


# A *Mathematica* Notebook for Computing the Homology of Iterated Products of Groups

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**Abstract.** Let  $G$  be a group which admits the structure of an iterated product of central extensions and semidirect products of abelian groups  $G_i$  (both finite and infinite). We describe a *Mathematica 4.0* notebook for computing the homology of  $G$ , in terms of some homological models for the factor groups  $G_i$  and the products involved. Computational results provided by our program have allowed the simplification of some of the formulae involved in the calculation of  $H_n(G)$ . Consequently the efficiency of the method has been improved as well. We include some executions and examples.

## 1 Introduction

The calculation of the homology of a given group is in general a difficult task. From a theoretical point of view, spectral sequences and resolutions have been traditionally used to solve the question. But calculations could not be carried out in practice, due to the complexity of the processes involved in the computations.

In the past decade, the interest in explicitly compute both the (co)homology and the correspondent representative (co)cycles of a group, has increased surprisingly, accordingly to their applications to very different fields, such as coding theory and cryptography.

This situation has motivated that most of Computer Algebra Systems (CAS) are now concerned about the design of functions for achieving homological calculations.

For instance, GAP (*Group, algorithms, programming* [9]) includes a *homology package* [7], which is concerned with the calculation of the homology of simplicial complexes and the Smith's normal forms of (preferably sparse) integer matrices. MAGMA [18] includes a routine for computing the homology of  $p$ -groups [5]. The KENZO system [6] provides an environment for achieving calculations in the framework of *effective homology* [21]. More recently, Ellis is developing a homological algebra library for use with the GAP computer algebra system (termed HAP [11], *homological algebra programming*), which intends to be a complete tool to make basic calculations in the cohomology of finite and infinite groups.

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Though the KENZO system has incorporated some routines for the calculation of the homology of central extensions (in light of the work in [20]), as far as we know, none of these packages handles the class of groups that we are concerned with. We intend to cover this gap with the *Mathematica* notebook [1] which is presented here, for computing the homology of iterated products of central extensions and semidirect products of abelian groups.

The main reason for which we have decided to work on *Mathematica* instead of other CAS is that all the systems cited above work in terms of resolutions. We prefer to work in terms of “contractions”, that is, at the level of reduced complexes of resolutions, following the philosophy in [8,21]. In most cases, both ways turn out to be equivalent [4]. Our preferences come from the fact that we want to elude the recursive formulae provided by the comparison theorem for resolutions. We prefer to work with explicit formulae from the first moment. For instance, the formulae in [19] for the maps involved in the Eilenberg-Zilber theorem, will allow us to perform explicit formulae for the maps describing a homological model for the iterated products that we are concerned with.

Furthermore, our method may be extended to cover iterated products of other groups for which homological models are known, such as finitely generated torsion-free nilpotent groups [17,15], finite  $p$ -groups [16,10], finitely generated two-step nilpotent groups [12] and metacyclic groups [13].

We have programmed the formulae of [3] for constructing a homological model for a semidirect product of abelian groups, and the formulae implicitly described in [20] for constructing a homological model for a central extension of abelian groups.

The term *homological model* refers to a contraction  $\phi: \bar{B}(\mathbf{Z}[G]) \xrightarrow{f} hG$  from the *reduced bar construction* of the group  $G$  (i.e. the reduced complex associated to the standard bar resolution) to a differential graded module of finite type  $hG$ , so that

$$H_*(G) = H_*(\bar{B}(\mathbf{Z}[G])) = H_*(hG)$$

and the homology of  $hG$  may be effectively computed by means of Veblen’s algorithm [22] (involving the Smith’s normal forms of the matrices representing the differential operator).

Here  $\phi: \bar{B}(\mathbf{Z}[G]) \xrightarrow{f} hG$  denotes a *contraction*, a special type of homotopy equivalence, where apart from the usual relations  $fg = 1$ ,  $1 - gf = d\phi + \phi d$ , the annihilation properties  $f\phi = 0$ ,  $\phi g = 0$ ,  $\phi\phi = 0$  are satisfied.

We must note that a routine in *Mathematica* which calculates the Smith’s normal form of a matrix over the integers was provided a decade ago by David Jabon [14].

Using this package and the formulae above, the notebook finally computes the homology of the input group.

We organize the paper as follows. Section 2 is devoted to describe the formulae concerning the homological models for central extensions and semidirect

products of abelian groups described in [20] and [3], respectively. The notebook itself is described in Section 3. In Section 4 we prove some results about simplifications on the formulae, which had been previously conjectured, attending to some output data provided by our program. Section 5 is devoted to show some executions and examples.

## 2 Describing Homological Models for the Factors

Let  $G$  be an iterated product of central extensions and semidirect products of abelian groups  $G_i$ ,  $1 \leq i \leq n$ . In this section, we describe a homological model for  $G$  in terms of some homological models for each of the factor groups  $G_i$ .

### 2.1 A Homological Model for $\mathbf{Z}$

Let  $E(u)$  denote the free DGA-algebra endowed with trivial differential and generators 1 (at degree 0) and  $u$  (at degree 1), so that  $u \cdot u = 0$ .

The comparison theorem for resolutions provides a homological model for  $\mathbf{Z}$  (see [4] for details),  $\phi_{\mathbf{Z}}: \bar{B}(\mathbf{Z}[\mathbf{Z}]) \xrightarrow[\mathbf{g}_{\mathbf{Z}}]{f_{\mathbf{Z}}} E(u)$ , which is a subtle modification of that in [8] (they differ just in the homotopy operator  $\phi_{\mathbf{Z}}$ ).

Here  $\mathbf{g}_{\mathbf{Z}}(u) = [1]$ ,  $f_{\mathbf{Z}}([n_1 | \dots | n_q]) = \begin{cases} n_1 u, & \text{if } q = 1 \\ 0, & \text{if } q > 1 \end{cases}$  and

$$\phi_{\mathbf{Z}}[n_1 | \dots | n_k] = \begin{cases} (-1)^k \sum_{i=1}^{n_k-1} [n_1 | \dots | n_{k-1} | i | 1], & \text{if } n_k > 1 \\ 0, & \text{if } n_k = -1, 0, 1 \\ (-1)^k \sum_{i=1}^{-n_k-1} [n_1 | \dots | n_{k-1} | -i | 1], & \text{if } n_k < 1 \end{cases} \quad (1)$$

### 2.2 A Homological Model for $\mathbf{Z}_n$

Let  $\Gamma(v)$  denote the free DGA-algebra endowed with trivial differential and generators  $\gamma_k(v)$  (at degree  $2k$ ,  $k \geq 0$ ,  $\gamma_0(v) = 1$ ), such that

$$\gamma_k(v)\gamma_h(v) = \frac{(k+h)!}{k!h!} \gamma_{k+h}(v)$$

In [8] a homological model  $\phi_{\mathbf{Z}_n}: \bar{B}(\mathbf{Z}[\mathbf{Z}_n]) \xrightarrow[\mathbf{g}_{\mathbf{Z}_n}]{f_{\mathbf{Z}_n}} (E(u) \otimes \Gamma(v), d)$  for  $\mathbf{Z}_n$  is also described, such that  $d(u) = 0$ ,  $d(u \otimes v) = n \cdot u$  and  $\mathbf{g}_{\mathbf{Z}_n}(u) = [1]$ ,

$$\mathbf{g}_{\mathbf{Z}_n}(\gamma_k(v)) = \sum_{x_i \in \mathbf{Z}_n} [1|x_1 | \dots | 1|x_k], \quad \mathbf{g}_{\mathbf{Z}_n}(u\gamma_k(v)) = \sum_{x_i \in \mathbf{Z}_n} [1|x_1 | \dots | 1|x_k | 1]$$

$$f_{\mathbf{z}_n}[x_1|y_1|\cdots|x_m|y_m] = \left[ \prod_{i=1}^m \delta_{x_i, y_i} \right] \gamma_m(v),$$

$$f_{\mathbf{z}_n}[x_1|y_1|\cdots|x_m|y_m|z] = \left[ z \prod_{i=1}^m \delta_{x_i, y_i} \right] u \gamma_m(v),$$

$$\text{for } \delta_{x_i, y_i} = \begin{cases} 0, & x_i + y_i < n \\ 1, & x_i + y_i \geq n \end{cases}$$

and  $\phi_{\mathbf{z}_n}([x_1|\cdots|x_k]) = -\varphi_{\mathbf{z}_n}([x_1|\cdots|x_k])$ , for  $\varphi_{\mathbf{z}_n}[\ ] = 0$ ,  $\varphi_{\mathbf{z}_n}[x] = \sum_{i=1}^{x-1} [1|i]$ ,

$$\varphi_{\mathbf{z}_n}[x|y|\sigma] = \sum_{i=1}^{x-1} [1|i|\sigma] + \delta_{x,y} \sum_{k=1}^{n-1} [1|k|\varphi_{\mathbf{z}_n}\sigma] \quad (2)$$

### 2.3 A Homological Model for a Central Extension

Here  $A \kappa_\alpha G$  denotes the central extension of  $A$  and  $G$  by means of the 2-cocycle  $\alpha : G \times G \rightarrow A$ , such that  $(a, g)(a', g') = (a + a' + \alpha(g, g'), g + g')$ . In case that  $\alpha$  is a 2-coboundary, then  $A \kappa_\alpha G$  is isomorphic to the direct product  $A \times G$ .

A homological model  $\phi_{AG} : \bar{B}(\mathbf{Z}[A \kappa_\alpha G]) \xrightarrow{f_{AG}} (hAG, d_{AG})$  for a central extension  $A \kappa_\alpha G$ , for  $A$  being an abelian group, is described in [20], in terms of homological models  $\phi_A : \bar{B}(\mathbf{Z}[A]) \xrightarrow{f_A} (hA, d_A)$  and  $\phi_G : \bar{B}(\mathbf{Z}[G]) \xrightarrow{f_G} (hG, d_G)$  of the factor groups  $A$  and  $G$ . Explicitly,

$$\begin{aligned} f_{AG} &= (f_A \otimes f_G)_{t\cap} AW_\delta \psi \varphi \\ g_{AG} &= \varphi \psi EML_\delta (g_A \otimes g_G)_{t\cap} \\ \phi_{AG} &= \varphi^{-1} \psi^{-1} (SHI_\delta + EML_\delta (1 \otimes \phi_G + \phi_A \otimes g_G f_G) t \cap AW_\delta) \psi \varphi \\ d_{AG} &= d_A \otimes 1 + 1 \otimes d_G + (f_A \otimes f_G) t \cap \sum_{i \geq 0} (-1)^i [(1 \otimes \phi_G + \phi_A \otimes g_G f_G) t \cap]^i (g_A \otimes g_G) \\ (f_A \otimes f_G)_{t\cap} &= (f_A \otimes f_G) (1 - t \cap \sum_{i \geq 0} (-1)^i [(1 \otimes \phi_G + \phi_A \otimes g_G f_G) t \cap]^i) (1 \otimes \phi_G + \phi_A \otimes g_G f_G) \\ (g_A \otimes g_G)_{t\cap} &= \sum_{i \geq 0} (-1)^i [(1 \otimes \phi_G + \phi_A \otimes g_G f_G) t \cap]^i (g_A \otimes g_G) \\ (1 \otimes \phi_G + \phi_A \otimes g_G f_G)_{t\cap} &= \sum_{i \geq 0} (-1)^i [(1 \otimes \phi_G + \phi_A \otimes g_G f_G) t \cap]^i (1 \otimes \phi_G + \phi_A \otimes g_G f_G) \\ t\cap &= AW_\delta \sum_{i \geq 0} (-1)^i (SHI_\delta)^i EML \\ AW_\delta &= AW (1 - \delta \sum_{i \geq 0} (-1)^i (SHI_\delta)^i SHI) \\ EML_\delta &= \sum_{i \geq 0} (-1)^i (SHI_\delta)^i EML \\ SHI_\delta &= \sum_{i \geq 0} (-1)^i (SHI_\delta)^i SHI \\ \delta((a_{n-1}, \dots, a_0), (g_{n-1}, \dots, g_0)) &= -((a_{n-2}, \dots, a_0), (g_{n-2}, \dots, g_0)) + \\ &\quad (-\alpha(g_{n-2}, g_{n-1}) + a_{n-2}, -\alpha(g_{n-3}, g_{n-2} + g_{n-1}) + \alpha(g_{n-3}, g_{n-2}) + a_{n-3}, \dots, \\ &\quad \dots, -\alpha(g_0, g_1 + \dots + g_{n-1}) + \alpha(g_0, g_1 + \dots + g_{n-2}) + a_0), (g_{n-2}, \dots, g_0)) \end{aligned}$$

$$\begin{aligned}
 \varphi([(a_0, g_0), \dots, (a_n, g_n)]) &= \begin{cases} ((a_0, g_0), \dots, (a_n, g_n)), & \text{if } A \rtimes_{\alpha} G \text{ is abelian} \\ (-1)^{\lceil \frac{n}{2} \rceil + 1} ((a_n, g_n), \dots, (a_0, g_0)), & \text{otherwise} \end{cases} \\
 \psi[(a_{n-1}, g_{n-1}), \dots, (a_0, g_0)] &= \\
 & \quad ([a_{n-1}, a_{n-2} + \alpha(g_{n-2}, g_{n-1}), \dots, a_0 + \alpha(g_0, g_1 + \dots + g_{n-2} + g_{n-1})], [g_{n-1}, \dots, g_0]) \\
 \psi^{-1}([a_{n-1}, \dots, a_0], [g_{n-1}, \dots, g_0]) &= [(a_{n-1}, g_{n-1}), (a_{n-2} - \alpha(g_{n-2}, g_{n-1}), g_{n-2}), \dots, \\
 & \quad \dots, (a_{n-i} - \alpha(g_{n-i}, g_{n-i+1} + \dots + g_{n-2} + g_{n-1}), g_{n-i}), \\
 & \quad \dots, (a_0 - \alpha(g_0, g_1 + \dots + g_{n-2} + g_{n-1}), g_0)] \\
 AW((a_{n-1}, \dots, a_0), (g_{n-1}, \dots, g_0)) &= \sum_{i=0}^n (a_{n-1}, \dots, a_{n-i}) \otimes (g_{n-i-1}, \dots, g_0) \\
 EML((a_{p-1}, \dots, a_0) \otimes (g_{q-1}, \dots, g_0)) &= (a_{p-1}, \dots, a_0) \star (g_{q-1}, \dots, g_0) \\
 SHI((a_{n-1}, \dots, a_0), (g_{n-1}, \dots, g_0)) &= \\
 & \quad \sum_{q=0}^{n-1} \sum_{p=0}^{n-p-q} ((a_{n-1}, \dots, a_{p+q+1}, 0), (g_{n-1}, \dots, g_{p+q+1}, g_{p+q} + \dots + g_q)) || \\
 & \quad (a_{p+q}, \dots, a_q) \star (g_{q-1}, \dots, g_0)
 \end{aligned}$$

The symbol  $\star$  refers to the shuffle product, so that the output of

$$(a_{p-1}, \dots, a_0) \star (g_{q-1}, \dots, g_0)$$

consists in the sum of all the different shuffles of the tuples, such that the inner order in the lists is preserved. The sign correspondent to a particular shuffle depends on the number of positions that elements  $a_i$  have got ahead of elements  $g_j$ .

## 2.4 A Homological Model for a Semidirect Product

Here  $A \rtimes_{\alpha} G$  denotes the semidirect product of  $A$  and  $G$  by means of the homomorphism  $\alpha : G \rightarrow \text{Aut}(A)$ , such that  $(a, g) \cdot (a', g') = (a + \alpha(g)(a'), g + g')$ . In case that  $\alpha$  is the zero map, then  $A \rtimes_{\alpha} G$  consists in the direct product  $A \times G$ .

A homological model  $\phi_{AG} : \bar{B}(\mathbf{Z}[A \rtimes_{\alpha} G]) \xrightarrow{f_{AG}}_{g_{AG}} (hAG, d_{AG})$  for a semidirect product  $A \rtimes_{\alpha} G$ , for  $G$  being an abelian group, is described in [3], in terms of homological models  $\phi_A : \bar{B}(\mathbf{Z}[A]) \xrightarrow{f_A}_{g_A} (hA, d_A)$  and  $\phi_G : \bar{B}(\mathbf{Z}[G]) \xrightarrow{f_G}_{g_G} (hG, d_G)$  of the factor groups  $A$  and  $G$ .

The formulae for central extensions given before also apply for a semidirect product, with the following exceptions,

$$\begin{aligned}
 \delta((a_{n-1}, \dots, a_0), (g_{n-1}, \dots, g_0)) &= -((a_{n-2}, \dots, a_0), (g_{n-2}, \dots, g_0)) + \\
 & \quad + ((\alpha(g_{n-1})(a_{n-2}), \dots, \alpha(g_{n-1})(a_0)), (g_{n-2}, \dots, g_0)) \\
 \psi[(a_{n-1}, g_{n-1}), \dots, (a_0, g_0)] &= \\
 & \quad ((\alpha(g_{n-1}^{-1})(a_{n-1}), \dots, \alpha(g_{n-1}^{-1} + \dots + g_0^{-1})(a_0)), (g_{n-1}, \dots, g_0)) \\
 \psi^{-1}((a_{n-1}, \dots, a_0), (g_{n-1}, \dots, g_0)) &= \\
 & \quad ((\alpha(g_{n-1})(a_{n-1}), g_{n-1}), \dots, (\alpha(g_0 + \dots + g_{n-1})(a_0), g_0))
 \end{aligned}$$

### 2.5 A Homological Model for an Iterated Product

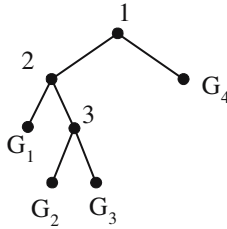
Let  $G$  be an iterated product of central extensions and semidirect products of abelian groups, which admits the form  $A \times_{\alpha} G$ , for  $A$  and  $G$  also being possibly iterated products. Here  $A \times_{\alpha} G$  denotes a single central extension or a single semidirect product, as it is the case. This way, a homological model for  $G$  has been already described in the subsections above, provided some homological models for  $A$  and  $G$  are known. In fact, we do know these homological models for  $A$  and  $G$ . It suffices to iterate this scheme, until we arrive to homological models for  $\mathbb{Z}$  or  $\mathbb{Z}_n$ .

## 3 The Notebook

### 3.1 Codifying the Group

An iterated product  $G$  of central extensions and semidirect products of abelian groups  $G_i$ ,  $1 \leq i \leq n$ , is codified as a rooted binary tree, such that every inner vertex represents a product, their sons being the correspondent factor groups. As usual, a inner vertex contributes two sons in the level immediately below. This way, the number of leaf vertices coincides with the number of factor groups  $G_i$ . In order to obtain the group  $G$ , the inner vertices (i.e. single products) of the tree must be chosen from bottom to the top level (root vertex), from left to the right while staying at the same level.

For instance, the binary tree of Figure 1 represents an iterated product.



**Fig. 1.** A binary tree representing the product  $(G_1 \times (G_2 \times G_3)) \times G_4$

We use a list for representing this binary tree, which we term *tree-list*. In fact, *Mathematica* is one of the most appropriate systems for handling with lists. Proceeding level by level, from top to the bottom, from left to the right, every vertex (but those placed on the last level) will be codified as an integer, attending to the correspondences in the table below.

Label	Meaning
0	leaf vertex
1	direct product
2	semidirect product $A \rtimes_{\alpha} G$
3	semidirect product $G_{\alpha} \ltimes A$
4	central extension $A \ltimes_{\alpha} G$
5	central extension $G_{\alpha} \rtimes A$

*Example 1.* Consider the dihedral group  $D_{4t} = \mathbb{Z}_{2t} \rtimes_{\alpha} \mathbb{Z}_2$ , for  $\alpha : \mathbb{Z}_2 \rightarrow \text{Aut}(\mathbb{Z}_{2t})$  such that  $\alpha(0)(x) = x$  and  $\alpha(1)(x) = -x$ . The list codifying this group consists in  $\{\{2\}\}$ . There is no need to add the list  $\{0, 0\}$  corresponding to the last level.

*Example 2.* Consider the central extension  $\mathbb{Z}_{2t} \ltimes_{\alpha} \mathbb{Z}_2$ , for  $\alpha : \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_{2t}$  being the 2-cocycle

$$\alpha(g_i, g_j) = \begin{cases} \lceil \frac{t}{2} \rceil + 1 & \text{if } g_i = g_j = 1 \\ 0 & \text{otherwise} \end{cases}$$

The list codifying this group consists in  $\{\{4\}\}$ . Once again, we do not take into account the list  $\{0, 0\}$  corresponding to the last level.

*Example 3.* Consider the iterated product  $(\mathbb{Z}_t \ltimes_{\alpha_2} \mathbb{Z}_2) \rtimes_{\alpha_1} \mathbb{Z}_2$ , for  $\alpha_2$  being the 2-cocycle  $\alpha_2 : \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_t$  defined as

$$\alpha_2(g_i, g_j) = \begin{cases} \lceil \frac{t}{2} \rceil + 1 & \text{if } g_i = g_j = 1 \\ 0 & \text{otherwise} \end{cases}$$

and  $\alpha_1$  being the dihedral action

$$\alpha_1(a, b) = \begin{cases} -b & \text{if } a = 1 \\ b & \text{if } a = 0 \end{cases}$$

The list codifying this group consists in  $\{\{2\}, \{4, 0\}\}$ .

### 3.2 Codifying the Homological Models

The homological models described in the precedent section involve several structures, such as:

- Linear combinations.
- Products of exterior and divided power algebras.
- Elements in a cartesian simplicial product  $X \times Y$ , linear combinations of tuples  $((x_1, y_1), \dots, (x_n, y_n))$ .
- Elements in a tensor product  $X \otimes Y$ , linear combinations of tuples  $(x_1, \dots, x_p) \otimes (y_1, \dots, y_q)$ .
- Formal series.

We now describe the way in which these structures are codified.

- The elimination of the attribute `Listable` on the addition and product functions provided by *Mathematica* supplies at once the possibility of making linear combinations with lists.
- Each of the groups  $G_i = \mathbb{Z}_{n_i}$  gives rise to a product  $P_i = E(u_i) \otimes \Gamma(w_i)$ , whereas each group  $G_j = \mathbb{Z}$  gives rise to a single  $P_j = E(u_j)$ . The elements in  $hG = \prod_{i=1}^n P_i$  are codified as linear combinations of tuples of length  $n$ , such that if  $m$  is the  $j^{\text{th}}$  entry of a tuple, it refers to the generator  $u^{m \pmod{2}} \otimes \gamma_{\lfloor \frac{m}{2} \rfloor}(w)$  (notice that only 0 and 1 entries are permitted in case of  $P_j = E(u_j)$  factors coming from  $G_j = \mathbb{Z}$ ). These tuples are ordered as numbers of  $n$  digits. For instance, if  $G = \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}_2$ , a basis for  $hG$  on degree 3 is given by  $\{(0, 0, 3), (0, 1, 2), (1, 0, 2), (1, 1, 1), (2, 0, 1), (2, 1, 0), (3, 0, 0)\}$ .
- Tuples are codified as lists, in a natural way.
- Formal series, which are always finite when applied on a concrete element, are codified in terms of the command `NestList`.

### 3.3 Calculating the Homology

As Veblen's algorithm indicates [22], in order to compute the homology  $H_i(G)$  it is necessary to calculate the Smith's normal forms of the matrices  $M_i$  and  $M_{i+1}$  representing the differential operators  $d_i$  and  $d_{i+1}$ .

Our program firstly computes the matrices  $M_i$  and  $M_{i+1}$ . Afterwards, we use the `SmithNormalForm.na` package [14] and finally compute  $H_i(G)$ . Though the actual version of the notebook does not provide representative  $i$ -cycles, it could be straightforwardly adapted to this end. In fact, such an option was available in an earlier (not published) version of the notebook, which ran only over finite groups.

### 3.4 Input and Output Data

In these circumstances, we may now determine exactly what the input and output data are.

INPUT DATA:

- The correspondent tree-list for the group  $G$ .
- The cardinalities of each elementary factor group  $\mathbb{Z}$  or  $\mathbb{Z}_n$ . The notation is 0 for  $\mathbb{Z}$  and  $n > 1$  for  $\mathbb{Z}_n$ . Notice that  $k$  factor groups correspond to  $k - 1$  products, and vice versa.
- The maps  $\alpha_i$  involved in the  $i^{\text{th}}$  product,  $J_i \times_{\alpha_i} K_i$ . The user should attend to the tree-list for the group  $G$ , in order to identify the index  $i$  corresponding to each product, as well as the syntax to use for codifying the elements in  $J_i$  and  $K_i$ , since they could be in turn iterated groups themselves.
- Finally, the desired degree  $k$ , in order to compute  $H_k$ .



OUTPUT DATA:

- The homology  $H_k(G)$ .

As soon as the computation has finished, the user is asked for going on computing  $H_{k+1}$ , since half of the computations (those corresponding to  $d_{k+1}$ ) may be reused.

### 4 Simplifications on the Formulae

Calculations achieved with our program have provided some evidences of annihilation properties on some summands on the maps characterizing the homological models of the precedent sections. We include here the results that we have finally proved.

**Proposition 1.** *In the case of a homological model for a semidirect product, the morphism  $SHI_\delta$  reduces to  $SHI$ , as well as  $EML_\delta$  reduces to  $EML$ , and  $AW_\delta$  reduces to  $AW - AW\delta SHI$ .*

**Proposition 2.** *In the case of a homological model for a semidirect product, the morphism  $t\cap$  reduces to  $t\cap ([a_{m-1}, \dots, a_0] \otimes [g_{n-1}, \dots, g_0]) = (-1)^m([\alpha(g_{n-1})(a_{m-1}), \dots, \alpha(g_{n-1})(a_0)] - [a_{m-1}, \dots, a_0]) \otimes [g_{n-2}, \dots, g_0]$*

### 5 Executions and Examples

We compute here the matrices  $M_2$  and  $M_3$  corresponding to  $d_2$  and  $d_3$ , as well as the homology groups  $H_1$  and  $H_2$  of some finite groups. These and other examples have provided essential information in order to calculate the total number of cocyclic Hadamard matrices on the correspondent groups, some of which seems to be new [2].

*Example 4.* Consider the family of groups  $\mathbb{Z}_{2t} \ltimes_\alpha \mathbb{Z}_2$ , for  $t \in \mathbb{N}$  and  $\alpha$  being the 2-cocycle  $f(1, 1) = \lceil \frac{t}{2} \rceil + 1$ .

$t$	1	2	3	4	5
$M_2$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ 0 & 0 \\ 0 & 4 \end{pmatrix}$	$\begin{pmatrix} 2 & 3 \\ 0 & 0 \\ 0 & 6 \end{pmatrix}$	$\begin{pmatrix} 2 & 5 \\ 0 & 0 \\ 0 & 8 \end{pmatrix}$	$\begin{pmatrix} 2 & 6 \\ 0 & 0 \\ 0 & 10 \end{pmatrix}$
$M_3$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 6 & 0 \\ 0 & -2 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$H_1$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	$\mathbb{Z}_{12}$	$\mathbb{Z}_{16}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{10}$
$H_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$

*Example 5.* Consider the dihedral groups  $D_{4t} = \mathbb{Z}_{2t} \rtimes_{\chi} \mathbb{Z}_2$ , for  $1 \leq t \leq 5$ .

$t$	1	2	3	4	5
$M_2$	$\begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & -2 \\ 0 & 4 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & -4 \\ 0 & 6 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & -6 \\ 0 & 8 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 \\ 0 & -8 \\ 0 & 10 \end{pmatrix}$
$M_3$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & -2 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & -4 \\ 0 & 6 & 4 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -8 & -6 \\ 0 & 8 & 6 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -10 & -8 \\ 0 & 10 & 8 \\ 0 & 0 & 0 \end{pmatrix}$
$H_1$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$H_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

*Example 6.* Consider the family of iterated products  $(\mathbb{Z}_t \rtimes_f \mathbb{Z}_2) \rtimes_{\chi} \mathbb{Z}_2$ , for  $2 \leq t \leq 5$ ,  $f$  being the 2-cocycle  $f(1,1) = \lceil \frac{t}{2} \rceil + 1$  and  $\chi$  being the dihedral action  $\chi(1,b) = -b$ .

$t$	2	3	4	5
$M_2$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$
$M_3$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -5 & 0 & -3 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 5 & 0 & 3 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
$H_1$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$H_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

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