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# WELL-BALANCED FINITE VOLUME SCHEMES: SOME STABILITY AND CONVERGENCE RESULTS

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**Abstract.** We report a stability and convergence analysis for some simplified well-balanced Finite Volume solvers of Hyperbolic Systems of Conservation Laws. These are specific solvers, recently introduced, that balance all steady solutions up to second order of accuracy by means of an additional numerical source term. We prove the stability and convergence of some of these solvers for scalar hyperbolic equations under reasonable conditions on the additional term.

*Keywords:* Finite Volumes, well-balanced solvers, stability, convergence

*AMS classification:* AMS 65N06, 76B15, 76M20, 76N99

## §1. Introduction and motivation

This work deals with the analysis of a class of solvers for 1D Non-homogeneous Hyperbolic Systems of Conservation Laws, designed to solve all steady solutions with high order accuracy. This class of solvers is based upon adding a numerical correcting term to the source term, that allows to balance up to second order all smooth steady solutions. Combined with efficient discretizations of the hyperbolic operator, it provides accurate and stable solutions of one and two-layers shallow water flows (See [3], [4], [6]).

For Non-homogeneous Hyperbolic Systems of Conservation Laws, it is well-known the relevance of computing the steady solutions with high accuracy. This allows to obtain accurate transient solutions. Otherwise, the numerical solution is affected by large errors, particularly in the speed of waves, yielding unacceptable solutions from the physical point of view.

This has been remarked by several authors since Roe, in 1986, observed the convenience of using 2nd order accurate quadrature formulae to approximate the contribution of the source terms (See [9]). This was formalised as the so-called “C-property" by Bermúdez and Vázquez in 1994 (See [1]), that asks for the numerical solution to balance either exactly or at least up to 2nd order of accuracy some actual steady solution of the system. In 1996 Greenberg and Leroux call this property as “well-balanced" solver, introducing a scheme for which they prove the convergence to the entropy solution for scalar equations. In a recent paper (See [3]), Chacón, Domínguez and Fernández Nieto relax this property to the concept of “asymptotically well-balanced" scheme, still asking that the steady solutions are computed up to second order, on all the domain but on a set whose measure tends to zero as the grid size tends to zero. In that paper a systematic deduction of a general class of solvers satisfying this property is performed.

This paper focuses on analysis aspects of the solvers introduced in [3]. A relevant result on this subjects was reported in 2003 by Perthame and Simeoni (See [8]). An extension of the Lax-Wendroff theorem to non-homogeneous scalar equations is proved. This result holds provided certain equilibria of the equations are balanced by the solver. We shall also analyze scalar equations. Up to our knowledge, there is still no complete proof of convergence for hyperbolic systems.

We start by motivating these schemes from a methodological point of view. Let us consider the standard upwind Finite Volume solution of the linear scalar equation,

$$\frac{\partial w}{\partial t} + a\,\frac{\partial w}{\partial x} = g(x), \quad \text{in} \quad ]0,L[\times]0,T[, \quad w(x,0) = w_0(x), \tag{1}$$

where $a$ is a constant speed and $w_0$ is the initial datum.

Let us consider a steady smooth solution $w(x)$ of (1). We consider conservative Finite Volume solvers, based upon integrating the equation on the control volumes $(x_{i-1/2}, x_{i+1/2})$, where $x_i = i\Delta x$. The simplest upwinding scheme corresponds to the discretization

$$a\,\frac{\partial w}{\partial x}(x_i) \simeq \begin{cases} a\dfrac{w_i - w_{i-1}}{\Delta x} & if \quad a > 0, \\ a\dfrac{w_{i+1} - w_i}{\Delta x} & if \quad a < 0; \end{cases}$$

that we may re-writte as the sum of a centered plus a de-centered part, as

$$a\,\frac{\partial w}{\partial x}(x_i) \simeq a\frac{w_{i+1} - w_{i-1}}{2\Delta x} - \frac{1}{2}\,|a|\,\Delta x \frac{w_{i+1} - 2w_i + w_{i-1}}{(\Delta x)^2}.$$

The consistency error is given by

$$a\frac{w_{i+1} - w_{i-1}}{2\Delta x} = a\,\frac{\partial w}{\partial x}(x_i) + O(\Delta x)^2;$$

$$\frac{1}{2}\,|a|\,\Delta x \frac{w_{i+1} - 2w_i + w_{i-1}}{(\Delta x)^2} = \frac{1}{2}\,|a|\,\Delta x\,\frac{\partial^2 w}{\partial x^2}(x_i) + O(\Delta x)^3.$$

If we consider a centered approximation of the source term, for instance

$$g(x_i) = \frac{g(x_{i+1/2}) + g(x_{i-1/2})}{2},$$

then the consistency error is second order accurate. As a consequence, there exist a first-order error, due to the upwinding of the flux, which is not compensated. However, if we perform a de-centered discretization of the source term,

$$g(x_i) \simeq \begin{cases} g(x_{i-1/2}) & if \quad a < 0, \\ g(x_{i+1/2}) & if \quad a > 0, \end{cases}$$

we obtain (for instance, when $a > 0$),

$$g(x_{i-1/2}) = a\,\frac{\partial w}{\partial x}(x_i) - \frac{1}{2}\,a\,\Delta x\,\frac{\partial^2 w}{\partial x^2}(x_i) + O(\Delta x)^2.$$

Consequently, the consistency error is $O(\Delta x^2)$, so the de-centered scheme is well-balanced up to second order. For homogeneous equations, this second order accuracy for steady solutions is automatically achieved by this upwind scheme.

The key point is that centered discretizations have a second order consistency error. However, a specific upwinding of the source term is needed to compensate the first order error introduced by the upwinding of the flux.

The resulting scheme, including the case $a < 0$, is

$$w_i^{n+1} = w_i^n - \Delta t \left[ \frac{\phi(w_i^n, w_{i+1}^n) - \phi(w_{i-1}^n, w_i^n)}{\Delta x} + \mathcal{G}_i \right], \tag{2}$$

where the numerical flux function is

$$\phi(w_i^n, w_i^{n+1}) = a\, \frac{w_i^n + w_i^{n+1}}{2} - \frac{1}{2}\, |a|\, (w_{i+1} - w_i), \tag{3}$$

and the numerical source term is

$$\mathcal{G}_i = \mathcal{G}_i^c + \mathcal{G}_i^d, \text{ with } \mathcal{G}_i^c = \frac{g(x_{i+1/2}) + g(x_{i-1/2})}{2}, \text{ and } \mathcal{G}_i^d = -\frac{1}{2}\, sgn(a)\, \big( g(x_{i+1/2}) - g(x_{i-1/2}) \big), \tag{4}$$

assuming that $sgn(a) = 0$ when $a = 0$.

We may re-interpret this scheme as a centered discretization of an equivalent equation with a specific additional part to the source term. Indeed, observe that

$$\mathcal{G}_i^d = -\frac{1}{2}\, sgn(a)\, \Delta x\, \frac{\partial g}{\partial x}(x_i) + O(\Delta x)^3.$$

Then, scheme (2)-(4) is a second order approximation at $x = x_i$ of the equivalent differential (parabolic) PD equation

$$\frac{\partial w}{\partial t} + a\, \frac{\partial w}{\partial x} - \frac{1}{2}\, sgn(a)\, \Delta x\, \frac{\partial^2 w}{\partial x^2} = g(w) - \frac{1}{2}\, sgn(a)\, \Delta x\, \frac{\partial g}{\partial x}.$$

Second order approximations of this equation automatically are balanced up to second order for all steady solutions of the original hyperbolic equation (1). This is achieved by adding the “numerical source term"

$$c(w) = -\frac{1}{2}\, sgn(a)\, \Delta x\, \frac{\partial g}{\partial x}$$

to the standard equivalent equation associated to the upwinding scheme we are considering,

$$\frac{\partial w}{\partial t} + a\, \frac{\partial w}{\partial x} - \frac{1}{2}\, sgn(a)\, \Delta x\, \frac{\partial^2 w}{\partial x^2} = g(w).$$

For numerical solvers other than the pure upwinding considered here, specific additional source terms must be added to achieve this purpose.

## §2. Balanced schemes for general hyperbolic systems

In this section we describe the derivation of the schemes introduced in [3] and [4] for general hyperbolic equations.

We consider a general non-homogeneous hyperbolic system

$$\frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} = G(x, W), \tag{5}$$

where $W : [0, L] \mapsto \mathbb{R}^n$ is the n-th dimensional unknown, $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ is the flux and $G : [0, L] \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is the source term.

The general conservative upwind Finite Volume solvers of this system are second order approximations of the equivalent parabolic system

$$\frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} - \nu \frac{\partial}{\partial x}(\mathcal{D}(W)\frac{\partial W}{\partial x}) = G(x, W), \tag{6}$$

where $\nu = \Delta x/2$ and $\mathcal{D}$ is the characteristic diffusion matrix of the scheme. Both Flux-Difference and Flux-Splitting schemes fit into this general framework.

Following the main ideas mentioned in the introduction above, [3] and [4] propose to add a numerical source term to this system,

$$\frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} - \nu \frac{\partial}{\partial x}(\mathcal{D}(W)\frac{\partial W}{\partial x}) = G(x, W) + C(x, W), \tag{7}$$

in such a way that all smooth steady solutions of the original system (5) are also solutions of this modified equivalent system. To do this, let us consider a steady solution $W(x)$ of (5). It satisfies

$$\frac{\partial F(W)}{\partial x} = G(x, W). \tag{8}$$

Thus, it is enough to have

$$-\nu \frac{\partial}{\partial x}(\mathcal{D}(W)\frac{\partial W}{\partial x}) = C(x, W).$$

Due to (8), if we denote by $A(W)$ the Jacobian matrix of $F$, we have

$$A(W)\,\frac{\partial W}{\partial x} = G(x, W).$$

Then, if $A(W)$ is non-singular at some point x, we have

$$\frac{\partial W}{\partial x} = A^{-1}(W)\,G(x, W)$$

at that point. This suggests to define the additional term as

$$C(x, W) = -\nu \frac{\partial}{\partial x}\left(\mathcal{D}(W)A^{-1}(W)\,G(x, W)\right).$$

Now, to build balanced schemes for the hyperbolic system (5), it is enough to use approximations of 2nd order in space of the equivalent equation

$$\frac{\partial W}{\partial t} + \frac{\partial \tilde{F}(W)}{\partial x} = G(x, W), \tag{9}$$

with

$$\tilde{F}(W) = F(W) - \nu\,\mathcal{D}(W)\,\left(\frac{\partial W}{\partial x} - A^{-1}(W)\,G(x,W)\right).$$

To ensure the conservation property, [3] and [4] propose to consider schemes with the structure

$$\frac{W_i^{n+1} - W_i^n}{\Delta t} + \frac{\phi_{i+1/2}^n - \phi_{i-1/2}^n}{\Delta x} = \mathcal{G}_{c,i}^n, \tag{10}$$

where $\phi_{i+1/2}^n$ and $\mathcal{G}_{c,i}^n$ respectively are second order approximations of $\tilde{F}(W)$ and $G(x,W)$ at $x = x_{i+1/2}$ and $x = x_i$ at time $t = t_n$. Concretely,

$$\phi_{i+1/2} = F_c(W_i, W_{i+1}) - \nu\, D(W_i, W_{i+1})\,\left(\frac{W_{i+1} - W_i}{\Delta x} - \widetilde{A^{-1}}(W_i, W_{i+1}) G_d(x_i, x_{i+1}, W_i, W_{i+1})\right),$$

where $F_c$ is a second order approximation of $F$ and $D$, $\widetilde{A^{-1}}$, $G_d$ are first order approximations of $\mathcal{D}$, $A^{-1}$ and $G$, respectively, all of them at $W_{i+1/2}$.

To define $\widetilde{A^{-1}}$ when $A$ is singular, assume that there exist $N$ continuous surfaces $\gamma_j$, $j = 1, \cdots, N$ such that the $j$-th eigenvalue $\lambda_j$ of $A$ is non-zero in $\mathbb{R}^N - \gamma_j$. The following definition of $\widetilde{A^{-1}}$ is proposed:

$$\widetilde{A^{-1}}(U,V) = X(\widetilde{W}(U,V))\,\widetilde{\Lambda^{-1}}(W(U,V))\,X^{-1}(\widetilde{W}(U,V)),$$

where $X$ is the matrix formed by the eigenvectors of $A$, $\tilde{W}$ is some intermediate state between $U$ and $V$, and $\widetilde{\Lambda^{-1}}$ is the diagonal matrix defined by

$$\widetilde{\Lambda^{-1}}(W(U,V)) = Diag(\widetilde{\lambda_1^{-1}}(W(U,V), \cdots, \widetilde{\lambda_N^{-1}}(W(U,V))$$

$$\text{with } \widetilde{\lambda_j^{-1}} = \begin{cases} \lambda_j^{-1}(\widetilde{W}) & \text{if } L[U,V] \subset \mathbb{R}^N - \gamma_j, \\ 0 & \text{otherwise}, \end{cases}$$

where $L[U,V]$ is the segment in $\mathbb{R}^N$ that connects $U$ and $V$.

With this definition, no upwinding of the source term is performed in the $j$-th characteristic field if the associated eigenvalue $\lambda_j$ vanishes at some point in the segment $L[U,V]$. This is coherent with the physical intuition, as in this case there is no a clear upwind direction. As we shall see below, this guarantees the stability of the scheme, at least for scalar equations.

This general method is proved to be asymptotically well-balanced up to second order of accuracy for steady solutions in several relevant situations: This occurs if either the states $W$ at which $A(W)$ is singular are isolated, or the eigenvalues of $\mathcal{D}$ and $A$ vanish at the same points. By "asymptotically well-balanced" is meant that here exist an increasing sequence of compact sets $\{K_n\}_{n\geq 1}$ such that

1. The measure of $]0, L] - \bigcup_{n\geq 1} K_n$ is zero,
2. All steady solutions satisfy the numerical scheme on $K_n$, provided $\Delta x$ is below a (small) positive value depending on $K_n$.

## §3. Convergence and stability analysis

This section is devoted to the analysis of scheme (2)-(4) for scalar equations ($N = 1$). In this case we denote the unknown by $w$, the flux by $f$, its Jacobian matrix (its derivative, in fact) by $A$ and the source term by $g$. For the sake of brevity, we shall consider the simplest upwind scheme, that may be written as

$$w_i^{n+1} = w_i^n - \Delta t \left[ \frac{\phi(w_i^n, w_{i+1}^n) - \phi(w_{i-1}^n, w_i^n)}{\Delta x} + \mathcal{G}_i \right], \tag{11}$$

where

$$\phi(w_i, w_{i+1}) = a(w_{i+1/2})\, w_{i+1/2} - \frac{1}{2}\, |a(w_{i+1/2})|\, (w_{i+1} - w_i),$$

and

$$\begin{aligned} \mathcal{G}_i^n &= \frac{1}{2}\, [g(x_{i-1/2}, w_{i-1/2}^n) + g(x_{i+1/2}, w_{i+1/2}^n)] - \\ &- \frac{1}{2}\, [sgn(a(w_{i+1/2}^n))g(x_{i+1/2}, w_{i+1/2}^n) - sgn(a(w_{i-1/2}^n))g(x_{i-1/2}, w_{i-1/2}^n)]. \end{aligned}$$

We shall denote by $W_h$ the piecewise constant function that takes the value $w_i^n$ on the cell $[x_{i-1/2}, x_{i+1/2}[\times[t_n, t_{n+1}[$. We shall also denote by $W^n$ the function $W_h(\cdot, t_n)$.

We may now state our main result :

**Theorem 1.** *Assume $f \in C^2(\mathbb{R})$, $a \in L^\infty(\mathbb{R})$, $g \in W^{1,\infty}(\mathbb{R} \times \mathbb{R})$, and the initial condition $w_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ with compact support. Then, under the CFL condition*

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{2 \max\limits_{\mathbb{R}} |a|}$$

*the scheme is $L^\infty$ and $BV$ stable. Concretely, the following estimates hold,*

$$\max_i |w_i^n| \leq \|w_0\|_\infty + t_n\, \|g\|_\infty, \quad 0 \leq t_n \leq T;$$

$$TV(w_n) \leq e^{A\, t_n}\, TV(w_0) + \frac{B}{A}\, \left(e^{A\, t_n} - 1\right), \quad 0 \leq t_n \leq T;$$

*where*

$$A = \|\partial_w g\|_\infty, \quad B = M(T)\, \|\partial_x g\|_\infty,$$

*for some bounded increasing function $M(T)$.*

*As a consequence, the sequence $\{W_h\}_{h>0}$ converges strongly in $L^1_{loc}(\mathbb{R}\times]0,T[)$ and weak-* in $L^\infty(\mathbb{R}\times]0,T[)$ to the entropy solution of the Cauchy problem*

$$\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} = g(x, w) \quad \textit{in } \mathbb{R}\times]0,T[; \quad w(x,0) = w_0(x) \quad \textit{in } \mathbb{R} \tag{12}$$

*Proof.* For brevity, we shall prove the theorem for the non-linear equation (1), when $g = g(w)$. Most of the complexity of the proof is in this case. At first, we write the scheme (11) as

$$w_i^{n+1} = [1 - (c + d)]\, w_i^n + c\, w_{i-1}^n + d\, w_{i+1}^n + \Delta t\, (\alpha\, g_{i-1/2}^n + \beta\, g_{i+1/2}^n),$$

where

$$c = \frac{\Delta t}{\Delta x}\,\frac{1}{2}\,[|a| - a], \quad d = \frac{\Delta t}{\Delta x}\,\frac{1}{2}\,[|a| + a],$$
$$\alpha = \frac{1}{2}\,(1 + sgn(a)), \quad \beta = \frac{1}{2}\,(1 - sgn(a)).$$

Observe that under the above CFL condition,

$$0 \le c, \quad 0 \le d, \quad c + d \le 1.$$

Also,

$$0 \le \alpha, \quad 0 \le \beta, \quad \alpha + \beta = 1.$$

Then,

$$\begin{aligned} |w_i^{n+1}| &\le [1 - (c+d)]\,|w_i^n| + c\,|w_{i-1}^n| + d\,|w_{i+1}^n| + \Delta t\,(\alpha\,|g_{i-1/2}^n| + \beta\,|g_{i+1/2}^n|), \\ &\le \max_i |w_i^n| + \Delta t\,\|g\|_\infty. \end{aligned}$$

From here we deduce the $L^\infty$ estimate,

$$\max_i |w_i^n| \le \max_i |w_i^0| + t_n\,\|g\|_\infty.$$

To obtain the bounded variation estimate, observe that

$$TV(w^n) = \sum_i |w_{i+1}^n - w_i^n|.$$

We have

$$\begin{aligned} w_{i+1}^{n+1} - w_i^{n+1} &= [1 - (c+d)]\,(w_{i+1}^n - w_i^n) + c\,(w_i^n - w_{i-1}^n) + d\,(w_{i+2}^n - w_{i+1}^n) \\ &+ \Delta t\,[\alpha\,(g_{i+1/2}^n - g_{i-1/2}^n) + \beta\,(g_{i+3/2}^n - g_{i+1/2}^n)\,]. \end{aligned}$$

By the mean value theorem, there exists some $z_i^n \in (w_{i-1/2}^n, w_{i+1/2}^n)$ such that

$$g_{i+1/2}^n - g_{i-1/2}^n = g'(z_i^n)\,(w_{i+1/2}^n - w_{i-1/2}^n) = \frac{1}{2}\,g'(z_i^n)\,\left[\,(w_{i+1}^n - w_i^n) + (w_i^n - w_{i-1}^n)\,\right].$$

Then,

$$\begin{aligned} \sum_i |w_{i+1}^{n+1} - w_i^{n+1}| &\le [1 - (c+d)]\,\sum_i |w_{i+1}^n - w_i^n| + c\,\sum_i |w_i^n - w_{i-1}^n| + d\,\sum_i |w_{i+2}^n - w_{i+1}^n| \\ &+ \frac{\Delta t}{2}\,\|g'\|_\infty\,\Bigg[\alpha \sum_i \left(|w_{i+1}^n - w_i^n| + |w_i^n - w_{i-1}^n|\right) \\ &+ \beta \sum_i \left(|w_{i+2}^n - w_{i+1}^n| + |w_{i+1}^n - w_i^n|\right)\Bigg] \\ &\le TV(w^n)\ (1 + \|g'\|_\infty) \le \cdots \le TV(w^0)\ (1 + \|g'\|_\infty)^{n+1} \\ &\le TV(w^0)\,e^{t_{n+1}\,\|g'\|_\infty}. \end{aligned}$$

We next consider that our scheme is consistent with the scalar equation (12) and also consistent with the entropy condition, due to the numerical diffusion terms $(-\frac{1}{2}\,|a|\,(w_{i+1}-w_i))$ introduced by the upwinding in the numerical flux function $\phi$. This, together with the $L^\infty$ and $BV$ estimates, proves the convergence in $L^1_{loc}$ to the unique entropy condition, following the standard theory (Cf. Godlevski-Raviart [7]). The $L^\infty$ weak-* convergence follows from the uniform $L^\infty$ bounds. $\square$

The key point to obtain the essential estimates is the property $\alpha \geq 0,\ \beta \geq 0,\ \alpha+\beta = 1$ satisfied by the upwinding coefficients of the numerical source term. This follows (in both the linear and the non-linear cases) from the definition of the matrix $\widetilde{A^{-1}}$, concretely from the fact that the approximate inverse eigenvalue $\widetilde{\lambda^{-1}}(U,V)$ vanishes if the eigenvalue $\lambda$ vanishes in the segment $L[U,V]$.

## §4. Application to bi-layer Shallow-Water equations

The bi-layer Shallow-Water equations model the flow of two non-miscible stratified (due to different densities) fluids. It is a system of non-homogeneous non-conservative hyperbolic equations whose unknowns are the wet section and the discharge of each layer. It may be written under the structure (See [2])

$$\frac{\partial W}{\partial t} + A(x,W)\frac{\partial W}{\partial x} = G(x,W), \tag{13}$$

but there is no flux function $F(x,W)$ such that $A(x,W)$ is its Jacobian matrix. Matrix $A$ has real eigenvalues –excepting for some states corresponding to interface Kelving-Helmoltz instabilities– and is diagonalizable.

The general structure of the scheme (10) allows a straightforward extension to equations with the structure (13). In [6] such extension is performed, and applied to simulate lock-exchange flows. These are steady-state flows corresponding to initial conditions in which two fluids of different densities are separated by a vertical wall, which is drop at the initial moment (See [2]). Such flows are usually considered as model flows for water exchange between seas

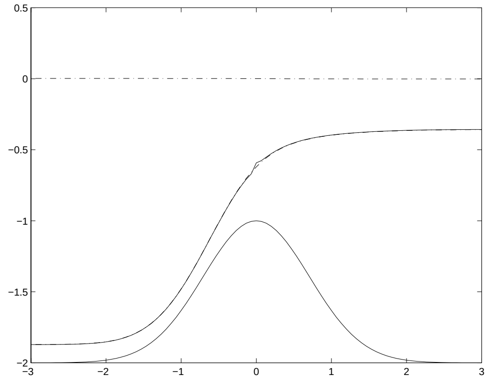


Figure 1: Steady solution for maximal lock-exchange flow. Variable topography. Solid line: Actual solver. Dotted line:Roe's solver.

of different densities separated by a straight. A specific treatment to stabilize the formation of interface Kelvin-Helmholtz instabilities is also introduced in [6].

In Figures 1 and 2 we represent the steady state for a particular lock-exchange experiment, corresponding to maximal exchange between the two adjacent seas (Armi-Farmer solutions, see [5]). The solution corresponding to the actual scheme is compared to that provided by the extension of Roe's scheme introduced in [2]. Both solutions coincide in all the domain for both layers, excepting on a rarefaction wave formed in the internal layer, where the actual scheme provides a more accurate solution.

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