruiz M.L. and Orenga P. *On a bi-layer shallow water problem* Nonlinear Anal., 4: 139-171, 2003.

[12] Chueskov I.D., Raugel G. and Rekalo A.M. *Interface boundary value problem for the Navier-Stokes equations in thin domains.* J. Differential Equations 208, 449-493, (2005).

[13] Desjardins B., Esteban M.J. *On weak solutions for fluid-rigid structure interaction: compressible and incompressible models.* Comm. Partial Diff. Eqs., 25, No 7-8, (2000) 1399-1413.

[14] Flori F., Orenga P. and Peybernes M. *Sur un problème de shallow water bicouche avec conditions aux limites de Dirichlet.* C.R. Acad. Sci. Paris. Ser. I 340, 2005.

[15] Ferrari S., Saleri F. *A new to dimensional Shallow-Water model including pressure effects and slow varying bottom topography.* M2AN, vol. 38, No 2, (2004), 211-234.

[16] Bresch D., Fernández-Nieto E.D., Narbona-Reina G. and Zabsonré J.D.D. *Derivation of a bi-layer model for Shallow-Water equation with viscosity. Numerical validation.* In preparation.

[17] Gerbeau J.F and Perthame B. *Derivation of viscous Saint-Venant system for laminar Shallow-Water; Numerical validation.* Disc. and Cont. Dynam. Syst. Series B, 1(1): 89-102, 2001.

[18] Ishii M. *Thermo-Fluid Dynamic Theory of Two-Phase Flow.* Eyrolles, Paris (1975).

[19] Lucas C. *Cosine effects of Shallow Water equations and mathematical properties.* In preparation.

[20] Marche F. *Theoretical and numerical study of Shallow Water models. Applications to Nearshore hydrodynamics.* Thèse à la Université de Bordeaux, 2005.

[21] Mellet A., Vasseur A. *On the barotropic compressible Navier-Stokes equations.* To appear in Comm. Partial. Diff. Eqs., 2007.

[22] Muñoz-Ruiz M.L. *Análisis de un modelo bicapa de aguas poco profundas.* Tesis Doctoral de la Universidad de Málaga, 2002.

[23] Muñoz-Ruiz M.L. *On a non-homogeneous bi-layer shallow water problem: smoothness and uniqueness results*. Nonlinear Anal., 59(3): 253-600, 2003.

[24] Orenga P. *Un théorème d'existence de solutions d'un problème de shallow water*. Arch. Rational Mech. Anal., 130: 183-204, 1995.

[25] Peybernes M. *Analyse de problème mathématiques de la mécanique des fluides de type bi-couche et à frontière libre*. Thèse à la Université de Pascal Paoli, 2006.

[26] Simon J. *Compact set in the space* $l^p(0,t;b)$. Ann. Mat. Pure Appl., 146(4): 65-96, 1987.

[27] Zabsonré J.D.D., Lucas C., E. Fernández-Nieto. *An energetically consistent viscous sedimentation model.*(submitted).