

# On the Cohomology of 3D Digital Images

Rocio Gonzalez–Diaz, Pedro Real

*Depto. de Matemática Aplicada I, Escuela Superior de Ingeniería Informática,  
Universidad de Sevilla, Avda. Reina Mercedes, s/n, 41012, Sevilla (Spain)*

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## Abstract

We propose a method for computing the cohomology ring of three–dimensional (3D) digital binary–valued pictures. We obtain the cohomology ring of a 3D digital binary–valued picture  $I$ , via a simplicial complex  $K(I)$  topologically representing (up to isomorphisms of pictures) the picture  $I$ . The usefulness of a simplicial description of the “digital” cohomology ring of 3D digital binary–valued pictures is tested by means of a small program visualizing the different steps of the method. Some examples concerning topological thinning, the visualization of representative (co)cycles of (co)homology generators and the computation of the cup product on the cohomology of simple pictures are showed.

*Keywords:* Digital topology, chain complexes, cohomology ring.

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## 1. Introduction

The homology groups (given in terms of number of connected components, holes and cavities in the 3D digital picture), the digital Euler characteristic or the digital fundamental group are well–known operations in Digital Topology [16, 11]. All of them can be considered as translations into the discrete setting of classical continuous topological invariants. In order to prove that a digital topology operation  $\pi_D$  (associated with a continuous operation  $\pi_C$ ) correctly reflects the topology of digital pictures considered as Euclidean spaces, the main idea is to associate a “continuous analog”  $C(I)$  with the digital picture  $I$ . In most cases, each digital picture  $I$  is associated with a polyhedron

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*Email addresses:* [rogodi@us.es](mailto:rogodi@us.es) (Rocio Gonzalez–Diaz), [real@us.es](mailto:real@us.es) (Pedro Real)

*URL:* <http://www.personal.us.es/rogodi> (Rocio Gonzalez–Diaz),

<http://www.pdipas.us.es/r/real> (Pedro Real)

$C(I)$  [11, 12, 10, 1]. It is clear that  $C(I)$  “fills the gaps” between the black points of  $I$  in a way that strongly depends on the grid and the adjacency relations chosen for the digital picture  $I$ . Recent attempts to enrich the list of computable digital topological invariants in such a way can be found in [9].

In this paper, starting from a 3D digital binary-valued picture  $I$ , a simplicial complex  $K(I)$  associated with  $I$  is constructed, in such a way that an isomorphism of pictures is equivalent to a simplicial homeomorphism of the corresponding simplicial complexes. Therefore, we are able to define the digital cohomology ring of  $I$  with coefficients in a commutative ring  $G$ , as the classical cohomology ring of  $K(I)$  with coefficients in  $G$  (see [15]). In order to compute this last algebraic object, it is crucial in our method to “connect” the chain complex  $C(K(I))$  canonically associated to  $K(I)$  and its homology  $H(K(I))$ , via an special chain equivalence [15]: a chain contraction [14]. We will obtain this goal in several steps. Using the technique of simplicial collapses [6], we topologically thin  $K(I)$ , obtaining a smaller simplicial complex  $M_{top}K(I)$  (with the same homology as  $K(I)$ ) and a chain contraction connecting their respective chain complexes  $C(K(I))$  and  $C(M_{top}K(I))$ . The following step is the construction of a chain contraction from  $C(M_{top}K(I))$  to its homology  $H(M_{top}K(I))$ . Having all this information at hand, it is easy to compute the digital cohomology ring of  $I$  for a given commutative ring  $G$ . In this way, cohomology rings are computable topological invariants which can be used for “topologically” classifying (up to isomorphisms of pictures) and distinguishing (up to cohomology ring level) 3D digital binary-valued pictures.

A small program for visualizing these cohomology aspects in the case  $G = \mathbf{Z}/\mathbf{Z}2$  has been designed by the authors and developed by others<sup>1</sup>. This software allows us to test in some simple examples the potentiality and topological acuity of the method.

We deal with digital pictures derived from a tessellation of three-space by truncated octahedra. This is equivalent to using a body-centered-cubic-grid whose grid points are the points  $(x, y, z) \in \mathbf{Z}^3$  in which  $x \equiv y \equiv z \pmod{2}$  (see [13]). The only Voronoi adjacency relation on this grid is 14-adjacency. Using this adjacency, it is straightforward to associate to a digital picture

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<sup>1</sup> The 1st version was programmed by J.M. Berrio, F. Leal and M.M. Maraver [3]; the 2nd version by F.Leal. <http://www.us.es/gtocoma/editcup.zip>.

$I$ , a unique simplicial complex  $K(I)$  (up to isomorphisms of pictures) with the same topological information as  $I$ . This grid is important in medical imaging applications due to its outstanding topological properties and its higher contents of symmetries. One advantage of the voxels in this grid is that they are more “sphere-like” than the cube, so that the volumetric data represented on this grid need fewer samples than on Cartesian cubic grid [17].

Since the objects considered in this paper are embedded in  $\mathbf{R}^3$  then the homology groups vanishes for dimensions greater than 3 and they are torsion-free for dimensions 0, 1 and 2 (see [2, ch.10]). The  $q$ th Betti number is defined as the rank of the  $q$ th homology group. In general, the 0th Betti number is the number of connected components, the 1st and 2nd Betti numbers have intuitive interpretations as the number of independent non-bounding loops and the number of independent non-bounding shells. According to the Universal Coefficient Theorem for Homology, the Betti numbers are independent of the group of coefficients (see [15, ch. 7]). Moreover, since the homology groups are torsion-free, the cohomology groups with coefficients in  $G$  are isomorphic to the homology groups with coefficients also in  $G$  (see [15, ch. 5]). Therefore, for simplicity we can consider that the ground ring is  $\mathbf{Z}/\mathbf{Z}2$  throughout the paper. Nevertheless, all the procedure we explain here, is valid for any commutative ring  $G$ .

The paper is organized as follows. In Section 2, the technique associating a simplicial complex to a 3D digital binary-valued picture is detailed. In Section 3, we explain a procedure for computing the cohomology ring of general simplicial complexes. In Section 4, we introduce the notion of digital cohomology ring of a 3D digital binary-valued picture and we show some examples concerning the visualization of representative (co)cycles of (co)homology generators and the computation of the cup product on the cohomology of simple pictures. Finally, Section 5 is devoted to conclusions and comments.

## 2. From Digital Images to Simplicial Complexes

**Digital Images.** We follow the terminology given in [12] for representing digital pictures. A *3D digital binary-valued picture space* (or, briefly, DPS) is a triple  $(V, \beta, \omega)$ , where  $V$  is the set of grid points in a 3D grid and each of  $\beta$  and  $\omega$  is a set of closed straight line segments joining pairs of points of  $V$ . The set  $\beta$  (resp. the set  $\omega$ ) determines the neighbourhood relations between black points (resp. white points) in the grid. A *3D digital binary-valued*

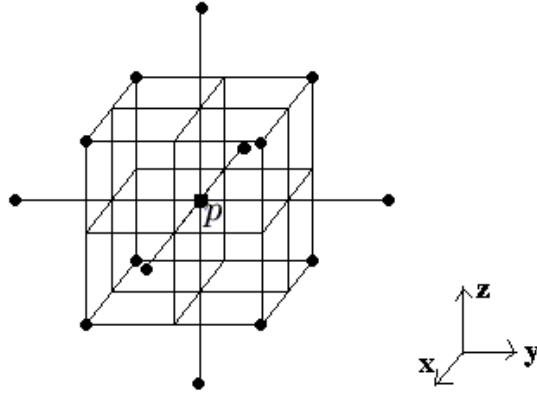


Figure 1: The 14-neighbours of a grid point  $p$  of the BCC grid.

*picture* is a quadruple  $I = (V, \beta, \omega, B)$ , where  $(V, \beta, \omega)$  is a DPS and  $B$  (the set of black points) is a finite subset of  $V$ .

An *isomorphism* of a DPS  $(V_1, \beta_1, \omega_1)$  to a DPS  $(V_2, \beta_2, \omega_2)$  is a homeomorphism  $h$  of the Euclidean 3-space to itself such that  $h$  maps  $V_1$  onto  $V_2$ , each  $\beta_1$ -adjacency onto a  $\beta_2$ -adjacency and each  $\omega_1$ -adjacency onto an  $\omega_2$ -adjacency, and  $h^{-1}$  maps each  $\beta_2$ -adjacency onto a  $\beta_1$ -adjacency and each  $\omega_2$  adjacency onto an  $\omega_1$ -adjacency. An *isomorphism of a picture*  $I_1 = (V_1, \beta_1, \omega_1, B_1)$  to a picture  $I_2 = (V_2, \beta_2, \omega_2, B_2)$  is an isomorphism of the DPS  $(V_1, \beta_1, \omega_1)$  to the DPS  $(V_2, \beta_2, \omega_2)$  that maps  $B_1$  onto  $B_2$ .

The DPS used in this paper, is the 3D *body-centered cubic grid* (BCC grid) [12]: The grid points  $\mathcal{V}$  are the points  $(a, b, c) \in \mathbf{Z}^3$  such that  $a \equiv b \equiv c \pmod{2}$ . The 14-neighbours of a grid point  $p$  with coordinates  $(a, b, c)$  are:  $(a \pm 2, b, c)$ ,  $(a, b \pm 2, c)$ ,  $(a, b, c \pm 2)$ ,  $(a \pm 1, b \pm 1, c \pm 1)$ .

**Simplicial Complexes.** The four types of non-empty simplices in  $\mathbf{R}^3$  are: a 0-simplex which is a vertex, a 1-simplex which is an edge, a 2-simplex which is a triangle and a 3-simplex which is a tetrahedron. In general, considering an ordering on a vertex set  $V$ , a  $q$ -simplex with  $q + 1$  affinely independent vertices  $v_0 < \dots < v_q$  of  $V$  is the convex hull of these points, denoted by  $\langle v_0, \dots, v_q \rangle$ . If  $i < q$ , an  $i$ -face of  $\sigma$  is an  $i$ -simplex whose vertices are in the set  $\{v_0, \dots, v_q\}$ . A *facet* of  $\sigma$  is a  $(q - 1)$ -face of it. A simplex is *shared* if it is a face of more than one simplex. Otherwise, the simplex is *free* if it belongs to one higher dimensional simplex, and *maximal* if it does not belong to any.

A *simplicial complex*  $K$  is a collection of simplices such that every face of a simplex of  $K$  is in  $K$  and the intersection of any two simplices of  $K$  is a face of each of them or empty. The set of all the  $q$ -simplices of  $K$  is denoted by  $K^{(q)}$ . A subset  $K' \subseteq K$  is a *subcomplex* of  $K$  if it is a simplicial complex itself.

Let  $K$  and  $L$  be simplicial complexes and let  $|K|$  and  $|L|$  be the subsets of  $\mathbf{R}^d$  that are the union of simplices of  $K$  and  $L$ , respectively. Let  $f : K^{(0)} \rightarrow L^{(0)}$  be a map such that whenever the vertices  $v_0, \dots, v_n$  of  $K$  span a simplex of  $K$ , the points  $f(v_0), \dots, f(v_n)$  are vertices of a simplex of  $L$ . Then  $f$  can be extended to a continuous map  $g : |K| \rightarrow |L|$  such that if  $x = \sum t_i v_i$  then  $g(x) = \sum t_i f(v_i)$ . The map  $g$  is called a *simplicial homeomorphism* if  $f$  is bijective and the points  $f(v_0), \dots, f(v_n)$  always span a simplex of  $L$ .

**Simplicial Representations.** Given a 3D digital binary-valued picture  $I = (\mathcal{V}, 14, 14, B)$  on the BCC grid, there is a process to uniquely associate a 3-dimensional simplicial complex  $K(I)$ . This simplicial complex is constructed on the triangulation of the Euclidean 3-space determined by the previous 14-neighbourhood relation. The *simplicial representation*  $K(I)$  of the digital picture  $I$  is described as follows: consider the lexicographical ordering on  $\mathcal{V}$  (if  $v = (a, b, c)$  and  $w = (x, y, z)$  are two points of  $\mathcal{V}$ , then  $v < w$  if  $a < x$ , or  $a = x$  and  $b < y$ , or  $a = x$ ,  $b = y$  and  $c < z$ ). The vertices (or 0-simplices) of  $K(I)$  are the points of  $I$ . The  $i$ -simplices of  $K(I)$  ( $i \in \{1, 2, 3\}$ ) are constituted by the different sorted sets of  $i$  14-neighbour black points of  $I$  (analogously, we could construct another simplicial complex whose  $i$ -simplices are the different sets of  $i$  14-neighbour white points of  $I$ ).

**Example 2.1.** Consider the digital picture  $J = (\mathcal{V}, 14, 14, B)$  where  $B$  is the set  $\{v_0 = (-1, -1, 1), v_1 = (-1, 1, 1), v_2 = (0, 0, 0), v_3 = (0, 0, 2), v_4 = (0, 2, 0)\}$ ; then  $K(J)$  is the simplicial complex with set of maximal simplices  $\{\langle v_0, v_1, v_2, v_3 \rangle, \langle v_1, v_2, v_4 \rangle\}$  (see Figure 2).

In the next section, we give a satisfactory algorithmic solution to the problem of the computation of the cohomology ring of finite simplicial complexes. This positive solution together with the naive simplicial construction described above will allow us to “cohomologically control” 3D digital binary-valued pictures (up to isomorphisms of pictures), since the following result holds.

**Theorem 2.2.** Two digital binary-valued pictures,  $I_1 = (\mathcal{V}, 14, 14, B_1)$  and

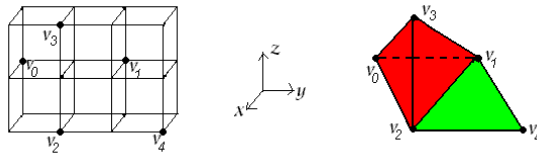


Figure 2: On the left, the black points of the digital picture  $J$  and, on the right, the simplicial representation  $K(J)$ .

$I_2 = (\mathcal{V}, 14, 14, B_2)$ , are isomorphic if and only if their simplicial representations  $K(I_1)$  and  $K(I_2)$  are simplicially homeomorphic.

The proof of this theorem is straightforward and left to the reader.

### 3. Computing the Cohomology Ring of Simplicial Complexes

First of all, we briefly explain the main concepts from Algebraic Topology we use in this paper. Our terminology follows Munkres book [15]. In the next subsections, we reinterpret classical methods in Algebraic Topology in terms of chain contractions [14] that will enable us to design an algorithm for computing the cohomology ring of general simplicial complexes.

**Chains and Homology.** Let  $K$  be a simplicial complex. A  $q$ -chain  $a$  is a formal sum of simplices of  $K^{(q)}$ . Since the group of coefficient is  $\mathbf{Z}/\mathbf{Z}2$ , a  $q$ -chain can be seen as a subset of  $q$ -simplices of  $K$ ; the sum of two  $q$ -chains  $c$  and  $d$  is the symmetric difference of the two sets  $c \cup d$  and  $c \cap d$ . The  $q$ -chains form a group with respect to the component-wise addition mod 2; this group is the  $q$ th chain group of  $K$ , denoted by  $C_q(K)$ . There is a chain group for every integer  $q \geq 0$ , but for a complex in  $\mathbf{R}^3$ , only the ones for  $0 \leq q \leq 3$  may be non-trivial. The boundary of a  $q$ -simplex  $\sigma = \langle v_0, \dots, v_q \rangle$  is the collection of all its facets which is a  $(q-1)$ -chain:  $\partial_q(\sigma) = \sum_{i=0}^q \langle v_0, \dots, \hat{v}_i, \dots, v_q \rangle$ , where the hat means that  $v_i$  is omitted. By linearity, the boundary operator  $\partial_q$  can be extended to  $q$ -chains. The collection of boundary operators connect the chain groups  $C_q(K)$  into the chain complex  $C(K)$ :  $\dots \xrightarrow{\partial_4} C_3(K) \xrightarrow{\partial_3} C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0$ . A  $q$ -chain  $a \in C_q(K)$  is called a  $q$ -cycle if  $\partial_q(a) = 0$ . If  $a = \partial_{q+1}(a')$  for some  $a' \in C_{q+1}(K)$  then  $a$  is called a  $q$ -boundary. We denote the groups of  $q$ -cycles and  $q$ -boundaries by  $Z_q$  and  $B_q$  respectively. An essential property

of the boundary operators is that the boundary of every boundary is empty,  $\partial_q \partial_{q+1} = 0$ . This implies that  $B_q \subseteq Z_q$  for  $q \geq 0$ . Define the  $q$ th *homology group* to be the quotient group  $Z_q/B_q$ , denoted by  $H_q(K)$ . Given  $a \in Z_q$ , the coset  $a + B_q$  is the *homology class* of  $H_q(K)$  determined by  $a$ . We denote this class by  $[a]$ . For a complex  $K$  in  $\mathbf{R}^3$ , only  $H_q(K)$  for  $0 \leq q \leq 2$  may be non-trivial.

**Cochains and Cohomology.** With each simplicial complex  $K$ , we have associated a sequence of abelian groups called its homology groups. Now, we associate with  $K$  another sequence of abelian groups called its *cohomology groups*. They are geometrically much less natural than the homology groups. Their origins lie in algebra rather than in geometry; in a certain algebraic sense, they are “dual” to the homology groups.

Let  $K$  be a simplicial complex. The group of  $q$ -cochains of  $K$  with coefficients in  $\mathbf{Z}/\mathbf{Z}2$  is the group  $C^q(K) = \{c : C_q(K) \rightarrow \mathbf{Z}/\mathbf{Z}2 \text{ such that } c \text{ is a homomorphism}\}$ . Observe that a  $q$ -cochain  $c$  can be defined on the  $q$ -simplices of  $K$  and it is naturally extended to  $C_q(K)$ . Therefore, a  $q$ -cochain can be expressed as a formal sum of elementary cochains  $\sigma^* : C_q(K) \rightarrow \mathbf{Z}/\mathbf{Z}2$  whose value is 1 on the  $q$ -simplex  $\sigma \in K$  and 0 on all other  $q$ -simplices of  $K$ . The boundary operator  $\partial_{q+1}$  on  $C_{q+1}(K)$  induces the *coboundary operator*  $\delta_q : C^q(K) \rightarrow C^{q+1}(K)$  via  $\delta_q(c) = c\partial_{q+1}$ , so that  $\delta_q$  raises dimension by one. The collection of coboundary operators connect the cochain groups  $C^q(K)$  into the *cochain complex*  $C^*(K)$ :  $C^0(K) \xrightarrow{\delta_0} C^1(K) \xrightarrow{\delta_1} C^2(K) \xrightarrow{\delta_2} C^3(K) \xrightarrow{\delta_3} \dots$ . We define  $Z^q(K)$  to be the kernel of  $\delta_q$  and  $B^{q+1}(K)$  to be its image. These groups are called the group of  $q$ -cocycles and  $q$ -coboundaries, respectively. Noting that  $\delta_q^2 = 0$  because  $\partial_q^2 = 0$ , define the  $q$ th *cohomology group*,  $H^q(K) = Z^q(K)/B^q(K)$  for  $q \geq 0$ .

The cochain complex  $C^*(K)$  is an algebra with the *cup product*  $\smile : C^p(K) \times C^q(K) \rightarrow C^{p+q}(K)$  given by:

$$(c \smile c')(\sigma) = c(\langle v_0, \dots, v_p \rangle) \bullet c'(\langle v_p, \dots, v_{p+q} \rangle)$$

where  $\sigma = \langle v_0, \dots, v_{p+q} \rangle$  is a  $(p+q)$ -simplex and  $\bullet$  is the natural product defined on  $\mathbf{Z}/\mathbf{Z}2$  [15, p. 292]. It induces an operation  $\smile : H^p(K) \times H^q(K) \rightarrow H^{p+q}(K)$ , via  $[c] \smile [c'] = [c \smile c']$ , that is bilinear, associative, commutative (up to a sign if the ground ring is not  $\mathbf{Z}/\mathbf{Z}2$ ), independent of the ordering of the vertices of  $K$  and topological invariant (more concretely, homotopy-type invariant) [15, p. 289], since the coboundary formula  $\delta_{p+q}(c \smile c') = \delta_p(c) \smile c' + c \smile \delta_q(c')$  holds for any  $c \in C^p(K)$  and  $c' \in C^q(K)$ .

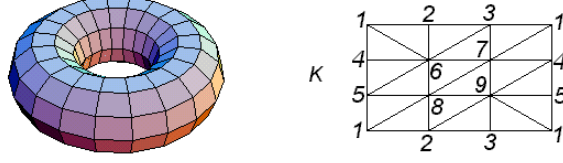


Figure 3: A triangulation of a torus.

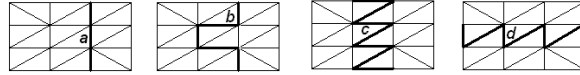


Figure 4: On the left, the 1-cycles  $a$  and  $b$ , and on the right, the 1-cocycles  $c$  and  $d$ .

**Example 3.1.** Consider the complex  $K$  pictured in Figure 3 which is obtained from a triangulation of a torus.

It is easy to check that the two 1-chains  $a = \langle 3, 7 \rangle + \langle 7, 9 \rangle + \langle 3, 9 \rangle$  and  $b = \langle 3, 7 \rangle + \langle 6, 7 \rangle + \langle 6, 8 \rangle + \langle 8, 9 \rangle + \langle 3, 9 \rangle$  (see Figure 4) are 1-cycles. For example,  $\partial_1(a) = \langle 3 \rangle + \langle 7 \rangle + \langle 7 \rangle + \langle 9 \rangle + \langle 3 \rangle + \langle 9 \rangle = 0$ . Moreover,  $a$  and  $b$  are homologous:  $\partial_2(\langle 6, 7, 8 \rangle + \langle 7, 8, 9 \rangle) = a + b$ . On the other hand,  $c = \langle 2, 3 \rangle^* + \langle 3, 6 \rangle^* + \langle 6, 7 \rangle^* + \langle 7, 8 \rangle^* + \langle 8, 9 \rangle^* + \langle 2, 9 \rangle^*$  and  $d = \langle 4, 5 \rangle^* + \langle 5, 6 \rangle^* + \langle 6, 8 \rangle^* + \langle 7, 8 \rangle^* + \langle 7, 9 \rangle^* + \langle 4, 9 \rangle^*$  are two 1-cocycles. To check this, we have to verify that  $\delta_1(c)$  and  $\delta_1(d)$  vanishes on all the 2-simplices of  $K$ . For example  $\delta_1(c)(\langle 2, 3, 6 \rangle) = c(\partial_2(\langle 2, 3, 6 \rangle)) = c(\langle 2, 3 \rangle) + c(\langle 2, 6 \rangle) + c(\langle 3, 6 \rangle) = 0$ . To check that both 1-cocycles are not coboundaries is a more difficult task since we have to verify that  $\delta_2(f) \neq c, d$  for any  $f$  being a 2-cochain.

The cup product of  $c$  and  $d$  is a new 2-cocycle  $c \smile d$ . By direct computation, we have that  $c \smile d = \langle 6, 7, 8 \rangle^*$ . We obtain this by applying it on all the 2-simplices of  $K$ . For example,  $(c \smile d)(\langle 6, 7, 8 \rangle) = c(\langle 6, 7 \rangle) \bullet d(\langle 7, 8 \rangle) = 1$  and  $(c \smile d)(\langle 7, 8, 9 \rangle) = c(\langle 7, 8 \rangle) \bullet d(\langle 8, 9 \rangle) = 0$ .

The example illustrates that while we can think of a 1-cycle as being a closed curve, the best way to think of a 1-cocycle is a picket fence.

**Chain Contractions.** In a more general framework, a chain complex  $\mathcal{C}$  is a sequence  $\cdots \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$  of abelian groups  $C_q$  and homomorphisms  $\partial_q$ , indexed with the non-negative integers, such that



for all  $q$ ,  $\partial_q \partial_{q+1} = 0$ . The  $q$ th homology group is the quotient group  $\text{Ker } \partial_q / \text{Im } \partial_{q+1}$ , denoted by  $H_q(\mathcal{C})$ . Let  $\mathcal{C} = \{C_q, \partial_q\}$  and  $\mathcal{C}' = \{C'_q, \partial'_q\}$  be two chain complexes. A chain map  $f : \mathcal{C} \rightarrow \mathcal{C}'$  is a family of homomorphisms  $\{f_q : C_q \rightarrow C'_q\}_{q \geq 0}$  such that  $\partial'_q f_q = f_{q-1} \partial_q$ . A chain map  $f : \mathcal{C} \rightarrow \mathcal{C}'$  induces a homomorphism  $f_* : H(\mathcal{C}) \rightarrow H(\mathcal{C}')$ .

Let us emphasize that a fundamental notion here is that of chain contraction.

**Definition 3.1.** [14] A chain contraction of a chain complex  $\mathcal{C}$  to another chain complex  $\mathcal{C}'$  is a set of three homomorphisms  $(f, g, \phi)$  such that:

- $f : \mathcal{C} \rightarrow \mathcal{C}'$  and  $g : \mathcal{C}' \rightarrow \mathcal{C}$  are chain maps.
- $fg$  is the identity map of  $\mathcal{C}'$ .
- $\phi : \mathcal{C} \rightarrow \mathcal{C}$  is a chain homotopy of the identity map  $id_{\mathcal{C}}$  of  $\mathcal{C}$  to  $gf$ , that is,  $\phi \partial + \partial \phi = id_{\mathcal{C}} + gf$ .

Important properties of chain contractions are:

- $\mathcal{C}'$  has fewer or the same number of generators than  $\mathcal{C}$ .
- $\mathcal{C}$  and  $\mathcal{C}'$  have isomorphic homology groups [15, p. 73].

Let us recall that the key of our method for computing the cohomology ring of chain complexes, is the construction of chain contractions  $(f, g, \phi)$  of a given chain complex  $\mathcal{C}$  to another chain complex  $\mathcal{H}$  (isomorphic to the homology of  $\mathcal{C}$ ). In this case, for each cycle  $a \in \mathcal{C}$ , the chain  $f(a) \in \mathcal{H}$  determines the homology class of  $a$ . Conversely, for each  $\alpha \in \mathcal{H}$  (which corresponds to a homology class of  $H(\mathcal{C})$ ),  $g(\alpha) \in \mathcal{C}$  determines a representative cycle of it. Finally, if  $a \in \mathcal{C}$  is a boundary, then  $a' = \phi(a)$  is a chain in  $\mathcal{C}$  such that  $\partial(a') = a$ .

### 3.1. Topological Thinning

Topological thinning is an important preprocessing operation in Image Processing. The aim is to shrink a digital picture to a smaller, simpler picture which retains a lot of the significant information of the original. Then, further processing or analysis can be performed on the shrunken picture.

There is a well-known process for thinning a simplicial complex using simplicial collapses [4]. Suppose  $K$  is a simplicial complex,  $\sigma \in K$  is a

maximal simplex and  $\sigma'$  is a free facet of  $\sigma$ . Then,  $K$  *simplicially collapses* onto  $K - \{\sigma', \sigma\}$ . An important property of this process is that there exists an explicit chain contraction of  $C(K)$  to  $C(K - \{\sigma', \sigma\})$  [6]. More generally, a *simplicial collapse* is any sequence of such operations. A *thinned* simplicial complex  $M_{top}K$  is a subcomplex of  $K$  with the condition that all the faces of the maximal simplices of  $M_{top}K$  are shared. Then, it is obvious that it is no longer possible to collapse.

The following algorithm computes  $M_{top}K$  (first step) and a chain contraction  $(f_{top}, g_{top}, \phi_{top})$  of  $C(K)$  to  $C(M_{top}K)$  (second step). In particular, recall that this means that the (co)homology of  $K$  and  $M_{top}K$  are isomorphic. Each step of the algorithm runs in time at most  $O(m^2)$  if  $K$  has  $m$  simplices.

**Algorithm 3.2.** *Topological Thinning Algorithm.*

*First step: Simplicial collapses.*

INPUT: A simplicial complex  $K$ .

Initially,  $M_{top}K := K$ , collapse := ( ), pair := True.

**While** pair is True **do**

pair := False.

**For each**  $\sigma \in M_{top}K$  **do**

**If**  $\sigma$  is maximal with a free facet  $\sigma'$  in  $M_{top}K$  **then**

$M_{top}K := K - \{\sigma', \sigma\}$ ,

collapse :=  $(\sigma', \sigma) \cup$  collapse,

pair := True.

**End if.**

**End for.**

**End while.**

OUTPUT: the simplicial complex:  $M_{top}K$

and the sorted set of simplices: collapse.

*Second step: the computation of the chain contraction.*

INPUT: The simplicial complexes  $K$  and  $M_{top}K$

and the sorted set collapse =  $(\sigma'_1, \sigma_1, \dots, \sigma'_n, \sigma_n)$ .

Initially,  $f_{top}(\sigma) := \sigma$ ,  $\phi_{top}(\sigma) := 0$  for each  $\sigma \in K$ ;

and  $g_{top}(\sigma) := \sigma$  for each  $\sigma \in M_{top}K$ .

**For**  $i = 1$  **to**  $i = n$  **do**

$f_{top}(\sigma'_i) := f_{top}(\partial\sigma_i + \sigma'_i)$ ,

$\phi_{top}(\sigma'_i) := \sigma_i + \phi_{top}(\partial\sigma_i + \sigma'_i)$ ,

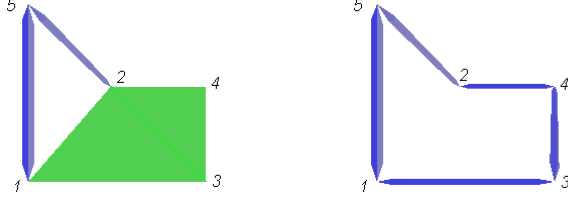


Figure 5: The simplicial complexes  $L$  (on the left) and  $M_{top}L$  (on the right).

$$f_{top}(\sigma_i) := 0.$$

**End for.**

OUTPUT: the chain contraction  $(f_{top}, g_{top}, \phi_{top})$  of  $C(K)$  to  $C(M_{top}K)$ .

**Example 3.3.** Consider the simplicial complex  $L$  whose set of maximal simplices is  $\{\langle 1, 5 \rangle, \langle 2, 5 \rangle, \langle 1, 2, 3 \rangle, \langle 2, 3, 4 \rangle\}$  (see Figure 5). Applying the first part of the algorithm above we have that  $M_{top}L = \{\langle 1, 3 \rangle, \langle 3, 4 \rangle, \langle 2, 4 \rangle, \langle 1, 5 \rangle, \langle 2, 5 \rangle\}$  and  $\text{collapse} = (\langle 2, 3 \rangle, \langle 2, 3, 4 \rangle, \langle 1, 2 \rangle, \langle 1, 2, 3 \rangle)$ . The stages of the second part of the algorithm is showed in the following table:

| $\sigma$                  | $f_{top}(\sigma)$   | $\phi_{top}(\sigma)$   |
|---------------------------|---|--|
| $\langle 2, 3 \rangle$    | $f_{top}\langle 2, 4 \rangle + f_{top}\langle 3, 4 \rangle$<br>$= \langle 2, 4 \rangle + \langle 3, 4 \rangle$                        | $\langle 2, 3, 4 \rangle + \phi_{top}\langle 2, 4 \rangle + \phi_{top}\langle 3, 4 \rangle$<br>$= \langle 2, 3, 4 \rangle$                           |
| $\langle 2, 3, 4 \rangle$ | 0   | 0  |
| $\langle 1, 2 \rangle$    | $f_{top}\langle 1, 3 \rangle + f_{top}\langle 2, 3 \rangle$<br>$= \langle 1, 3 \rangle + \langle 2, 4 \rangle + \langle 3, 4 \rangle$ | $\langle 1, 2, 3 \rangle + \phi_{top}\langle 1, 3 \rangle + \phi_{top}\langle 2, 3 \rangle$<br>$= \langle 1, 2, 3 \rangle + \langle 2, 3, 4 \rangle$ |
| $\langle 1, 2, 3 \rangle$ | 0   | 0  |

### 3.2. “Algebraic Thinning”

Having obtained the thinned complex  $M_{top}K$ , we next construct a chain contraction  $(f_{alg}, g_{alg}, \phi_{alg})$  of the chain complex  $C(M_{top}K)$  to its homology. This step can be considered as a thinning at algebraic level (for this reason we call it “algebraic thinning”). We compute  $(f_{alg}, g_{alg}, \phi_{alg})$  interpreting the “incremental algorithm” [5] for computing homology groups in  $\mathbf{R}^3$ , in terms of chain contractions. As we will see later, the design of an algorithm for computing the cohomology ring of  $K$ , will be possible thanks to the information saved in the chain contraction of  $C(K)$  to its homology constructed before.

Let  $(\sigma_1, \dots, \sigma_m)$  be a sorted set of all the simplices of  $K$  with the property that any subset  $\{\sigma_1, \dots, \sigma_i\}$ ,  $i \leq m$ , is a subcomplex of it. Algorithm 3.4 computes a chain complex  $\mathcal{H}$  with a set of generators  $h$ , and a chain contraction  $(f_{alg}, g_{alg}, \phi_{alg})$  of  $C(K)$  to  $\mathcal{H}$ . Initially,  $h$  is empty. In the  $i$ th step of the algorithm, the simplex  $\sigma_i$  is added to the subcomplex  $\{\sigma_1, \dots, \sigma_{i-1}\}$  and then, a homology class is created or destroyed. If  $f_{alg}\partial(\sigma_i) = 0$  then  $\sigma_i$  “creates” a homology class. Otherwise,  $\sigma_i$  “destroys” one homology class “involved” in the expression of  $f_{alg}\partial(\sigma_i)$ . At the end of the algorithm,  $\mathcal{H}$  is a chain complex isomorphic to the homology of  $K$ .

**Algorithm 3.4.** *Algebraic Thinning Algorithm*

INPUT: The sorted set  $(\sigma_1, \dots, \sigma_m)$ .

Initially,  $f_{alg}(\sigma) := 0$ ,  $\phi_{alg}(\sigma) := 0$  for each  $\sigma \in K$ ; and  $h := \{ \}$ .

**For**  $i = 1$  to  $i = m$  do

**If**  $f_{alg}\partial(\sigma_i) = 0$  then

$h := h \cup \{\sigma_i\}$

$f_{alg}(\sigma_i) := \sigma_i$ .

**Else** take any one  $\sigma_j$  of  $f_{alg}\partial(\sigma_i)$ , then

$h := h - \{\sigma_j\}$ ,

**For**  $k = 1$  to  $k = m$  do

**If**  $\sigma_j$  appears in the expression of  $f_{alg}(\sigma_k)$  then

$f_{alg}(\sigma_k) := f_{alg}(\sigma_k) + f_{alg}\partial(\sigma_i)$

$\phi_{alg}(\sigma_k) := \phi_{alg}(\sigma_k) + \sigma_i + \phi_{alg}\partial(\sigma_i)$

**End if.**

**End for.**

**End if.**

**End for.**

**For** each  $\sigma \in h$  do

$g_{alg}(\sigma) := \sigma + \phi_{alg}\partial(\sigma)$ .

**End for .**

OUTPUT: the chain contraction  $(f_{alg}, g_{alg}, \phi_{alg})$  of  $C(K)$  to  $\mathcal{H}$ .

The output of the algorithm allows us to determine both a representative cycle for each homology class and the homology class for each cycle. Moreover, for any  $q$ -boundary  $a$  on  $K$  we can obtain a  $(q + 1)$ -chain  $a' = \phi_{alg}(a)$  on  $K$  such that  $a = \partial(a')$ .

Concerning to the complexity, suppose  $K$  has  $m$  simplices. In the  $i$ th step of the algorithm ( $1 \leq i \leq m$ ), we have to evaluate  $\partial\sigma_i$ . The number of simplices involved in  $\partial\sigma_i$  is fewer or the same than the dimension of  $\sigma_i$  which is at most 3. On the other hand, the number of elements involved in the formulae for  $f_{alg}\partial\sigma_i$  and  $\phi_{alg}\partial\sigma_i$  is  $O(3m) = O(m)$ . If  $\partial\sigma_i \neq 0$ , we have to update  $f_{alg}\partial\sigma_k$  and  $\phi_{alg}\partial\sigma_k$  for  $1 \leq k \leq m$ , so the total cost of these operations is  $O(m^2)$ . Therefore, the total algorithm runs in time at most  $O(m^3)$ .

Let us observe that composing the chain contraction  $(f_{top}, g_{top}, \phi_{top})$  of  $C(K)$  to  $C(M_{top}K)$ , described in the previous subsection, with  $(f_{alg}, g_{alg}, \phi_{alg})$  of  $C(M_{top}K)$  to  $\mathcal{H}$  (isomorphic to  $H(K)$ ), we get a new chain contraction  $(f_{alg}f_{top}, g_{top}g_{alg}, \phi_{top} + g_{top}\phi_{alg}f_{top})$  of  $C(K)$  to  $\mathcal{H}$ .

**Example 3.5.** Let  $L$  be the simplicial complex showed in Figure 6. The intermediate stages of the algorithm are:

| $i$                                   | 0 | 1                   | 2                   | 3                   | 4                      | 5                   | 6   | 7                      | 8                      | 9                      | 10                        |
|---------------------------------------|---|---------------------|---------------------|---------------------|------------------------|---------------------|---|------------------------|------------------------|------------------------|---------------------------|
| $\sigma$                              |   | $\langle 1 \rangle$ | $\langle 2 \rangle$ | $\langle 3 \rangle$ | $\langle 2, 3 \rangle$ | $\langle 4 \rangle$ | $\langle 3, 4 \rangle$                        | $\langle 1, 4 \rangle$ | $\langle 1, 2 \rangle$ | $\langle 2, 4 \rangle$ | $\langle 2, 3, 4 \rangle$ |
| $f_{alg}(\langle 1 \rangle)$          | 0 | $\langle 1 \rangle$ |                     |                     |                        |                     |   | $\langle 4 \rangle$    |                        |                        |                           |
| $\phi_{alg}(\langle 1 \rangle)$       | 0 |                     |                     |                     |                        |                     |   | $\langle 1, 4 \rangle$ |                        |                        |                           |
| $f_{alg}(\langle 2 \rangle)$          | 0 |                     | $\langle 2 \rangle$ |                     | $\langle 3 \rangle$    |                     | $\langle 4 \rangle$                           |                        |                        |                        |                           |
| $\phi_{alg}(\langle 2 \rangle)$       | 0 |                     |                     |                     | $\langle 2, 3 \rangle$ |                     | $\langle 2, 3 \rangle + \langle 3, 4 \rangle$ |                        |                        |                        |                           |
| $f_{alg}(\langle 3 \rangle)$          | 0 |                     |                     | $\langle 3 \rangle$ |                        |                     | $\langle 4 \rangle$                           |                        |                        |                        |                           |
| $\phi_{alg}(\langle 3 \rangle)$       | 0 |                     |                     |                     |                        |                     | $\langle 3, 4 \rangle$                        |                        |                        |                        |                           |
| $f_{alg}(\langle 2, 3 \rangle)$       | 0 |                     |                     |                     |                        |                     |   |                        |                        |                        |                           |
| $\phi_{alg}(\langle 2, 3 \rangle)$    | 0 |                     |                     |                     |                        |                     |   |                        |                        |                        |                           |
| $f_{alg}(\langle 4 \rangle)$          | 0 |                     |                     |                     |                        | $\langle 4 \rangle$ |   |                        |                        |                        |                           |
| $\phi_{alg}(\langle 4 \rangle)$       | 0 |                     |                     |                     |                        |                     |   |                        |                        |                        |                           |
| $f_{alg}(\langle 3, 4 \rangle)$       | 0 |                     |                     |                     |                        |                     |   |                        |                        |                        |                           |
| $\phi_{alg}(\langle 3, 4 \rangle)$    | 0 |                     |                     |                     |                        |                     |   |                        |                        |                        |                           |
| $f_{alg}(\langle 1, 4 \rangle)$       | 0 |                     |                     |                     |                        |                     |   |                        |                        |                        |                           |
| $\phi_{alg}(\langle 1, 4 \rangle)$    | 0 |                     |                     |                     |                        |                     |   |                        |                        |                        |                           |
| $f_{alg}(\langle 1, 2 \rangle)$       | 0 |                     |                     |                     |                        |                     |   |                        | $\langle 1, 2 \rangle$ |                        |                           |
| $\phi_{alg}(\langle 1, 2 \rangle)$    | 0 |                     |                     |                     |                        |                     |   |                        |                        |                        |                           |
| $f_{alg}(\langle 2, 4 \rangle)$       | 0 |                     |                     |                     |                        |                     |   |                        |                        | $\langle 2, 4 \rangle$ |                           |
| $\phi_{alg}(\langle 2, 4 \rangle)$    | 0 |                     |                     |                     |                        |                     |   |                        |                        |                        | $0$                       |
| $f_{alg}(\langle 2, 3, 4 \rangle)$    | 0 |                     |                     |                     |                        |                     |   |                        |                        |                        | $\langle 2, 3, 4 \rangle$ |
| $\phi_{alg}(\langle 2, 3, 4 \rangle)$ | 0 |                     |                     |                     |                        |                     |   |                        |                        |                        |                           |

Finally,  $h = \{\langle 4 \rangle, \langle 1, 2 \rangle\}$ ,  $g_{alg}(\langle 4 \rangle) := \langle 4 \rangle$  and  $g_{alg}(\langle 1, 2 \rangle) := \langle 1, 2 \rangle + \langle 1, 4 \rangle + \langle 2, 3 \rangle + \langle 2, 4 \rangle$ .

Summing up, the output is a chain contraction  $(f_{alg}, g_{alg}, \phi_{alg})$  of  $C(L)$  to the chain complex  $\mathcal{H}_L$  (isomorphic to  $H(L)$ ) with set of generators  $h = \{\langle 4 \rangle, \langle 1, 2 \rangle\}$ . In particular, we obtain that  $H_0(L) \simeq \mathbf{Z}/\mathbf{Z}2$ ,  $H_1(L) \simeq \mathbf{Z}/\mathbf{Z}2$  and  $H_2(L) = 0$ .

### 3.3. Computing the Cohomology Ring

After applying in order topological and algebraic thinning to the simplicial complex  $K$ , we are able to compute the multiplication table on the

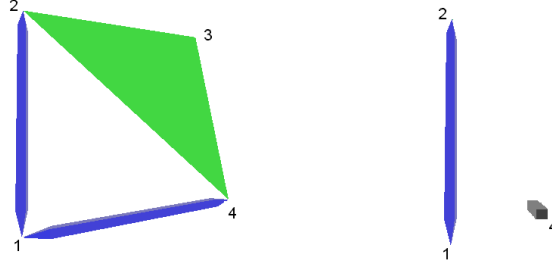


Figure 6: The complex  $L$  (on the left) and the generators of  $\mathcal{H}_L$  (on the right).

cohomology. Since the ground ring is a field, the homology and cohomology groups of  $K$  are always isomorphic.

Let  $(f, g, \phi)$  be a chain contraction of  $C(K)$  to  $\mathcal{H}$  (where  $\mathcal{H}$  is a chain complex isomorphic to  $H(K)$  and to  $H^*(K)$ ), and let  $h = \{\alpha_1, \dots, \alpha_p\}$  be a set of generators of  $\mathcal{H}$  obtained using the algorithms explained before. Then  $\alpha_i^* f : C_q(K) \rightarrow \mathbf{Z}/\mathbf{Z}2$  (where  $\alpha_i^*(\alpha_j) = 1$  if  $j = i$  and 0 otherwise) is a representative cocycle of the cohomology class corresponding to  $\alpha_i$ , for  $1 \leq i \leq p$ .

Let  $\alpha$  and  $\beta$  be two elements of  $h$ , then the cup product of the cohomology classes corresponding to  $\alpha$  and  $\beta$  can be computed as follows:

**Algorithm 3.6.** *The cup product of two classes of cohomology.*

INPUT: the elements  $\alpha$  and  $\beta$

and the set of generators  $h = \{\alpha_1, \dots, \alpha_p\}$  of  $\mathcal{H}$ .

Initially,  $\lambda_k := 0$  for  $1 \leq k \leq p$  and  $cup := 0$ .

For  $k = 1$  to  $k = p$  do

$\lambda_k := (\alpha^* f \smile \beta^* f)g(\alpha_k)$ .

End for.

$cup := \sum_{k=1}^p \lambda_k \alpha_k$ .

OUTPUT: the chain  $cup$ .

Observe that the complexity of this algorithm for computing  $\alpha \smile \beta$  is  $O(m^4)$ . Moreover, if we are interested in computing the cohomology ring of  $K$ , we have to apply the algorithm above to all the pairs  $(\alpha_i, \alpha_j)$ ,  $1 \leq i \leq j \leq p$  (since the cup product is commutative,  $\alpha_i \smile \alpha_j = \alpha_j \smile \alpha_i$ ). Then, the

algorithm for computing the cohomology ring of  $K$  will run in time at most  $O(m^6)$  if  $K$  has  $m$  simplices.

Let us note that the cohomology ring of  $K$  is not suitable in general for topological classification tasks. This is due to the fact that determining whether two rings are isomorphic or not by means of their respective multiplication tables is an extremely difficult computational question. In order to avoid this problem, we will put the information of the cup product table into a different form.

If we restrict our interest in simplicial complexes are embedded in  $\mathbf{R}^3$ , observe that the possible non-trivial cup products are the ones  $\alpha \smile \beta$  where both  $\alpha$  and  $\beta$  are elements of  $h$  corresponding to cohomology classes of dimension 1.

In order to design a new algorithm for computing the cup product in a way that we can determine whether two cohomology rings are isomorphic or not by means of their respective multiplication tables, we need to define a new concept. Given a chain contraction  $(f, g, \phi)$  of  $C(K)$  to a chain complex  $\mathcal{H}$  (isomorphic to  $H^*(K)$ ) with set of generators  $h$ , and a simplex  $\sigma = \langle v_0, v_1, v_2 \rangle$  of dimension 2, suppose that  $\{\alpha_1, \dots, \alpha_p\}$  is the set of elements of  $h$  of dimension 1,  $f(\langle v_0, v_1 \rangle) = \sum_{i \in I} \alpha_i$  and  $f(\langle v_1, v_2 \rangle) = \sum_{j \in J} \alpha_j$  where  $I$  and  $J$  are subsets of the set  $\{1, 2, \dots, p\}$ . Define  $(f \odot f)(\sigma) = \sum_{i \in I} \sum_{j \in J} (\alpha_i, \alpha_j)$ . This definition can be extended to 2-chains by linearity.

**Algorithm 3.7.** *Cup Product Algorithm*

INPUT: A simplicial complex  $K$   
and a chain contraction  $(f, g, \phi)$  of  $C(K)$   
to a chain complex  $\mathcal{H}$  (isomorphic to  $H^*(K)$ )  
with set of generators  $h$ .

Initially,  $q :=$  the number of elements of  $h$  of dimension 2,  
 $b_i := 0$  for  $1 \leq i \leq q$  and  $M := ()$ .

**For**  $i = 1$  to  $i = q$  do  
 $b_i := (f \odot f)g(\alpha_i)$ .

**End for.**

$M := (b_1, \dots, b_q)$ .

OUTPUT: The sorted set  $M$ .

Let  $\{\alpha_1, \dots, \alpha_p\}$  be the set of elements of  $h$  of dimension 1 and  $\{\beta_1, \dots, \beta_q\}$  the ones of dimension 2. Each  $b_i$ ,  $1 \leq i \leq q$ , is of the form  $\sum \lambda_{jk}^i (\alpha_j, \alpha_k)$  where the sum is taken over the set  $\{(j, k) : 1 \leq j, k \leq p\}$  and  $\lambda_{j,k}^i = (\alpha_j^* f \smile$



$\alpha_k^* f)g(\beta_i)$  could be 0 or 1. Since the cup product is commutative, we have that  $\lambda_{jk}^i = \lambda_{kj}^i$ . Therefore, the output of this algorithm can be put into a matrix form  $\mathcal{M}$  of (cohomology classes  $\beta_i, 1 \leq i \leq q$ )  $\times$  (pairs of cohomology classes  $(\alpha_j, \alpha_k), 1 \leq j \leq k \leq p$ ). The column of  $\mathcal{M}$  corresponding to the pair  $(\alpha_j, \alpha_k), 1 \leq j \leq k \leq p$ , gives the value of the cup product  $\alpha_j \smile \alpha_k$ . This algorithm for computing the matrix  $\mathcal{M}$  runs in time at most  $O(m^4)$  if  $K$  has  $m$  simplices.

From the diagonalization  $D$  of the matrix  $\mathcal{M}$ , a first cohomology invariant  $HB_1(K)$  for distinguishing non-homeomorphic simplicial complexes with isomorphic (co)homology groups appears. We define this cohomology number in order to have a handy numerical tool for distinguishing 3D digital pictures.

**Definition 3.2.** *Given a simplicial complex  $K$ , the integer  $HB_1(K)$  is a cohomology invariant defined as the rank of the matrix  $\mathcal{M}$ .*

#### 4. A First Approach to the Digital Cohomology Ring

Since an isomorphism of pictures on the 3D body centered cubic grid is equivalent to a simplicial homeomorphism of the corresponding simplicial representations, we are able to define the digital cohomology ring of  $I$  with coefficients in  $\mathbf{Z}/\mathbf{Z}2$  as the cohomology ring of  $K(I)$  with coefficients in  $\mathbf{Z}/\mathbf{Z}2$ . Moreover, the following definitions hold:

**Definition 4.1.** *Given a digital picture  $I = (\mathcal{V}, 14, 14, B)$ , the digital cohomology ring of  $I$  with coefficients in  $\mathbf{Z}/\mathbf{Z}2$  is defined as the cohomology ring of  $K(I)$  with coefficients in  $\mathbf{Z}/\mathbf{Z}2$ . The cohomology invariant  $HB_1(I)$  is defined as  $HB_1(K(I))$ .*

In the previous sections, we have showed that it is possible to compute the digital cohomology ring of  $I$  with coefficients in  $\mathbf{Z}/\mathbf{Z}2$ . The steps of the method are: first, we construct the simplicial complex  $K(I)$ . Second, we topologically thin  $K(I)$ , obtaining a smaller simplicial complex  $M_{top}K(I)$  and a chain contraction  $(f_{top}, g_{top}, \phi_{top})$  of  $C(K(I))$  to  $C(M_{top}K(I))$ . Third, we compute  $\mathcal{H}$  which is isomorphic to  $H(I)$  and a chain contraction  $(f_{alg}, g_{alg}, \phi_{alg})$  of  $C(M_{top}K(I))$  to  $\mathcal{H}$ . Four, we calculate the cohomology ring of  $I$  via the cohomology ring of  $M_{top}K(I)$  and the invariant  $HB_1(I)$  via  $HB_1(M_{top}K(I))$ , using the chain contractions constructed before. All the information obtained in this way is useful for topologically classifying and distinguishing binary 3D digital pictures.

#### 4.1. Some Examples

In order to show examples of the computation and visualization of the cohomology ring of simple pictures, we expose a small prototype called EditCup. We use a free program for building 3D worlds. In our case, a world is a particular 3D simplicial complex  $K$  representing a digital picture  $I$  considering the 14-adjacency. A way for distinguishing the different maximal simplices of a simplicial representation is by using different colours: red for tetrahedra, green for triangles, blue for edges, and black for vertices.

All the computations are done considering  $\mathbf{Z}/\mathbf{Z}2$  as the ground ring. For visualizing (co)chains, the simplices on which a given (co)chain is non-null, are lighted in a different color. On the other hand, the “visualization” of any (co)homology class on  $K$  is given by lighting the simplices of  $K$  on which the representative cochain of this class is non-null. Moreover, the “visualization” of any (co)homology class on the original 3D digital binary-valued picture  $I$  could be given by lighting the points of  $I$  such that the corresponding vertices span simplices on which the representative cochain of this class is non-null.

The DPS used in these examples, that we call (14, 14)-DPS, is  $(\mathbf{Z}^3, 14, 14)$ , in which the underlying grid is the set of points with integer coordinates in the Euclidean 3-space  $E^3$  and the 14-neighbours of a grid point (black or white) with integer coordinates  $(x, y, z)$  are:  $(x \pm 1, y, z)$ ,  $(x, y \pm 1, z)$ ,  $(x, y, z \pm 1)$ ,  $(x + 1, y - 1, z)$ ,  $(x - 1, y + 1, z)$ ,  $(x + 1, y, z - 1)$ ,  $(x - 1, y, z + 1)$ ,  $(x, y + 1, z - 1)$ ,  $(x, y - 1, z + 1)$ ,  $(x + 1, y + 1, z - 1)$ ,  $(x - 1, y - 1, z + 1)$  (see Figure 7). The (14, 14)-DPS and the BCC grid are isomorphic DPSs: each grid point  $(x, y, z)$  of the (14, 14)-DPS can be associated to a point  $(a, b, c)$  via the formula:  $(a, b, c) = (x + y - 2z, -x + y, -x - y)$ .

Let us consider now the following pictures: a torus  $I = (\mathbf{Z}^3, 14, 14, B_I)$  and a wedge of two topological circles and a topological 2-sphere  $J = (\mathbf{Z}^3, 14, 14, B_J)$  (see Figure 8). In the volumetric representation of the picture  $I$  (resp.  $J$ ), we use voxels with centres the points  $B_I$  (resp.  $B_J$ ). It is clear that the (co)homology groups of  $I$  are isomorphic to those of  $J$ . They are  $\mathbf{Z}/\mathbf{Z}2$ ,  $\mathbf{Z}/\mathbf{Z}2 \oplus \mathbf{Z}/\mathbf{Z}2$  and  $\mathbf{Z}/\mathbf{Z}2$  of dimension 0, 1 and 2, respectively. So, the (co)homology information is not enough for topological distinguishing both pictures.

Let  $K(I)$  and  $K(J)$  be the simplicial representations of  $I$  and  $J$  respectively (see Figure 9). In order to compare the cohomology ring of both pictures, the first step is the computation of chain contractions  $(f_I, g_I, \phi_I)$  of  $C(K(I))$  to  $\mathcal{H}_I$  and  $(f_J, g_J, \phi_J)$  of  $C(K(J))$  to  $\mathcal{H}_J$  using the topological

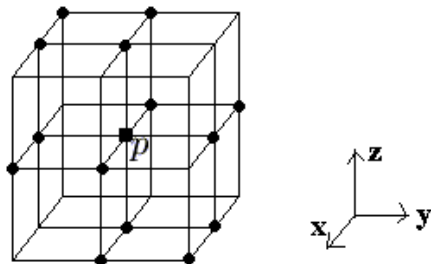


Figure 7: The 14-neighbours of a grid point  $p$  of the (14,14)-DPS.



Figure 8: The volumetric representation of the pictures  $I$  (on the left) and  $J$  (on the right).

and algebraic thinning algorithms explained before, where  $\mathcal{H}_I$  (resp.  $\mathcal{H}_J$ ) is a chain complex isomorphic to the (co)homology of  $I$  (resp.  $J$ ).

Let us denote by  $\alpha_1$  and  $\alpha_2$  (resp.  $\alpha'_1$  and  $\alpha'_2$ ) the generators of  $\mathcal{H}_I$  (resp.  $\mathcal{H}_J$ ) of dimension 1 and  $\alpha_3$  (resp.  $\alpha'_3$ ) the generator of  $\mathcal{H}_I$  (resp.  $\mathcal{H}_J$ ) of dimension 2. Let us also denote by  $a_i$  the representative cycles of the generators of  $H(I)$  (that is,  $a_i = g_I(\alpha_i)$ ); and by  $a'_i$  the same of  $H(J)$ . We visualize these cycles on  $K(I)$  and  $K(J)$  in Figure 10. In Figure 11, the representative cocycles  $b_i$  (resp.  $b'_i$ ) obtained via the formula  $b_i = \alpha_i^* f_I$  (resp.  $b'_i = \alpha_i^* f_J$ ) of the generators of  $H^*(I)$  (resp.  $H^*(J)$ ) are shown. Recall that we do it by lighting the simplices on which the cochains is non-null.

The output of Algorithm 3.7 for  $\mathcal{H}_I$  and  $\mathcal{H}_J$  are  $M_I = ((\alpha_1, \alpha_2) + (\alpha_2, \alpha_1))$  and  $M_J = (0)$ , respectively. The matrices corresponding to the cohomology rings of the pictures  $I$  and  $J$  are:

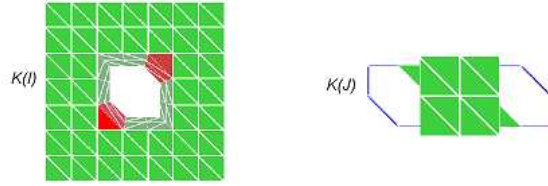


Figure 9: The simplicial complexes  $K(I)$  (on the left) and  $K(J)$  (on the right).

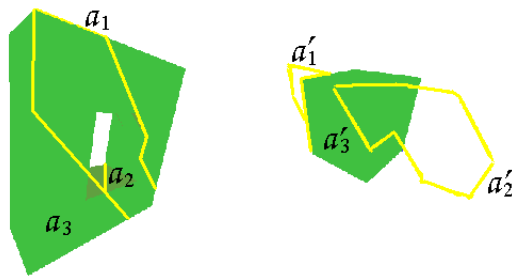


Figure 10: The cycles  $a_1, a_2$  and  $a'_1, a'_2$  (in yellow); and  $a_3$  and  $a'_3$  (in green).



Figure 11: The cocycles  $b_1, b_2$  and  $b'_1, b'_2$  (in yellow); and  $b_3$  and  $b'_3$  (in green).

$$\begin{array}{c|ccc} I & (\alpha_1, \alpha_1) & (\alpha_1, \alpha_2) & (\alpha_2, \alpha_2) \\ \hline \alpha_3 & 0 & 1 & 0 \end{array} \quad \begin{array}{c|ccc} J & (\alpha'_1, \alpha'_1) & (\alpha'_1, \alpha'_2) & (\alpha'_2, \alpha'_2) \\ \hline \alpha'_3 & 0 & 0 & 0 \end{array}$$

Therefore,  $HB_1(I) = 1$  and  $HB_1(J) = 0$ . We conclude that  $K(I)$  and  $K(J)$  are not homeomorphic (more precisely, we conclude that they are not homotopy equivalent), then  $I$  and  $J$  are not isomorphic.

Let us expose another example (see Figure 12): the picture  $A$  is a wedge of two torus; the picture  $B$  consists in a wedge of a sphere and a genus-2 torus

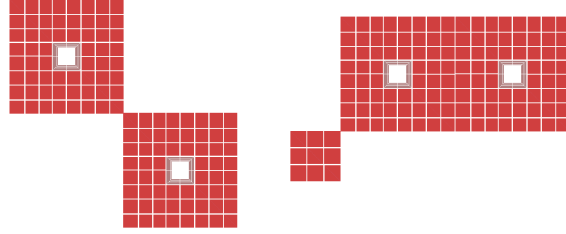


Figure 12: The pictures  $A$  (on the left) and  $B$  (on the right).

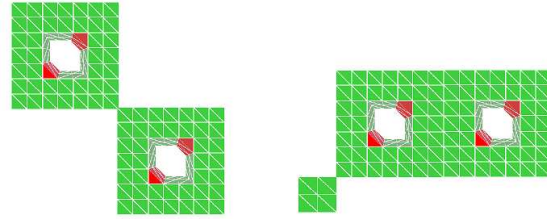


Figure 13: The simplicial complexes  $K(A)$  (on the left) and  $K(B)$  (on the right).

(a sphere with two handles and two holes). Both pictures have 1 connected component, 4 holes and 2 cavities.

The simplicial representations of  $A$  and  $B$ ,  $K(A)$  and  $K(B)$  are showed in Figure 13. In Figure 14 (resp. Figure 15), the representative cycles and cocycles of the generators of the (co)homology of  $A$  (resp.  $B$ ) are showed.

Let us denote by  $\alpha_i$  (resp.  $\alpha'_i$ ),  $i = 1, 2, 3, 4$ , the generators of  $\mathcal{H}_A$  (resp.  $\mathcal{H}_B$ ) of dimension 1; and by  $\beta_i$  (resp.  $\beta'_i$ ),  $i = 1, 2$ , the generators of  $\mathcal{H}_A$  (resp.  $\mathcal{H}_B$ ) of dimension 2. All of them are obtained using the algebraic thinning algorithm explained in the previous section. The output of Algorithm 3.7 for  $\mathcal{H}_A$  and  $\mathcal{H}_B$  is  $M_A = ((\alpha_1, \alpha_2) + (\alpha_1, \alpha_3) + (\alpha_2, \alpha_1) + (\alpha_3, \alpha_1), (\alpha_3, \alpha_4) + (\alpha_4, \alpha_3))$  and  $M_B = (0, (\alpha_1, \alpha_4) + (\alpha_2, \alpha_3) + (\alpha_3, \alpha_2) + (\alpha_4, \alpha_1))$ . Therefore, the matrices corresponding to the cohomology rings of  $A$  and  $B$  are:

| $A$       | (1, 1) | (1, 2) | (1, 3) | (1, 4) | (2, 2) | (2, 3) | (2, 4) | (3, 3) | (3, 4) | (4, 4) |
|-----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\beta_1$ | 0      | 1      | 1      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| $\beta_2$ | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 1      | 0      |

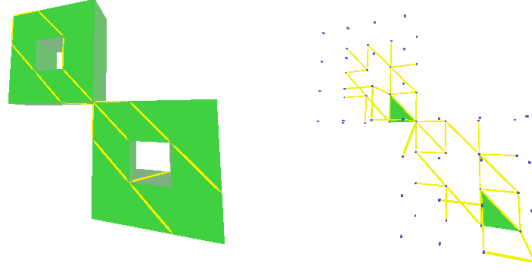


Figure 14: On the left (resp. on the right), the representative cycles (resp. cocycles) of the generators of  $H(A)$  (resp. of  $H^*(A)$ ).

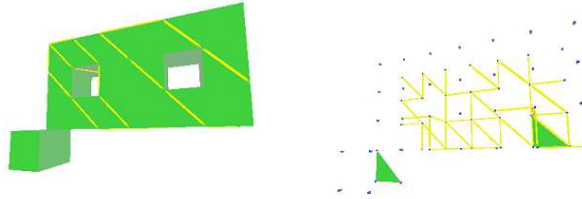


Figure 15: On the left (resp. on the right), the representative cycles (resp. cocycles) of the generators of  $H(B)$  (resp. of  $H^*(B)$ ).

| $B$        | (1, 1) | (1, 2) | (1, 3) | (1, 4) | (2, 2) | (2, 3) | (2, 4) | (3, 3) | (3, 4) | (4, 4) |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\beta'_1$ | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| $\beta'_2$ | 0      | 0      | 0      | 1      | 0      | 1      | 0      | 0      | 0      | 0      |

where  $(i, j)$  represents the pair  $(\alpha_i, \alpha_j)$  (resp.  $(\alpha'_i, \alpha'_j)$ ).

We conclude that  $HB_1(A) = 2$  and  $HB_1(B) = 1$ , and then  $K(A)$  and  $K(B)$  are not homeomorphic (more precisely, we conclude that they are not homotopy equivalent), therefore  $A$  and  $B$  are not isomorphic.

## 5. Conclusions and Future Work

We have seen that there is a true algorithm for computing the digital cohomology ring (with coefficients in  $\mathbf{Z}/\mathbf{Z}2$ ) of a 3D binary picture on the BCC grid. It is also possible to compute the digital cohomology ring of  $I$  with coefficients in any commutative ring  $G$ , thanks to the fact the simplicial complex

$K(I)$  is embedded in  $\mathbf{R}^3$  and, consequently, it have torsion-free homology. We deal here with  $\mathbf{Z}/\mathbf{Z}2$  coefficients, in order to simplify and avoiding signs in the explanation of our algorithmic formulation, to give an easy geometric interpretation of "digital" cohomology classes and to work with binary arithmetic. Moreover, there is no problem to define the cohomology ring of  $I$  with coefficients in a commutative ring  $G$  as the cohomology ring of  $K(I)$  with coefficients in  $G$ ; and the cohomology invariant  $HB_1(I; G)$  with coefficients in  $G$  as  $HB_1(K(I); G)$ . On the other hand, since  $HB_1(K(I); G)$  can be obtained from the first homology group of the reduced bar construction  $\bar{B}(C^*(K(I); G))$  [14] associated to the cochain complex  $C^*(K(I); G)$  with coefficients in  $G$ , we will confine ourselves to say that the rest of homology groups of this last algebraic object give rise to more complicated cohomology invariants for a digital binary-valued picture.

In this paper, we talk about topological and algebraic thinning. Concerning the first one, we do not use here well-known direct (without passing to simplicial framework) topological thinnings of digital binary-valued pictures because we are interested in constructing chain contractions which allow us to obtain cohomology results. Concerning the second one, the idea of computing a chain contraction of a chain complex to its homology has also been used in [7] for computing primary and secondary cohomology operations.

Another important question is to try to improve the complexity of the algorithm computing digital cohomology ring on the BCC grid detailed in this paper. We do not take advantage here neither of the particular simplicial structure of the simplicial complex  $K(I)$  (determined by the BCC grid) associated to  $I$ , nor of representing in a compressed form (without loss of information) the 3D digital picture (for example, in an octree format). To obtain positive results in these directions and to eliminate from our algorithm the intermediary simplicial objects will allow us to specify a more refined algorithm computing digital cohomology on the BCC grid.

Finally, another important question that it is necessary to deal with in a near future is to try to generalize this work to other natural digital picture spaces.

## References

- [1] Ayala R., Domínguez E., Francés A.R., Quintero A.: Homotopy in Digital Spaces. DGCI 2000. LNCS, Springer-Verlag **1953** (2000) 3–14
- [2] Alexandroff P., Hopf H.: Topologie I. Springer, Berlin 1935

- [3] Berrio J.M., González-Díaz R., Leal F., López M.M., Real P.: Visualizing Cohomology Aspects of 3D Objects. Proc. of the 6th Asian Tech. Conf. in Math. (2001) 459–468
- [4] Björner A.: Topological Methods. Handbook on Combinatorics. Elsevier Sci. **2** (1995) 1819–1872



- [5] Delfinado C.J.A., Edelsbrunner H.: An Incremental Algorithm for Betti Numbers of Simplicial Complexes on the 3-Sphere. *Comput. Aided Geom. Design* **12** (1995) 771–784
- [6] Forman R.: *Combinatorial Differential Topology and Geometry. New Perspective in Geom. Combinatorics.* MSRI Public. **8** (1999) 177–206
- [7] González-Díaz R., Real P.: Computation of Cohomology Operations on Finite Simplicial Complexes. *Homology, Homotopy and Applications* **5** (2) (2003) 83–93
- [8] González-Díaz R., Real P.: Towards Digital Cohomology. DGCI 2003, LNCS, Springer **2886** (2003) 92–101
- [9] Kenmochi Y., Imiya A.: Polyhedral Set Operations for 3D Discrete Object Deformation. *Scale-Space 2001*, LNCS, Springer-Verlag and IEEE/CS **2106** (2001) 417–425
- [10] Khalimsky E.D., Kopperman R.D., Meyer P.R.: Computer Graphics and Connected Topologies on Finite Ordered Sets. *Topology and Appl.* **36** (1990) 1–17
- [11] Kong T.Y.: A Digital Fundamental Group. *Computer Graphics* **13** (1989) 159–166
- [12] Kong T.Y., Roscoe A.W., Rosenfeld A.: Concepts of Digital Topology. *Topology and its Applications* **46** (1992) 219–262
- [13] Kovalevsky V.A.: Discrete Topology and Contour Definition. *Pattern Recognition Letter* **2** (1984) 281–288
- [14] MacLane S.: *Homology. Classic in Math., Springer-Verlag* 1995
- [15] Munkres J.R.: *Elements of Algebraic Topology.* Addison-Wesley Co. 1984
- [16] Rosenfeld A.: 3D Digital Topology. *Inform. and Control* **50** (1981) 119–127
- [17] Theussl T., Moller T., Groller M.E.: Optimal Regular Volume Sampling. *Proc. IEEE Conference on Visualization* (2001).