Reusing Integer Homology Information of Binary Digital Images⋆

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Abstract. In this paper, algorithms for computing integer (co)homology of a simplicial complex of any dimension are designed, extending the work done in [1, 2, 3]. For doing this, the homology of the object is encoded in an algebraic-topological format (that we call AM-model). Moreover, in the case of 3D binary digital images, having as input AM-models for the images I and J, we design fast algorithms for computing the integer homology of $I \cup J$, $I \cap J$ and $I \setminus J$.

1 Introduction

Efficient algorithms for computing topological information are powerful tools in the fields of Data Mining, Pattern Recognition, Geometric Modeling and nD Digital Image Processing. Nevertheless, topological notions (such as the cup product on cohomology, cohomology operations, fundamental group, homotopy groups, etc) are hard to adapt into an n-dimensional discrete framework; and the number of available computational tools are limited. It is a fact that the cup product on cohomology is a topological invariant which contains more information than homology groups when we deal with an object of dimension greater than or equal to 3. Since cohomology is essentially an algebraic notion, it seems reasonable to encode it using a classical algebraic-topological cover: chain homotopy equivalences. In the setting of Simplicial Topology, we use here this extra algebraic-topological information (that we will define as an AM-model for a simplicial complex) to compute the cup product on integer cohomology as well as cohomological numbers derived from it, extending the work developed in [1, 2]. Our computational approach follows the philosophy of the Effective Homology Theory developed by F. Sergeraert in [4, 5]. In particular, we prove that all the algorithms for computing integer homology based in the matrix reduction method to Smith normal form (for example [6, 7, 8, 9]) can be translated to our setting with no extra computational cost in time. Finally, we successfully apply this computational algebraic topological approach to 3D binary digital images

and we prove that a suitable extended notion of AM-model for binary 3D digital images can be reused under voxel-set operations (union, intersection and difference).

2   Integer Homology, Chain Contractions and AM-Models

In [9], an algorithm improving the efficiency of the classical integer reduction homology algorithm is described. Their technique is mainly based on the results of [8], in which a matrix reduction to integer Smith normal form is determined in an efficient way. There is no problem for translating this method to our framework since it consists in constructing a chain homotopy equivalence from the previously calculated Smith normal form, without additional computational cost. Moreover, our strategy of saving more algebraic information outperforms the previous algorithms for computing integer homology in several points such as: 1) cohomological features can be computed; 2) we can efficiently control the topological changes after addition or deletion of simplices.

First, we give a brief summary of concepts and notations. The terminology follows Munkres book [6]. We will consider that the ground ring is $\mathbf{Z}$.

Simplicial Complexes. Considering an ordering on a vertex set V, a q–simplex with $q + 1$ affinely independent vertices $v_0 < \cdots < v_q$ of V is the convex hull of these points, denoted by $\langle v_0, \ldots, v_q \rangle$. If $i < q$, an i–face of σ is an i–simplex whose vertices are in the set $\{v_0, \ldots, v_q\}$. A simplex is *maximal* if it does not belong to any higher dimensional simplex. A *simplicial complex* K is a collection of simplices such that every face of a simplex of K is in K and the intersection of any two simplices of K is a face of each of them or empty. The set of all the q–simplices of K is denoted by $K^{(q)}$. The *dimension of K* is the dimension of the highest dimensional simplex in K.

Chains and Homology. Let K be a simplicial complex. A q–*chain a* is a formal sum of simplices of $K^{(q)}$. The q–chains form the *qth chain group* of K, denoted by $C_q(K)$. The *boundary* of a q–simplex $\sigma = \langle v_0, \ldots, v_q \rangle$ is the $(q-1)$–chain: $\partial_q(\sigma) = \sum_{i=0}^{q} (-1)^i \langle v_0, \ldots, \hat{v}_i, \ldots, v_q \rangle$, where the hat means that v_i is omitted. By linearity, ∂_q can be extended to q–chains. The collection of boundary operators connect the chain groups $C_q(K)$ into the *chain complex $C(K)$*: $\cdots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0$. An essential property is that $\partial_q \partial_{q+1} = 0$. In a more general setting, a *chain complex $\mathcal{C}$* is a sequence $\cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$ of abelian groups C_q and homomorphisms d_q, such that for all q, $d_q d_{q+1} = 0$. The set of all the homomorphisms d_q is called the *differential* of $\mathcal{C}$. A q–chain $a \in C_q$ is called a q–*cycle* if $d_q(a) = 0$. If $a = d_{q+1}(a')$ for some $a' \in C_{q+1}$ then a is called a q–*boundary*. Denote the groups of q–cycles and q–boundaries by Z_q and B_q respectively. Define the *qth homology group* to be the quotient group Z_q/B_q, denoted by $H_q(\mathcal{C})$. We say that a is a *representative q–cycle* of a homology generator α if $\alpha = a + B_q$. We denote $\alpha = [a]$. The qth *betti number* β_q is the rank of the free part of $H_q(\mathcal{C})$. Intuitively, β_0 is the number of components of

connected pieces, β_1 is the number of independent "holes" and β_2 is the number of "cavities".

Chain Contractions. A *chain contraction* [10] of a chain complex $\mathcal{C}$ to a chain complex $\mathcal{C}'$ is a set of three homomorphisms (f, g, ϕ) such that: $f : \mathcal{C} \to \mathcal{C}'$ and $g : \mathcal{C}' \to \mathcal{C}$ are chain maps; fg is the identity map of $\mathcal{C}'$; and $\phi : \mathcal{C} \to \mathcal{C}$ is a chain homotopy of the identity map id of $\mathcal{C}$ to gf, that is, $\phi\partial + \partial\phi = id - gf$. In this case, $\mathcal{C}$ and $\mathcal{C}'$ have isomorphic homology groups [6, p. 73].

AM-Models. An *AM-model* for a simplicial complex K is the set (C, M, f, g, ϕ) where C is a basis of $C(K)$, M is a subset of generators of $C(K)$ and (f, g, ϕ) is a chain contraction from $C(K)$ to $M(K)$ where $M(K)$ is the chain complex generated by M with differential $\partial|_{M(K)}$ such that in each dimension q, the matrix A of the differential $\partial_q|_{M(K)}$ coincides with its Smith normal form and satisfies that any non-null entry of A is greater than 1. Moreover, if the homology is free or the ground ring is a field, then $M(K)$ is isomorphic to the homology of K. It is necessary to emphasize that given a simplicial complex K, it is possible to define different AM-models for K since the chain complex $M(K)$ and the morphisms f, g and ϕ can admit different formulae.

A translation of the integer reduction homology algorithm in terms of chain contractions has been made in [11]. Here we rewrite this work using a more algorithmic language. This algorithm consists in reducing the matrix A_q of the boundary operator in each dimension q, to its Smith normal form A'_q, relative to some basis $\{a_1, \ldots a_r\}$ of $C_q(K)$ and $\{e_1, \ldots, e_s\}$ of $C_{q-1}(K)$ such that $\{a_{\ell+1}, \ldots, a_r\}$ is a basis of $Z_q(K)$, and $\{\lambda_1 e_1, \ldots, \lambda_\ell e_\ell\}$ is a basis of $B_{q-1}(K)$ [6, pp. 56-61].

Algorithm 1. *Computing an AM-model for a Finite Simplicial Complex.*

INPUT: A simplicial complex K of dimension d.
Initially: $C_q := K^{(q)}$, $M_q := K^{(q)}$ and $C'_q = \{\ \}$ for $0 \leq q \leq d$,
$\qquad\quad f(\sigma) := \sigma$, $g(\sigma) := \sigma$, $\phi(\sigma) := 0$ for every $\sigma \in K$.
For $\ q = 1$ to $q = d$ do
$\qquad$ Reduce the matrix A_q of the boundary operator ∂_q relative to
$\qquad$ the basis C_q and M_{q-1} to its Smith normal form A'_q relative
$\qquad$ to some basis $\{a_1, \ldots, a_r\}$ of C_q and $\{e_1, \ldots, e_s\}$ of M_{q-1} where:
$\qquad$ $\partial_q(a_i) = e_i$, for $1 \leq i \leq t \leq$ min (r, s);
$\qquad$ $\partial_q(a_i) = \lambda_i e_i$, $\lambda_i \in \mathbf{R}$, for $t < i \leq \ell \leq$ min (r, s);
$\qquad$ and $\partial_q(a_i) = 0$ for $\ell < i \leq r$.
$\qquad$ Define $\ C_{q-1} := C'_{q-1} \cup \{e_1, \ldots, e_s\}$, $M_{q-1} := \{e_{t+1}, \ldots, e_s\}$,
$\qquad$ $C_q := \{a_1, \ldots, a_r\}$, $C'_q := \{a_1, \ldots, a_t\}$, $M_q := \{a_{t+1}, \ldots, a_r\}$,
$\qquad$ $f(a_i) := 0$, $f(e_i) := 0$ and $\phi(e_i) := a_i$ for $1 \leq i \leq t$.
OUTPUT: The set $(C_0 \cup \cdots \cup C_d, M_0 \cup \cdots \cup M_d, f, g, \phi)$.

The following result shows that although $M(K)$ is not isomorphic to the homology of K, we can directly obtain the integer homology from it.

Theorem 2. *Let K be a finite simplicial complex of any dimension. The set $(C_0 \cup \cdots \cup C_d, M_0 \cup \cdots \cup M_d, f, g, \phi)$ defines an AM-model for K, being $C =$*

$C_0 \cup \cdots \cup C_d$ a basis of $C(K)$, $M(K)$ the chain complex generated by $M = M_0 \cup \cdots \cup M_d$ and with differential $\partial|_{M(K)}$, and (f, g, ϕ) a chain contraction from $C(K)$ to $M(K)$. Moreover, the integer homology of K and integer homology generators can be directly obtained from M and $\partial|_{M(K)}$.

If K has m simplices, an AM-model for K can be computed in time and storage $\mathcal{O}(m^3)$.
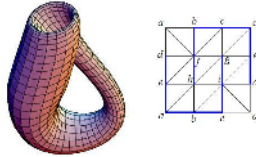

Fig. 1. The Klein bottle and a triangulation of it

Example 1. Consider the simplicial complex K in Figure 1 whose underlying space is the Klein bottle [6, p. 283]. Running the algorithm above, we obtain that the vertex $\langle a \rangle$ belongs to C_0 and $M_0(K)$, the cycles $\alpha_1 := \langle a, b \rangle + \langle b, c \rangle - \langle a, c \rangle$ and $\alpha_2 := \langle a, d \rangle + \langle d, e \rangle - \langle a, e \rangle$ belong to C_1 and M_1; and the 2-chain consisting in the sum of all the triangles in K, $\beta := -\langle a, b, f \rangle - \langle b, c, f \rangle + \langle a, c, g \rangle - \langle a, e, g \rangle + \langle e, g, i \rangle - \langle e, d, i \rangle + \langle c, d, i \rangle - \langle a, c, d \rangle + \langle b, c, i \rangle + \langle a, b, h \rangle - \langle a, e, h \rangle - \langle e, d, f \rangle + \langle a, d, f \rangle + \langle c, f, g \rangle - \langle f, g, h \rangle - \langle h, g, i \rangle - \langle b, h, i \rangle$, is an element of C_2 and M_2. The rest of the elements of C_0 are the boundaries of the 1-simplices marked in blue in Figure 1. These 1-simplices are also elements of C_1. Denote by x one of these 1-simplices. The rest of the elements of C_1 are the boundaries of all the 2-simplices except for $\langle f, g, h \rangle$. These 2-simplices belong to C_2. Denote by y one of these 2-simplices. The images of the maps (f, g, ϕ) on the generators of $C(K)$ and $M(K)$ are described in the table below:

C	M	f	g	ϕ
$\langle a \rangle$	$\langle a \rangle$	$\langle a \rangle$	$\langle a \rangle$	0
α_1	α_1	α_1	α_1	0
α_2	α_2	α_2	α_2	0
β	β	β	β	0
x		0		0
∂x		0		x
y		0		0
∂y		0		y

Summing up, $M_0 = \{\langle a \rangle\}$, $M_1 = \{\alpha_1, \alpha_2\}$ and $M_2 = \{\beta\}$. Moreover, $\partial(\langle a \rangle) = 0$, $\partial(\alpha_1) = 0$, $\partial(\alpha_2) = 0$ and $\partial(\beta) = 2\alpha_2$. Therefore we obtain that $H_0(K) \simeq \mathbf{Z}$, $H_1(K) \simeq \mathbf{Z} \oplus \mathbf{Z}/\mathbf{Z}2$ and representative cycles of the homology generators are $\langle a \rangle$ for $H_0(K)$, α_1 for the free part of $H_1(K)$ and α_2 for the torsion part.

3   Cohomology Computations with Integer Coefficients

In this section, we extend the work done in [11, 1, 2] (with coefficients in a field) for computing cohomology features over the coefficient domain $\mathbf{Z}$. The interest

for computing cohomology (the dual notion of homology) is that cohomology has an additional multiplicative structure, the cup product, from which we can derive finer invariants than homology. Observe that working with coefficients in a field, homology groups are free and isomorphic to cohomology groups. Nevertheless, working with coefficients in $\mathbf{Z}$, homology and cohomology of simplicial complexes can have torsion part and, in this case, they are not isomorphic.

Cochains and Cohomology. Let $\mathcal{C}$ be a chain complex. The *cochain complex* C^* in each dimension q is the group of q–*cochains* with coefficients in $\mathbf{Z}$, $C^q = \{c : C_q \rightarrow \mathbf{Z}$ such that c is a homomorphism$\}$. If $\{a_1, \ldots, a_n\}$ is a basis of C_q then a basis of C^q is $\{a_1^*, \ldots, a_n^*\}$, where $a_i^* : C_q \rightarrow \mathbf{Z}$ is given by $a_i^*(a_i) = 1$ and $a_i^*(a_j) = 0$ for $1 \leq i, j \leq n$ and $j \neq i$. For each q, the differential d_{q+1} on C_{q+1} induces the *codifferential* $\delta_q : C^q \rightarrow C^{q+1}$ via $\delta_q(c) = cd_{q+1}$, so that δ_q raises dimension by one. Define Z^q to be the kernel of δ_q and B^{q+1} to be its image. These groups are called the group of q–*cocycles* and q–*coboundaries*, respectively. Define the qth *cohomology group*, $H^q(\mathcal{C}) = Z^q/B^q$ for $q \geq 0$.

The following result shows that we can directly obtain the integer cohomology of K from an AM-model for it. This assertion is not given in [11,1,2].

Theorem 3. *Let K be a finite simplicial complex of any dimension. Given an AM-model (C, M, f, g, ϕ) for K, the integer cohomology of K and integer cohomology generators can be directly obtained from M and $\partial|_{M(K)}$.*

Example 2. Consider the AM-model (C, M, f, g, ϕ) obtained in Example 1 for the simplicial complex K whose underlying space is the Klein bottle. Starting from the chain complex $M(K)$ whose basis is $\{\langle a \rangle, \alpha_1, \alpha_2, \beta\}$ and differential $\partial|_{M(K)}$, we construct in an straightforward way the cochain complex $M^*(K)$ whose basis is $\{\langle a \rangle^*, \alpha_1^*, \alpha_2^*, \beta^*\}$ and codifferential δ given by: $\delta(\langle a \rangle^*) = \langle a \rangle^* \partial|_{M(K)} = 0$, $\delta(\alpha_1^*) = \alpha_1^* \partial|_{M(K)} = 0$, $\delta(\alpha_1^*) = \alpha_2^* \partial|_{M(K)} = 2\beta^*$, $\delta(\beta^*) = \beta^* \partial|_{M(K)} = 0$. Therefore we obtain that $H^0(K) \simeq \mathbf{Z}$, $H^1(K) \simeq \mathbf{Z}$ and $H^2(K) \simeq \mathbf{Z}/\mathbf{Z}2$; and the generators are: $\langle a \rangle^*$ for $H^0(K)$, α_1^* for $H^1(K)$ and β^* for $H^2(K)$.

Cup Product. The cochain complex $C^*(K)$ is a ring with the *cup product* $\smile$: $C^p(K) \times C^q(K) \rightarrow C^{p+q}(K)$ given by: $(c \smile c')(\langle v_0, \ldots, v_{p+q} \rangle) = c(\langle v_0, \ldots, v_p \rangle) \cdot c'(\langle v_p, \ldots, v_{p+q} \rangle)$. It induces an operation $\smile$: $H^p(K) \times H^q(K) \rightarrow H^{p+q}(K)$, via $[c] \smile [c'] = [c \smile c']$, that is bilinear, associative, commutative up to a sign, independent of the ordering of the vertices of K and homotopy-type invariant [6, p. 289].

Working with coefficients in $\mathbf{Z}/\mathbf{Z}2$, a new cohomology invariant called HB1 is obtained in [1, 2]. The idea is to put into a matrix form the multiplication table of the cup product of cohomology generators of dimension 1. The following algorithm compute HB1 working with integer coefficients. Assuming that K has m simplices, the complexity of this algorithm is $\mathcal{O}(m^6)$. This algorithm is an straightforward extension of that given in [1,2].

Algorithm 4. *Algorithm for computing HB1 with integer coefficients.*

INPUT: An AM-model (C, M, f, g, ϕ) for a simplicial complex K.
Let $\{\alpha_1, \ldots, \alpha_p\}$ and $\{\beta_1, \ldots, \beta_m\}$ be the set of 1 and 2-cycles in M.
For $i = 1$ to p do
    For $j = i$ to p do
        For $k = 1$ to m do
            $b_{((i,j),k)} := (\alpha_i^* f \smile \alpha_j^* f)(g(\beta_k))$.
HB1:= the rank of the 2D matrix of integers $B_{(p(p+1)/2) \times m} = (b_{((i,j),k)})$.
OUTPUT: The integer HB1.

The implementation of the algorithms described above working with coefficients in $\mathbf{Z/Z2}$ has been made by J. Sánchez-Peláez and P. Real. We have tested it on several 3D objects. We give here an example of the computation of the cohomology, cohomology generators and the invariant HB1.

Example 3. Consider the simplicial complex T whose underlying space is showed in Figure 2 (on the left). It consists in 11847 simplices. The running time for computing an AM-model for T and the homology of T using a Pentium 4, 3.2 GHz, 1Gb RAM was 2 seconds. We obtain that $\beta_0 = 1$, $\beta_1 = 4$ and $\beta_2 = 3$. The running time for computing the cup product was 1.5 seconds. In Figure 2 (on the center), the 1 and 2-simplices on which the representative cocycles are non-null are drawing. The table on the right of Figure 2 shows the results of the cup product of any two cohomology generators of dimension 1. Finally, HB1= 2.
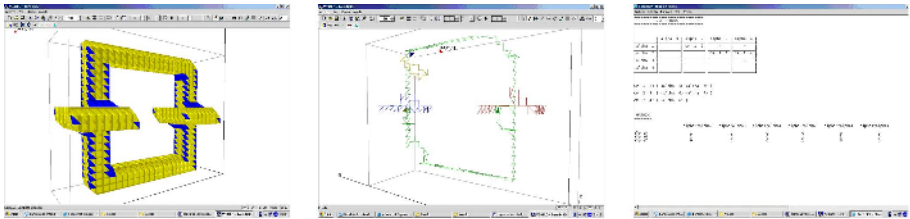

Fig. 2. The simplicial complex T, representative cocycles of the generators of $H^1(T)$ and $H^2(T)$ and the multiplication table of the cup product

4   AM-Models for 3D Digital Images

Three dimensional digital images are usually captured into the cubic grid or computed from 2D projections. There are, however, capturing techniques such as CT or MRI to produce images into other grids, such as the face-centered cubic (fcc) and the body-centered cubic (bcc) grids [12]. An important issue in Digital Volume Processing is to design efficient algorithms for analysis and processing in these grids, since it is very easy to obtain data structures for the fcc and bcc grids. On the other hand, the only Voronoi adjacency relation on the bcc grid is the 14–adjacency. Using this adjacency, it is straightforward to associate to a digital image I, a unique simplicial complex $K(I)$ (up to isomorphism) with the

same topological information as I. The i–simplices of $K(I)$ ($i \in \{0, 1, 2, 3\}$) are constituted by the different sets of i mutually 14–neighbor black points in I.

Definition 1. *Let I be a 3D binary digital image. An* AM-model *for I is defined as an AM-model for its simplicial representation $K(I)$.*

Since simplicial complexes considered in this section are embedded in $\mathbf{R}^3$, their homology groups vanishes for dimensions greater than 3 and they are torsion–free for dimensions 0, 1 and 2 (see [13, ch.10]). Therefore the chain complex $M(K(I))$ is isomorphic to the homology of I.

In the following table we present the running time for computing integer homology generators of the 3D digital images showed in figure 3. We have to say that these images have been created in a cubic grid. For these reason, we consider a special 14-adjacency in the cubic grid in the way that it is isomorphic to the bcc grid.

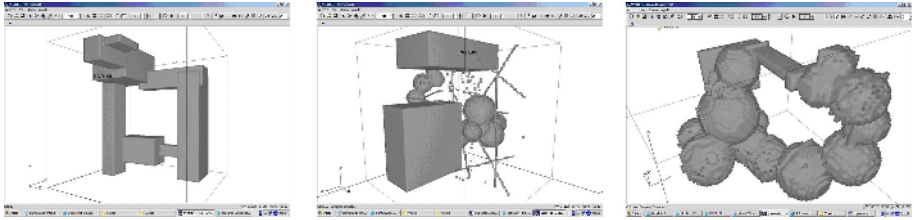| Image I	Number of voxels in I	Time for computing	β_0	β_1	β_2
A	26308	50 seconds	2	9	3
B	31012	38 seconds	138	419	13
C	18842	27 seconds	1	277	5

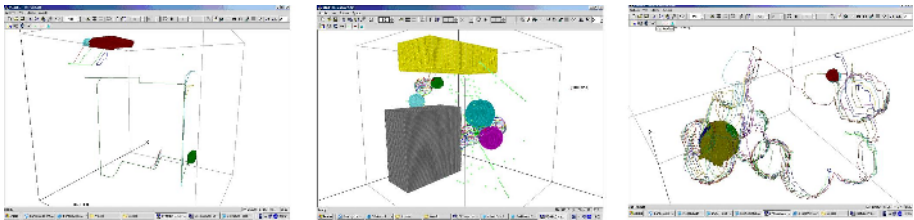**Fig. 3.** The 3D digital images A, B and C

Fig. 4. Representative cycles of the homology generators of the images A, B and C

4.1   Computing "Good" Homology Generators

In [14], algorithms for obtaining "optimal" generators of the first homology group
are developed using Dijkstra's shortest path algorithm for any oriented 2-mani-
folds. Here, in the context of digital volumes we sketch some techniques for
drawing "good" representative cycles of homology generators.

Given an AM-model $(K(I), M_I, f_I, g_I, \phi_I)$ for a 3D digital image I, we say
that x is the representative cycle of the generator $h \in M_I$ obtained by g_I if
$g_I(h) = x$. Our interest now is to get a new AM-model $(K(I), M_I', f_I', g_I', \phi_I')$
with "good" representative cycles of homology generators obtained by g_I'. This
means that each representative cycle obtained by g_I' belongs to the boundary of
the image ∂I (it is constituted by the set of black voxels in I with a 14-neighbor
white voxel). Moreover, it is required that in dimension 0, it must be a vertex;
in dimension 1, an *elementary cycle* (it is connected, each vertex is shared by
exactly two edges and two consecutive edges can not belong to the same triangle
in $K(I)$) and in dimension 2, an *elementary cavity* (it is a connected 2-cycle
with exactly one white connected component inside and three triangles can not
belong to the same tetrahedra).

Now, for obtaining good representative cycles of homology generators we can
use the following new result.

Lemma 1. *Let (K, M, f, g, ϕ) be an AM-model for a simplicial complex K. Let h
be a generator of M and x a chain in $C(K)$ such that $x = g(h)$. Let x' be a chain
in $C(K)$ such that $\partial(x') = 0$ and $f(x') = h$. Then, it is possible to define a new
AM-model (K, M, f', g', ϕ') for K such that $g'(h) := x'$ as follows: $g'(h) := x'$
and $g'(z) := z$ if $z \neq h$; $\phi'(x) := \phi(x')$, $\phi'(x') := \phi(x)$ and $\phi'(z) := \phi(z)$ for all
$z \neq x', x$.*

*If we change the basis of $M(K)$ and/or the basis of $C(K)$, it is straightforward
to obtain a new AM-model for K.*

Now, suppose we have an AM-model $(K(I), M_I, f_I, g_I, \phi_I)$ for I at hand. First of
all, we compute an AM-model for ∂I, $(K(\partial I), M_{\partial I}, f_{\partial I}, g_{\partial I}, \phi_{\partial I})$. If the elements
of $M_{\partial I}$ are denoted by $\{\alpha_1, \ldots, \alpha_n\}$ then the set of representative cycles obtained
by $g_{\partial I}$ is $S_{\partial I} = \{g_{\partial I}(\alpha_1), \ldots, g_{\partial I}(\alpha_n)\}$. Since all the homology generators of I
are homology generators of ∂I, find a subset M_I' of $\{f_I g_{\partial I}(\alpha_1), \ldots, f_I g_{\partial I}(\alpha_n)\}$,
which is a basis of $M(K(I))$. Obtain the new AM-model $(K(I), M_I', f_I', g_I', \phi_I')$
for I using Lemma 1. Now, denote by $S_I = \{c_1, \ldots, c_m\}$ the set of all the repre-
sentative cycles obtained by g_I' which is a subset of $S_{\partial I}$. Decompose and replace
each 0-cycles in S_I by its constitutive vertices, each 1-cycle by its elementary
cycles and each 2-cycle by its elementary cavities. Let $M_I'' := \{ \ \}$. For each cycle
s in S_I, if $\{f_I'(s)\} \bigcup M_I''$ is a linearly-independent set then $M_I'' := \{f_I(s)\} \bigcup M_I''$;
otherwise, $M_I'' := M_I''$. Obtain the new AM-model $(K(I), M_I'', f_I'', g_I'', \phi_I'')$ us-
ing Lemma 1. Then, $(K(I), M_I'', f_I'', g_I'', \phi_I'')$ is an AM-model for I with "good"
representative cycles of homology generators of I.

4.2   AM-Models After Adding or Deleting a Voxel

Now, we study the problem of topologically controlling a digital image using
AM-models when it suffers local changes (addition or deletion of one voxel).

More concretely, we show how to compute an AM-model for a digital image when a voxel is added or deleted using the algebraic-topological information computed before. Assuming that I has m voxels, the complexity of the algorithms in this subsection is $\mathcal{O}(m^2)$. The key idea for both algorithms is that when a q-simplex is added to or deleted from an AM-model of K, we only have to put into a Smith normal form the matrix of ∂_q for obtaining the new AM-model. Moreover, take into account that adding or deleting a voxel v of I means to add or delete a set of simplices of $K(I \cup \{v\})$ having v as a vertex. Since we work with simplicial complexes representing 3D digital images considering the 14-adjacency, the maximum number of simplices having v as a vertex is 74.

Let (K, M, f, g, ϕ) be an AM-model for a simplicial complex K. The differential of $M(K)$ is null since the homology is torsion free. Moreover, the value of all the possible non-null entries of the Smith normal form of the matrix of the differential of $C(K)$ in each dimension only can only be 1.

AM-Models After Adding a Voxel. As we have mentioned before, the addition of a voxel v to I means the addition to $K(I)$ of all the simplices of $K(I \cup \{v\}) \setminus K(I)$. In each step of the process, one simplex is added. Assuming that I has m voxels, the following algorithm computes an AM-model for the image $I \cup \{v\}$ with integer coefficients in $\mathcal{O}(m^2)$.

Algorithm 5. *Incremental Algorithm for computing an AM-model for a 3D Binary Digital Image.*

INPUT: An AM-model $AMM_I = (K(I), M, f, g, \phi)$ for I and a voxel $v \notin I$. Let $\{\sigma_1, \ldots, \sigma_n\}$ ($n \leq 74$) be the ordered-by-increasing-dimension set of all the simplices of $K(I \cup \{v\}) \setminus K(I)$.
$K_0 := K(I)$.
For $i = 1$ to $i = n$ do:
    Let q be the dimension of σ_i; let $C_q = \{a_1, \ldots, a_r\}$,
    $M_q = \{a_{t+1}, \ldots, a_r\}$, $C_{q-1} = \{e_1, \ldots, e_s\}$ and $M_{q-1} = \{e_{t+1}, \ldots, e_{s+1}\}$;
    let $\partial_q(a_j) = e_j$ for $1 \leq j \leq t$ and $\partial_q(a_j) = 0$ for $t < j \leq \min(r, s)$;
    let $\partial_q(\sigma_i) = \sum_{\ell=1}^s \lambda_\ell e_\ell$ where $\lambda_\ell \in \mathbf{R}$.
    Define $a := \sigma_i - \sum_{\ell=1}^t \lambda_\ell a_\ell$ and $C_q := \{a_1, \ldots, a_r, a\}$.
    If $\lambda_\ell = 0$ for $\ell > t$ then
        $f(a) := a$, $g(a) := a$, $\phi(a) := 0$ and $M_q := \{a_{t+1}, \ldots, a_r, a\}$.
    Else obtain the Smith normal form of the matrix of ∂_q
        relative to some base $\{e_1, \ldots, e_t, e'_{t+1}, \ldots, e'_s\}$ of $C_{q-1}(K)$.
        Define $f(a) := 0$, $\phi(a) := 0$, $\phi(e'_{t+1}) := a$, $f(e'_{t+1}) := 0$,
        $C_{q-1} := \{e_1, \ldots, e_t, e'_{t+1}, \ldots, e'_s\}$ and $M_{q-1} := \{e'_{t+2}, \ldots, e'_s\}$.
    $K_i := K_{i-1} \cup \{\sigma_i\}$.
OUTPUT: An AM-model (K_n, M, f, g, ϕ) for $I \cup \{v\}$.

AM-Models After Deleting a Voxel from a 3D Digital Image. The deletion of a voxel v from I means the deletion from $K(I)$ of all the simplices having v as a vertex. In each step of the process one simplex is deleted. Suppose that an AM-model for a digital image I with m voxeles has been computed and

after this a voxel is deleted. The following algorithm computes an AM-model for the image $I \setminus \{v\}$ with integer coefficients in $\mathcal{O}(m^2)$.

Algorithm 6. *Decremental Algorithm for computing an AM-model for a 3D Digital Image I.*

INPUT: An AM-model $(K(I), M, f, g, \phi)$ for I and a voxel $v \in I$.
Let $\{\mu_1, \ldots, \mu_n\}$ $(n \leq 72)$ be the ordered-by-decreasing-dimension set of all the simplices of $K(I)$ having v as a vertex.
$K_0 := K(I)$.
For $i = 1$ to $i = n$ do
    Let q be the dimension of σ_i; let $C_q = \{a_1, \ldots, a_r\}$,
    $M_q = \{a_{t+1}, \ldots, a_r\}$, $C_{q-1} = \{e_1, \ldots, e_s\}$ and $M_{q-1} = \{e_{t+1}, \ldots, e_{\ell+1}\}$;
    let $\partial_q(a_j) = e_j$ for $1 \leq j \leq t$ and $\partial_q(a_j) = 0$ for $t < j \leq \min(r, s)$
    Find the element $a_k \in C_q$ such that
    $C_q := \{a_1, \ldots, \hat{a}_k, \ldots, a_r\}$ is a base of $C_q(K_{i-1} \setminus \{\sigma_i\})$.
    If $1 \leq k \leq t$ then
        $M_{q-1} := \{e_k, e_{t+1}, \ldots, e_{\ell+1}\}$, $f(e_k) := e_k$, $g(e_k) := e_k$ and $\phi(e_k) := 0$.
    Else $M_q := \{a_{t+1}, \ldots, \hat{a}_k, \ldots, a_r\}$.
    $K_i := K_{i-1} \setminus \{\sigma_i\}$.
OUTPUT: An AM-model (K_m, M, f, g, ϕ) for $I \setminus \{v\}$.

4.3   AM-Models Under Voxel-Set Operations on 3D Digital Images

In this subsection, we efficiently reuse the AM-model information for digital images under voxel-set operations (union, intersection and difference).

Let I and J be two digital images. We will not consider these trivial cases: $I = \emptyset$, $J = \emptyset$, $I \cap J = \emptyset$, $I \subseteq J$ and $J \subseteq I$. Let $AMM_I := (K(I), M_I, f_I, g_I, \phi_I)$ and $AMM_J := (K(J), M_J, f_J, g_J, \phi_J)$ be AM-models for I and J, respectively.

We give now the pseudocode of the algorithms we have developed for computing AM-models for $I \cup J$, $I \cap J$ and $I \setminus J$ starting from AM-models for I and J. Denote by $Fr_I(J) = \{v_1, \ldots, v_m\}$ the set of all the voxels of $I \setminus J$ that are 14-neighbors of a voxel of J. Algorithm 7 is a common preprocessing to the three voxel-set operations treated here. In this algorithm, an AM-model for the image $I \setminus Fr_I(J)$ is calculated.

Algorithm 7. *Preprocessing.*

INPUT: The AM-model AMM_I and the set $Fr_I(J) = \{v_1, \ldots, v_m\}$.
$I_{m+1} := I$.
For $i = m$ to $i = 1$ do
    apply Algorithm 6 to v_i and the AM-model $(K(I_{i+1}), M_I, f_I, g_I, \phi_I)$.
    $I_i := I_{i+1} \setminus \{v_i\}$.
OUTPUT: An AM-model $(K(I_1), M_I, f_I, g_I, \phi_I)$ for $I \setminus Fr_I(J)$.

For computing an AM-model for $I \cup J$, we first compute an AM-model for $(I \cup J) \setminus Fr_I(J)$ using Algorithm 7 and after that we add the voxels of $Fr_I(J)$ using Algorithm 5.

Algorithm 8. *Computing an AM-model for $I \cup J$.*

INPUT: The AM-models AMM_I for I and AMM_J for J and the set $Fr_I(J)$.
Apply Algorithm 7 to AMM_I and $Fr_I(J) = \{v_1, \ldots, v_m\}$.
Define $I_0 := (I \cup J) \setminus Fr_I(J)$;
$f(\mu) := f_J(\mu), \quad \phi(\mu) := \phi_J(\mu)$ if $\mu \in K(J)$;
$f(\mu) := f_I(\mu), \quad \phi(\mu) := \phi_I(\mu)$ if $\mu \in K_0 \setminus K(J)$; $M := Imf$;
$g(\alpha) := g_J(\alpha)$ if $\alpha \in M_J$ and $g(\alpha) := g_I(\alpha)$ if $\alpha \in M \setminus M_J$.
For $i = 1$ to $i = m$ do
    apply Algorithm 5 to v_i and the AM-model $(K(I_{i-1}), M, f, g, \phi)$.
    $I_i := I_{i-1} \cup \{v_i\}$.
OUTPUT: an AM-model $(K(I_m), M, f, g, \phi)$ for $I \cup J$.

Algorithm 7 is also the essential step for computing an AM-model for $I \cap J$.

Algorithm 9. *Computing an AM-model for $I \cap J$.*

INPUT: The AM-model AMM_I for I and the set $Fr_I(J)$.
Apply Algorithm 7 to AM_I and $Fr_I(J)$.
Define $f(\mu) := f_I(\mu)$ and $\phi(\mu) := \phi_I(\mu)$ if $\mu \in K(I \cap J)$;
     $M := Imf$ and $g(\alpha) := g_I(\alpha)$ if $\alpha \in M$.
OUTPUT: an AM-model $(K(I \cap J), M, f, g, \phi)$ for $I \cap J$.

For computing an AM-model for $I \setminus J$, we first apply Algorithm 7. Second, we consider the voxels that are in $I \setminus (J \cup Fr_I(J))$. Finally, we add the voxels of $Fr_I(J)$ using Algorithm 5.

Algorithm 10. *Computing an AM-model for $I \setminus J$.*

INPUT: The AM-model AMM_I for I and the set $Fr_I(J)$.
Apply Algorithm 7 to AMM_I and $Fr_I(J) = \{v_1, \ldots, v_m\}$.
Define $I_0 := I \setminus (J \cup Fr_I(J)); \; f(\mu) := f_I(\mu)$ and $\phi(\mu) := \phi_I(\mu)$ if $\mu \in K(I_0)$
     $M := Imf$; $g(\alpha) := g_I(\alpha)$ if $\alpha \in M$.
For $i = 1$ to $i = m$ do
    apply Algorithm 5 to v_i and the AM-model $(K(I_{i-1}), M, f, g, \phi)$.
    $I_i := I_{i-1} \cup \{v_i\}$.
OUTPUT: an AM-model $(K(I_m), M, f, g, \phi)$ for $I \setminus J$.

5   Comments

The algebraic-topological representation of simplicial complexes of any dimension showed here, allows us to compute topological invariants derived from the integer cohomology ring. Moreover, we give a positive answer to the problem of efficiently reusing AM-models for determining homological information of new 3D binary digital images constructed from the previous ones using voxel-set operations.

There is considerable scope for further research: 1) To compute cohomology operations or homotopy groups of simplicial complexes using AM-models.

2) To suitably extend our method to nD binary digital images in any grid using simplicial analogous techniques [15, 16, 17, 18].

Potential applications of our particular method in computer vision and digital image processing involving not only 3D object but also higher dimensional structures can be encountered in Medical Imaging and Object Modeling. Our method seems to be especially well adapted to segmentation under topological constraints and elimination of small topological noise.

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