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COMPLETENESS OF SPACES WITH TOEPLITZ DECOMPOSITIONS[1]

(Decompositions/Completeness/Summability/Bases)

PEDRO J. PAÚL, CARMEN SÁEZ, JUAN M. VIRUÉS

Departamento de Matemática Aplicada II. Escuela Superior de Ingenieros. Universidad de Sevilla.

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ABSTRACT

We extend to the setting of Toeplitz decompositions of a locally convex space E into subspaces (E_k) a result about Schauder decompositions due to Kalton that links the completeness of E to the completeness of both the decomposition and the pieces (E_k). The proof is simplified by the application of a double limit technique. Then we study what we call the Garling topology of a space with a Toeplitz decomposition with respect to a matrix T in the framework of the β_T-duality of sequence spaces.

RESUMEN

En este trabajo extendemos al marco de las descomposiciones de Toeplitz de un espacio localmente convexo E en subespacios (E_k) un resultado sobre descomposiciones de Schauder debido a Kalton que liga la completitud de E con la completitud de la descomposición y las de las piezas (E_k); el uso de una técnica de límite doble nos permite simplificar la prueba original. Posteriormente estudiamos lo que llamamos topología de Garling de un espacio con una descomposición de Toeplitz con respecto a una matriz T en el marco de la β_T-dualidad de espacios de sucesiones.

INTRODUCTION

Up to what point can one substitute ordinary summability by a matrix summability method in the definition of a Schauder decomposition and, still, obtain nice results about the locally convex structure of the space in terms of the locally convex structure of its pieces? Our purpose here is to extend the characterization of the completeness of a Schauder decomposition obtained by Kalton (9), (10) to the setting of decompositions defined in terms of more general matrix summability methods. The proof is simplified by the application of a double limit technique.

Given an infinite matrix T one can consider its convergence field c_T and define, for a sequence space λ, its corresponding β_T-dual $\lambda^{\beta T}$. This dual pair has been studied by Buntinas (2), (3), Meyers (12) and Noll (14) as a continuation of the β-duality theory of Garling (5), (6). Here we use the theorem about completeness to study what we call the Garling topology, because it is a natural extension of the $\sigma\gamma$-topology defined in (5), of a space with a T-decomposition and, in particular, we give some applications to the dual pair $(\lambda, \lambda^{\beta T})$.

Terminology and Notation. Although our notation and terminology will be mostly standard, e.g. φ is the space of finitely nonzero sequences, c is the space of convergent sequences, $e^{[k]}$ stands for the k-th unit sequence (we refer the reader to (16), (17), (19) and (21)), let us recall a few facts from summability theory. Let $T = [t_{nk}]$ be an infinite matrix of scalars from the field $\mathbb{K}$ of real or complex numbers. The matrix T is said to be: *row-finite* if each row of T is in φ, an *Sp_1-matrix* if each column of T is convergent to 1, and *reversible* if for every sequence $y \in c$ the infinite system of linear equations $T \cdot x = y$ has unique solution. It is well-known (21, 5.4.5-5.4.9) that each row-finite and reversible T has a unique two-sided inverse matrix T^{-1} such that each row of T^{-1} is in l^1 and for each $y \in c$ the unique solution of $T \cdot x = y$ is $T^{-1} \cdot y$.

Let E be a locally convex space. The convergence field of T in E is the space $c_T(E)$ of all sequences (x_k) from E such that the product $T \cdot (x_k)$ is a convergent sequence in E. For $(x_k) \in c_T(E)$ the limit of the sequence $T \cdot (x_k)$ is called the T-limit of (x_k) and will be denoted by $T\text{-lim} \cdot x_k$, in other words

$$T - lim\, x_k := \lim_n \sum_k t_{nk}\, x_k.$$

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We simply denote by c_T the convergence field of T in $\mathbb{K}$. If T is a row-finite and reversible matrix then the norm $\|x\|_T := \|T\cdot x\|_\infty$ makes c_T a Banach space (isomorphic to c).

Definitions. Let $T = [t_{nk}]$ be a row-finite infinite matrix of scalars. A sequence (P_k) of non-trivial, mutually orthogonal and continuous linear projections defined on a locally convex space E is said to be a *Toeplitz decomposition* of E with respect to the matrix T or, shortly, a *T-decomposition* of E, if

$$x = T - lim\, P_k x \quad \text{for every } x \in E.$$

Alternatively, if we define the sequence of operators

$$T_n : x \in E \to T_n(x) := \sum_k t_{nk} P_k x \in E,$$

then (P_k) is a T-decomposition of E whenever $\lim_n T_n x = x$ for every $x \in E$.

It is important to note that the operators T_n's are not projections in general (they are increasing projections in the case of a Schauder decomposition), however we do have $T_n P_k = P_k T_n = t_{nk} P_k$ for all $n, k \in \mathbb{N}$. Note also that the sequence of operators (T_n) is precisely the product $T \cdot (P_k)$ hence saying that $\lim_n T_n x = x$ is the same as saying that the sequence $T \cdot (P_k x)$ converges to x. Call $E_k := P_k(E)$. Since E_k does not reduces to the zero subspace and for every $x_k \in E_k$ we have $x_k = \lim_n T_n x_k = \lim_n t_{nk} x_k$, it follows that $\lim_n t_{nk} = 1$, i.e., T is an *Sp_1-matrix.*

Still another way of looking at a Toeplitz decomposition is the following: Every E_k is a complemented subspace of E and we can identify every $x \in E$ with the vector-valued sequence $(P_k x) \in \prod E_k$, so that E becomes a linear subspace of $\prod E_k$ that, with the topology translated from E, has the set of all finite sequences as a dense subspace because $\lim_n T_n x = x$ for every $x \in E$ and T is row-finite.

A T-decomposition (P_k) of a locally convex space E is said to be: *finite-dimensional* if every E_k is finite-dimensional; *equicontinuous* if the sequence of operators (T_n) is equicontinuous; and *complete* if for each squence $(x_k) \in \prod E_k$ such that the product $T \cdot (x_k)$ is a Cauchy sequence in E there exists $x \in E$ such that $x_k = P_k x$ for every $k \in \mathbb{N}$ and, *a fortiori*, $T \cdot (x_k)$ converges to x.

Example 1. Every Schauder decomposition is a Toeplitz decomposition with respect to the ordinary summability matrix $\Sigma = [\sigma_{nk}]$, where $\sigma_{nk} = 1$ if $n \leq k$ and $\sigma_{nk} = 0$ otherwise.

Example 2. A Cesàro basis induces a one-dimensional Toeplitz decomposition with respect to $C^1 \cdot \Sigma$, where $C^1 = [c^1_{nk}]$ is the Cesàro matrix of order 1 defined by $c^1_{nk} = n^{-1}$ if $n \leq k$ and $c^1_{nk} = 0$ otherwise. (The matrix $C^1 \cdot \Sigma$ is sometimes called the series-to-sequence Cesàro matrix.) Decompositions of Banach spaces with respect to Cesàro matrices were firstly considered by Butzer and his collaborators in Aachen (see (19, pp. 785 and 801 of vol. II)).

Example 3. A K-space is a locally convex sequence space $\lambda \supset \varphi$ such that the k-th projection defined by $\pi_k((x_n)_n) := x_k e^{[k]}$ is continuous for every $k \in \mathbb{N}$. A K-space λ is said to have property *T-AK* if $x = T\text{-lim } x_k e^{[k]}$ for every sequence $x = (x_k) \in \lambda$. Thus, a sequence space λ has property *T-AK* if and only if the sequence (π_k) is a (one-dimensional) T-decomposition of λ or, in other words, the sequence of operators defined by $\tau_n := \sum_k t_{nk}\pi_k$, i.e. $(\tau_n) := T \cdot (\pi_k)$, satisfies $x = \lim_n \tau_n(x_k)$ for every sequence $x = (x_k) \in \lambda$ (see (2), (3) or (12)). (When dealing with scalar sequences, we shall keep the notations (π_k) and (τ_k) throughout the paper). In particular, λ has property Σ-AK means precisely that $(e^{[k]})$ is a Schauder basis of λ.

We shall be interested in matrices T such that c_T has property *T-AK*. These matrices were characterized by Buntinas (3, Thms. 8-10).

Buntinas's Theorem. *Let T be row-finite and reversible Sp_1-matrix. Then the following conditions are equivalent:*

(1) *The sequence of coordinate projections (π_k) is a T-decomposition of c_T.*

(2) *The sequence of operators (τ_n) is equicontinuous on c_T.*

(3) *If we denote T^{-1} by $[t^{-1}_{nk}]$ then*

$$\sup\left\{\sum_j\left|\sum_k t_{mk}t_{nk}t^{-1}_{kj}\right| : m, n \in \mathbb{N}\right\} < \infty.$$

(4) *The dual $(c_T)'$ can be identified with the multiplier space $(c_T \to c_T)$ formed by the sequences y such that the coordinatewise product xy is in c_T for every $x \in c_T$ and, in this case, the bilinear form of the dual pair is given by*

$$\langle x, y\rangle_{(c_T,(c_T)')} = T - \lim xy.$$

The first non-trivial examples of matrices T such that c_T has property *T-AK* are the series-to-sequence Cesàro matrices of order $\alpha \geq 0$; this was proved by Zeller (22). Therefore, to avoid clumsy repetitions, a row-finite and reversible Sp_1-matrix T such that c_T has property *T-AK* will be called a *Zeller-Buntinas matrix*. For such a matrix T we define $b(T) := \sup_n\|\tau_n\|$, where $\|\tau_n\|$ is the norm of τ_n as a bounded operator from the Banach space c_T into itself. Note also that if T is a Zeller-Buntinas matrix then c_T is a *sum space* in the sense of Ruckle (17).

Example 4. Let Ω be an open, bounded and balanced subset of $\mathbb{C}^m$ and let $A(\Omega)$ be the space of all functions that are holomorphic on Ω and can be extended continuosly to the closure of Ω endowed with the topology of uniform convergence on Ω. It is mentioned without proof in (15) that $A(\Omega)$ has the bounded approximation property. As a matter of fact, what happens is that $A(\Omega)$ has an equicontinuous and a finite-dimensional Toeplitz decomposition with respect to the series-to-sequence Cesàro matrix $C^1 \cdot \Sigma$. This decomposition is the natural one given by the Taylor series: Each $f \in A(\Omega)$ can be uniquely written as $f(\cdot) = \sum_{k=0}^{\infty} P_k(f)(\cdot)$ (pointwise convergence), where each $P_k(f)$ is a k-homogeneous polynomial. Now, the set of all homogeneous polynomials is dense in $A(\Omega)$ (for a proof see (1), the method utilized in Section 1 of that paper can easily be adapted to show the present result). On the other hand, the corresponding sequence of operators (T_n) is equicontinuous by (1, Lemma 1.1) or (13, 5.2 Proposition), where it is shown that $|T_n(f)(z)| \leq \|f\|_\infty$ for all $z \in \Omega$. Finally, a standard argument about equicontinuous sets (11, §39.4.(1)) shows that $f = T\text{-lim } P_k(f)$ for all $f \in A(\Omega)$.

COMPLETENESS OF SPACES WITH TOEPLITZ DECOMPOSITIONS

Our first purpose is to extend to the setting of Toeplitz decompositions a result due to Kalton (10) that links the completeness of E to the completeness of both the decomposition and the pieces E_k. We shall make use of a double limit technique that lies behind the proof given by Kalton for Schauder decompositions. The Double Limit Lemma is certainly well-known for double sequences but we need a reformulation in terms of a double net that can be proven analogously.

Double Limit Lemma. *Let E be a locally convex space and $\{x_{ij} : (i, j) \in I \times J\}$ be a double net in E such that for each $i \in I$ there exists the limit $y_i = \lim_j x_{ij}$ and for each $j \in J$ there exists the limit $z_j = \lim_i x_{ij}$. If the convergence of $(x_{ij})_j$ to y_i is uniform in I then the three nets (x_{ij}), (y_i) and (z_j) are Cauchy nets. If, in addition, E is complete then the three nets above are convergent to the same limit.*

Theorem 1. *Let (P_k) be an equicontinuous T-decomposition of a locally convex space E. Then the following are equivalent:*

(1) *E is complete (resp. quasi-complete or sequentially complete).*

(2) *(P_k) is complete and each E_k is complete (resp. quasi-complete or sequentially complete).*

Proof. It is clear that (1) implies (2), so we have to show that (2) implies (1). We shall deal only with the completeness case because the proofs for the three cases are essentially the same. Let $(z_i)_{i\in I}$ be a Cauchy net in E. For each $k \in \mathbb{N}$ there exists $x_k \in E_k$ such that $(P_k z_i)_{i\in I}$ converges to x_k because P_k is continuous and E_k is complete. Since T is row-finite, for every $n \in \mathbb{N}$ we have

$$\lim_i T_n z_i = \lim_i \sum_k t_{nk} P_k z_i = \sum_k t_{nk} x_k.$$

On the other hand, $\lim_n T_n z_i = z_i$ for every $i \in I$. To see that we can apply the Double Limit Lemma to the double net $\{T_n z_i : (i, n) \in I \times \mathbb{N}\}$, let us check that the convergence of $(T_n z_i)_{i\in I}$ is uniform in $\mathbb{N}$. Given a continuous seminorm q_1 on E, there exists a continuous seminorm q_2 such that

$$q_1(x) \leq q_2(x) \quad \text{and} \quad q_1(T_n x) \leq q_2(x) \quad \text{for all } n \in \mathbb{N} \text{ and } x \in E$$

because (P_k) is an equicontinuous T-decomposition. Since $(z_i)_{i\in I}$ is a Cauchy net, it follows that there exists some index $i_0 \in I$ such that $q_2(z_i - z_j) \leq 1$ whenever $i, j \geq i_0$. Therefore, $q_1(T_n z_i - T_n z_j) \leq 1$ for all $n \in \mathbb{N}$ and $i, j \geq i_0$. Take limits in j to obtain

$$q_1\left(T_n z_i - \lim_j T_n z_j\right) \leq 1 \quad \text{for all } n \in \mathbb{N} \text{ and } i \geq i_0.$$

This shows that the convergence of $(T_n z_i)_{i\in I}$ is uniform in $\mathbb{N}$. The Double Limit Lemma tells us that the product $T \cdot (x_k) = \left(\sum_k t_{nk} x_k\right)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (P_k) is a complete T-decomposition, there exists $x \in E$ such that $x_k = P_k x$ for every $k \in \mathbb{N}$ and $T \cdot (x_k)$ converges to x. Finally, since the net $(z_i)_{i\in I}$ is convergent in the completion of E, the Double Limit Lemma tells us now that $(z_i)_{i\in I}$ must converge to x as well. ■

Corollary 1. *Let (P_k) be an equicontinuous and finite-dimensional Toeplitz decomposition of a locally convex space E. Then the following are equivalent:*

(1) *E is complete.*

(2) *E is quasi-complete.*

(3) *E is sequentially complete.*

(4) *(P_k) is complete.*

Corollary 2. *If a barrelled and sequentially complete locally convex space has a finite-dimensional Toeplitz decomposition then it is complete.*

Remark. An *extended Schauder basis* of a locally convex space E is a family $(x_i)_{i\in I}$ with the property that for every $x \in E$ there is a unique family $(\alpha_i(x))_{i\in I}$ of scalars such that x can be written as $x = \sum_i \alpha_i(x) x_i$ and the functionals $x \to \alpha_i(x)$ are continuous. Webb (20) proved that a separable, non-complete, Montel locally convex space cannot have any extended Schauder basis; our Corollary 2 shows that it cannot have any finite-dimensional Toeplitz decomposition neither.

THE GARLING TOPOLOGY OF A SPACE WITH A TOEPLITZ DECOMPOSITION

Let E be a locally convex space with a T-decomposition (P_k). In this section we will see that there exists a coarsest E'-polar topology on E for which (P_k) is an equicontinuous Toeplitz decomposition. This topology turns out to be a natural generalization of the $\sigma\gamma$-topology introduced by Garling in his deep study of the β-duality between sequence spaces (5), (6) and, accordingly, will be called here the Garling topology of E. In the case of Schauder decompositions this topology has been studied by Kalton (9).

Using primes to denote adjoint operators, for every $x \in E$ and every $u \in E'$ we can write

$$\langle x, u\rangle = \lim_n \langle T_n x, u\rangle = \lim_n \sum_k t_{nk}\langle P_k x, u\rangle =$$

$$= \lim_n \sum_k t_{nk}\langle x, P_k' u\rangle = \lim_n \langle x, T_n' u\rangle.$$

This shows that (P_k') *is also a* T-decomposition of E' endowed with the weak topology $\sigma(E', E)$. If we call $E_k := P_k(E)$ and $E_k' := P_k'(E')$ then the dual of E_k can be identified with E_k'. The computation above also shows that the sequence $(T_n' u)$ is $\sigma(E', E)$-bounded.

Definition. Let (P_k) *be a* T-decomposition of a locally convex space E. The *Garling topology* of E is the polar topology $\gamma_T(E, E')$ of uniform convergence on the family $\{(T_n' u) : u \in E'\}$. Alternatively, $\gamma_T(E, E')$ is generated by the family of seminorms

$$x \in E \to \sup_n |\langle T_n x, u\rangle|, \quad (u \in E').$$

The Garling and the weak topology coincide on each E_k because for all $x_k \in E_k$ and $u \in E'$ we have

$$\sup_n |\langle T_n x_k, u\rangle| = \sup_n |\langle t_{nk} x_k, u\rangle| = \left(\sup_n |t_{nk}|\right) |\langle x_k, u\rangle|,$$

we shall make use of this fact a couple of times.

The properties of the Garling topology depend heavily on the *T-AK* property of the convergence field associated to the matrix T. Note that this is given for free in the case of a Schauder decomposition: $(e^{[k]})$ is a Schauder basis of the space cs ($= c_\Sigma$) of all summable sequences. To see how to connect the Garling topology with the properties of c_T, let F be the vector-valued sequence space defined by

$$F := \left\{(x_k) \in \prod_k E_k : T \cdot (x_k) \text{ is a } \sigma(E, E')\text{-Cauchy sequence}\right\}.$$

As we noted above, E can be identified with a subspace of F. Note that, using the terminology given in the previous section, E equals F if and only if (P_k) is a complete T-decomposition of $E[\sigma(E, E')]$; in this case, (P_k) is said to be β_T-*complete* by analogy with the Schauder decomposition case (10).

For each $u \in E'$ we define the operator

$$\Delta_u : (x_k) \in F \to \Delta_u(x_k) := (\langle x_k, u\rangle)_k \in c_T.$$

It is easy to see that Δ_u satisfies the following properties

(i) For every $n \in \mathbb{N}$ and $(x_k) \in F$ we have that the sequence $(T_n x_k)_k$ is also in F and $\Delta_u(T_n x_k)_k = \tau_n \Delta_u(x_k)_k$. In particular, $\Delta_u(T_n P_k x)_k = \tau_n \Delta_u(P_k x)_k$ for each $x \in E$.

(ii) If $[T \cdot (x_k)]_n$ stands for the n-th element of the sequence $T \cdot (x_k)$, then

$$\|\Delta_u(x_k)\|_T = \sup_n |\langle [T \cdot (x_k)]_n, u\rangle|$$

and, in particular, $\|\Delta_u(P_k x)\|_T = \sup_n |\langle T_n x, u\rangle|$ for each $x \in E$ so that the Garling topology is generated by the family of seminorms $x \to \|\Delta_u(P_k x)\|_T$ as $u \in E'$.

Proposition 1. *Let T be a Zeller-Buntinas matrix and* (P_k) *be a T-decomposition of a locally convex space E. Then* $\gamma_T(E, E')$ *is the coarsest E'-polar topology such that* (P_k) *is an equicontinuous T-decomposition of E.*

Proof. Using that the projections (P_k) are weakly continuous on E, that the Garling topology is stronger than the weak topology and that the Garling topology induces on each subspace E_k its own weak topology $\sigma(E_k, E_k')$, it follows that the projections (P_k) are continuous on E for the Garling topology. We now show that (P_k) is an equicontinuous T-decomposition of E endowed with $\gamma_T(E, E')$. For all $x \in E$, $u \in$ E', and $m \in \mathbb{N}$ we have, using (i) and (ii) above,

$$\|\Delta_u(P_k T_m x)_k\|_T = \|\Delta_u(T_m P_k x)_k\|_T =$$

$$= \|\tau_m \Delta_u(P_k x)_k\|_T \leq b(T)\, \|\Delta_u(P_k x)\|_T$$

so that (T_n) is a γ_T-equicontinuous sequence. (A remark is in order here: although $\gamma_T(E, E')$ is generated by the family of seminors $\sup_n |\langle T_n x, u\rangle|$, since the T_n's are not increasing projections —as it is the case of a Schauder decomposition— we cannot conclude directly that $\sup_n |\langle T_n T_m x, u\rangle| \leq$ $\leq \sup_n |\langle T_n x, u\rangle|$.)

The following computation shows that $x = \gamma_T(E, E')$-$\lim_n T_n x$ for all $x \in E$:

$$\|\Delta_u(P_k x - P_k T_m x)_k\|_T = \|\Delta_u(P_k x)_k - \Delta_u(T_m P_k x)_k\|_T$$

$$= \|\Delta_u(P_k x)_k - \tau_m \Delta_u(P_k x)_k\|_T \xrightarrow[m\to\infty]{} 0,$$

where, in the latter step, we have used that c_T has property *T-AK*. Now, let ν be an E'-polar topology such that (P_k) is an equicontinuous T-decomposition of E endowed with ν.

Fix $u \in E'$, then there is a ν-equicontinuous set $D \subset E'$ such that

$$\sup_n |\langle T_n x, u\rangle| \leq \sup_{v\in D} |\langle x, v\rangle|.$$

This shows that $\gamma_T(E, E')$ is coarser than ν. ■

Example 5. If T is a Zeller-Buntinas matrix then the Garling topology on c_T coincides with the $\|\cdot\|_T$-topology. To see this, consider the squence $e = (1, 1, \ldots) \in (c_T)'$. Then for every $x = (x_k) \in c_T$, we have

$$\|\Delta_e x\|_T = \sup_n \left|\langle \tau_n x, e\rangle_{(c_T,(c_T)')}\right| = \sup_n \left|\sum_k t_{nk} x_k\right| = \|x\|_T.$$

This shows that the norm topology is coarser than the Garling topology. The converse follows from Proposition 1. (For the case of the Cesàro series to sequence summability matrix $C^1 \cdot \sum$ this fact was proved by Florencio (4), using different techniques.)

The dual of $E[\gamma_T(E, E')]$ can be bigger than E' (see Remark 2 below), but we can characterize it in the following way: Given $u \in E'$ and $a \in (c_T)'$ we may define a linear functional au on E by

$$\langle x, au\rangle := \langle \Delta_u(P_k x), a\rangle_{(c_T,(c_T)')}.$$

If T is a Zeller-Buntinas matrix, so that $(c_T)'$ is also a sequence space and we write $a = (a_k)$ then we have

$$\langle x, au\rangle = \left\langle \left(\langle P_k x, u\rangle\right)_k, (a_k)_k\right\rangle_{(c_T,(c_T)')} = T-\lim_k a_k \langle P_k x, u\rangle.$$

We denote by $(c_T)' \cdot E'$ the space of all linear functionals thus obtained.

Proposition 2. *Let T be a Zeller-Buntinas matrix and let (P_k) be a T-decomposition of a locally convex space E. Then the dual space of E endowed with its Garling topology is $(c_T)' \cdot E'$. In particular, $\gamma_T(E, E')$ is compatible with the dual pair if and only if $E' = (c_T)' \cdot E'$.*

Proof. Given $u \in E'$ and $a \in (c_T)'$ we have

$$|\langle x, au\rangle| = |\langle \Delta_u(P_k x), a\rangle| \leq \|\Delta_u(P_k x)\|_T \|a\|.$$

Therefore, au is $\gamma_T(E, E')$-continuous. Conversely, let z be a $\gamma_T(E, E')$-continuous linear functional on E. Then there exists $u \in E'$ such that $|\langle x, z\rangle| \leq \|\Delta_u(P_k x)\|_T$ for all $x \in E$. Identify E with its image in F via the injection $x \to (P_k x)$. This enables us to define a linear functional a by

$$a : \left(\langle P_k x, u\rangle\right) \in \Delta_u(E) \to \left\langle \left(\langle P_k x, u\rangle\right), a\right\rangle := \langle x, z\rangle$$

which, obiously, is well-defined and $\|\cdot\|_T$-continuous. By using that the subspaces (E_k) are non-trivial, it is easy to see that $\varphi \subset \Delta_u(E)$ so that this is a dense subspace of c_T. Finally, extend a to all of c_T by continuity to obtain

$$\langle x, z\rangle = \left\langle \left(\langle P_k x, u\rangle\right), a\right\rangle = T-\lim_k a_k \langle P_k x, u\rangle = \langle x, au\rangle$$

and the proof is finished. ■

Remarks. (1) If B_T stands for the unit ball of $(c_T)'$, the equality $E' = (c_T)' \cdot E'$ is equivalent to $E' = B_T \cdot E'$, and if this equality holds then E' is said to be *B_T-invariant,* as in the sequence space case (5), (9).

(2) If E' is B_T-invariant then the Garling topology is compatible with the dual pair (E, E') and so the sequence of functionals $(uT_n)_n$ is $\beta(E', E)$-bounded in E', in which case the T-decomposition is said to be *simple.* As there are non-simple Schauder basis (see the remarks following Def. 2.3 in (9)), it follows that not all Garling topologies are compatible.

(3) It is easy to see that a Toeplitz decomposition is simple if and only if the weak and the Garling topologies have the same family of bounded sets.

We study now when is $E[\gamma_T(E, E')]$ a complete space.

Proposition 3. *Let T be a Zeller-Buntinas matrix and let (P_k) be a T-decomposition of a locally convex space E. Then $E[\gamma_T(E, E')]$ is complete (resp. quasi-complete or sequentially complete) if and only if (P_k) is β_T-complete and each $E_k[\sigma(E_k, E_k')]$ is complete (resp. quasi-complete or sequentially complete).*

Proof. As we pointed out above, the Garling and the weak topology coincide on each E_k. Hence, according to Theorem 1, we have to prove that (P_k) is a complete T-decomposition of $E[\gamma_T(E, E')]$ if and only if it is a complete T-decomposition of E endowed with its weak topology; i.e., β_T-complete. The «if» part follows easily from the fact that the weak topology is coarser than the Garling topology.

So, assume that (P_k) is a complete T-decomposition of $E[\gamma_T(E, E')]$ and let $(x_k) \in \Pi\, E_k$ be such that $T \cdot (x_k)$ is a weakly-Cauchy sequence in E, it suffices to show that $T \cdot (x_k)$ is also a $\gamma_T(E, E')$-Cauchy sequence. Denote by z_n the n-th element of $T \cdot (x_k)$, that is $z_n = \sum_k t_{nk} x_k$. It is clear that $P_k z_n = t_{nk} x_k$ for all $n, k \in \mathbb{N}$ so, using (i) above, we have

$$\Delta_u(P_k z_n)_k = \Delta_u(t_{nk} x_k)_k = \tau_n\left(\Delta_u(x_k)_k\right).$$

Finally, using the γ_T-continuous seminorms as given in (ii), we obtain

$$\left\|\Delta_u\left(P_k(z_n - z_m)\right)_k\right\|_T = \left\|\tau_n\left(\Delta_u(x_k)_k\right) - \tau_m\left(\Delta_u(x_k)_k\right)\right\|_T$$

and this latter expresion goes to zero as $m, n \to \infty$ because $\Delta_u(x_k)$ is in c_T and this space has property T-AK. ■

Corollary. *Let T be a Zeller-Buntinas matrix and let (P_k) be a T-decomposition of a Banach space E. Then $\gamma_T(E, E')$ is a complete topology (resp. quasi-complete) if and only if (P_k) is β_T-complete and each E_k is finite-dimensional (resp. reflexive).*

Example 6. Let T be a Zeller-Buntinas matrix and E be a locally convex space with an equicontinuous T-decomposition (P_k). If v stands for the topology of E, then Proposition 1 tells us that

$$\sigma(E, E') \leq \gamma_T(E, E') \leq v.$$

Example 5 shows that for $E = c_T$ the first inequality is strict, but the second is an equality.

On the other hand, if E is an infinite dimensional Banach space, then the space $c_0(E)$ formed by the null sequences in E is a Banach space with a natural infinite-dimensional Schauder decomposition (7) and its Garling topology, which cannot be complete, does not coincide with its norm topology so that both inequalities are strict.

APPLICATION TO THE β_T-DUALITY OF SEQUENCE SPACES

In what follows, T is a Zeller-Buntinas matrix and λ stands for a sequence space containing φ. The β_T*-dual* of λ is the space λ^{β_T} of all sequences y such that the coordinatewise product xy is T-convergent for every $x \in \lambda$.

If λ is a K-space with property T-AK then it is clear that $\lambda' \subset \lambda^{\beta_T}$. If, in addition, λ is sequentially barrelled then $\lambda' = \lambda^{\beta_T}$ (see (2), (3), (12) and (14)). On the other hand, if λ has property T-AK then $\lambda \subset (\lambda')^{\beta_T}$ and, by Proposition 3, the equality holds if and only if $\lambda[\gamma_T(\lambda, \lambda')]$ is sequentially complete.

Assume now that no topology is defined *a priori* on λ. The natural bilinear form $(x, y) \rightarrow T\text{-lim}\, x_k y_k$ makes $(\lambda, \lambda^{\beta_T})$ a separated dual pair and, clearly, both $\lambda[\sigma(\lambda, \lambda^{\beta_T})]$ and $\lambda^{\beta_T}[\sigma(\lambda^{\beta_T}, \lambda)]$ have property T-AK. We may ask if there is a stronger topology on λ having property T-AK and still compatible with the duality $(\lambda, \lambda^{\beta_T})$. We shall characterize this topology by extending and combining results given by Garling (5) and Schaefer (18) for the β-duality (the duality defined in terms of ordinary summability).

Lemma. *Let T be a Zeller-Buntinas matrix and λ be a sequence space containing φ. Then the space $\lambda^{\beta_T}[\gamma_T(\lambda^{\beta_T}, \lambda)]$ is complete and for a set $C \subset \lambda^{\beta_T}$ the following conditions are equivalent*

(1) *C is $\gamma_T(\lambda^{\beta_T}, \lambda)$-relatively compact.*

(2) *C is $\gamma_T(\lambda^{\beta_T}, \lambda)$-bounded and the convergence of the sequence $(\tau_n y)$ to y in the Garling topology $\gamma_T(\lambda^{\beta_T}, \lambda)$ is uniform with respect to $y \in C$.*

(3) C is *$\gamma_T(\lambda^{\beta_T}, \lambda)$-bounded and for every $x \in \lambda$ the convergence of the sequence $\tau_n(y)$ to xy in the $\|\cdot\|_T$-topology is uniform with respect to $y \in C$.*

Proof. According to Proposition 3, to prove that $\lambda^{\beta_T}[\gamma_T(\lambda^{\beta_T}, \lambda)]$ is complete we have to show that (π_k) is a complete T-decomposition of $\lambda^{\beta_T}[\sigma(\lambda^{\beta_T}, \lambda)]$, but this follows from the very definition of β_T-dual.

By Proposition 1, (π_n) is an equicontinuous T-decomposition of $\lambda^{\beta_T}[\gamma_T(\lambda^{\beta_T}, \lambda)]$; i.e., (τ_n) is a $\gamma_T(\lambda^{\beta_T}, \lambda)$-equicontinuous sequence of operators that converges pointwise to the identity on λ^{β_T} hence, by using (11, §39.4(1)), we have that (τ_n) converges uniformly on $\gamma_T(\lambda^{\beta_T}, \lambda)$-compact sets. Since $\tau_n(\lambda^{\beta_T})$ is finite-dimensional for every $n \in \mathbb{N}$, it follows that for every $n \in \mathbb{N}$, the set $\tau_n(C)$ is relatively compact provided that C is $\gamma_T(\lambda^{\beta_T}, \lambda)$-bounded. Then Mazur's Theorem (8, Thm. 1) implies that conditions (1) and (2) are equivalent. The equivalence of (2) and (3) is clear. ■

Theorem 2. *Let T be a Zeller-Buntinas matrix and λ be a sequence space containing φ. Then the stronger topology on λ that has property T-AK and is compatible with the duality $(\lambda, \lambda^{\beta_T})$ is the topology $k_T(\lambda, \lambda^{\beta_T})$ of uniform convergence on the absolutely convex and $\gamma_T(\lambda^{\beta_T}, \lambda)$-compact subsets of λ^{β_T}.*

Proof. That the dual of $\lambda[k_T(\lambda, \lambda^{\beta_T})]$ equals λ^{β_T} follows from the Mackey-Arens's Theorem and the fact that the Garling topology is stronger than the weak topology. That $\lambda[k_T(\lambda, \lambda^{\beta_T})]$ has property T-AK follows from the previous lemma by simply noting that $\langle x - \tau_n(x), y\rangle = \langle x, y - \tau_n(y)\rangle$ for all $x \in \lambda$, $y \in \lambda^{\beta_T}$ and $n \in \mathbb{N}$. Finally, if $C \subset \lambda^{\beta_T}$ is an absolutely convex and $\sigma(\lambda^{\beta_T}, \lambda)$-compact set such that

$$\lim_n \sup_{y \in C} |\langle x - \tau_n(x), y\rangle| = 0 \quad \text{fo all } x \in \lambda,$$

then, again by the lemma, C is $\gamma_T(\lambda^{\beta_T}, \lambda)$-compact. This implies that $k_T(\lambda, \lambda^{\beta_T})$ is the stronger topology satisfying the desired properties. ■

Corollary 1. *Let λ be a sequence space containing φ. Then the stronger topology on λ that has property AK and is compatible with the duality $(\lambda, \lambda^{\beta})$ is the topology $k_T(\lambda, \lambda^{\beta})$ of uniform convergence on the absolutely convex and $\sigma\gamma(\lambda^{\beta}, \lambda)$-compact subsets of λ^{β}.*

This corollary can also be obtained by combining (5, Prop. 11) with (18, remarks following Prop. 4).

Corollary 2. *Let T be a Zeller-Buntinas matrix and λ be a sequence space containing φ. Then $\lambda[\beta(\lambda, \lambda^{\beta_T})]$ has* *property T-AK if and only if every* $\sigma(\lambda^{\beta_T}, \lambda)$*-bounded set is* $\gamma_T(\lambda^{\beta_T}, \lambda)$*-relatively compact.*

Proof. Use the theorem plus the fact that if λ is a *K*-space having property *T-AK* then its topological dual is contained in its β_T-dual.

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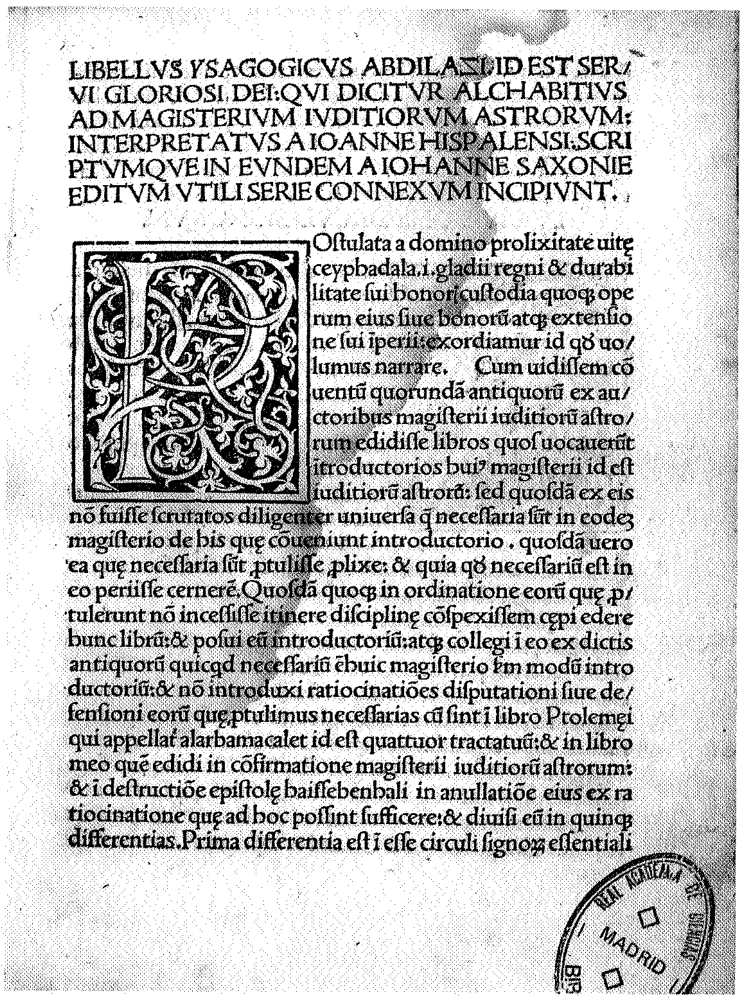

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