Integral Operators for Computing Homology Generators at Any Dimension⋆

Helena Molina-Abril⋆⋆, and Pedro Real

Applied Math Department, University of Seville, Spain
{rogodi,majiro,belenmg,habril,real}@us.es
http://alojamientos.us.es/gtocoma

Abstract. Starting from an nD geometrical object, a cellular subdivision of such an object provides an algebraic counterpart from which homology information can be computed. In this paper, we develop a process to drastically reduce the amount of data that represent the original object, with the purpose of a subsequent homology computation. The technique applied is based on the construction of a sequence of elementary chain homotopies (*integral operators*) which algebraically connect the initial object with a simplified one with the same homological information than the former.

Keywords: integer homology generators, chain homotopies.

1 Introduction

Algebraic Topology provides a great variety of tools that have potential applications to many fields such as Discrete Geometry, Data Analysis, Computer Graphics or nD Digital Image Processing. Some of these applications rely on the explicit computation of topological features. For instance, starting from an n-dimensional geometric or combinatorial object (mainly, $n = 3, 4, \ldots$), one can compute associated topological invariants, such as Betti numbers (which represent the number of connected components, holes or cavities), Euler characteristic, as well as other advanced features, such as homology groups. The information provided by the latter includes the former, so homology provides a stronger characterization of the topology.

When the ground ring is a field, the algebraic topological model given in [5,6] (inspired in the incremental technique of Delfinado-Edelsbrunner [3]) is a useful tool for computing homology and representative cycles of homology generators. Working in the integer domain, a chain homotopy version of the Smith Normal Form (SNF) method to compute homology [13,4,14] is given in [7,8].

In this paper, in order to reduce the number of generators of the initial chain complex $\mathcal{C}$, we define very simple and elementary operators, which give place

---

⋆⋆ Fellow associated to University of Seville under TIC-02268 grant.

to a sequence of chain homotopies from $\mathcal{C}$ to another chain complex $\mathcal{C}'$ with the same integer homology. Afterwards, the chain-homotopy version of the SNF given in [7,8] can be applied to obtain the integer homology of $\mathcal{C}'$, with the important property that the input data for such a computation has been significantly simplified.

The paper is organized as follows. In Section 2, we define the elementary chain homotopies *integral operators*; we provide both geometric and algebraic versions of such operators. Section 3 is devoted to describe the process of concatenating a sequence of such integral operators that guarantees the achievement of a smalle chain complex with the same homology than the original one. We also present an example of application of the method on a real binary volume image of a trabecular bone. In the last section, some conclusions are drawn.

2   Integral Operators

There exist several combinatorial structures which can model n-dimensional geometric objects, based on different cellular subdivisions (cells) of the object (see [11,2]). For example, up to dim 3, simplicial complexes are made up by vertices, edges, triangles and tetrahedra, while vertices, edges, squares and cubes constitute the collection of cells of a cubical complex. Adding a group structure to a cell complex, an algebraic structure, called chain complex, is obtained. Coefficients will be considered over **Z**.

Chain complexes and homology. [13,12]. Given a cell complex K, obtained by a subdivision of a geometric object into cells of different dimensions, one can define, for each dimension q, the chain group $\mathcal{C}_q$ whose elements, called q-*chain*, are linear combinations of cells of dimension q (q-cells). Then, a finite *chain complex* $\mathcal{C}$ is given by a couple $\{\mathcal{C}, \partial\}$, where $\mathcal{C} = \{\mathcal{C}_q\}_{0 \leq q \leq d}$ is a finite sequence of such a chain groups and the *differential* $\partial = \{\partial_q\}_{0 \leq q \leq d}$ is a sequence of homomorphisms $\partial_q : \mathcal{C}_q \to \mathcal{C}_{q-1}$, such that the composition of any two consecutive maps is zero: $\partial_q \partial_{q-1} = 0$ for all $0 < q \leq d$ (and $\partial_0 \equiv 0$). A chain complex $\mathcal{C}$ can be encoded as a pair (C, ∂), where $C = \bigcup_{0 \leq q \leq d} C_q$, each $C_q = \{u_q^1, \ldots, u_q^{n_q}\}$ is a basis of $\mathcal{C}_q$ (in general, C_q will be a set of q-cells) and the differentials are expressed with respect these basis. For simplicity, we sometimes omit the indices.

In an informal way, a chain complex can be seen as an algebraic generalization of directed multigraphs. Let $G = (V, E, i_G)$ be the directed multigraph drawn in Figure 1 where $V = \{(1), (2), (3)\}$ is the set of vertices, $E = \{a, b, c, d, e\}$ is the set of edges and $i_G : E \to V \times V$ maps each edge with its incident vertices (preserving the order given by the orientation of the edges). Adding a group structure to V and E, the map i_G can be redefined as an application ∂_G, preserving the group structure, from E to V (see Figure 1 on the right). The application ∂_G can be seen as the differential of the chain complex $\mathcal{C}$ where $\mathcal{C}_0$ (resp. $\mathcal{C}_1$) has V (resp. E) as a basis. Notice that e is a self-loop, so $\partial_G(e) = 0$.

Given a chain complex $\mathcal{C}$, a q-chain $a \in \mathcal{C}_q$ is called a q-*cycle* if $\partial_q a = 0$ (in the example, $\partial_G(b + d + c) = 0$). If $a = \partial_{q+1}b$ for some $b \in \mathcal{C}_{q+1}$ then a is called
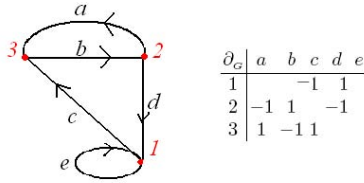
Fig. 1. The directed multigraph G and the matrix corresponding to ∂_G

a *q-boundary* (e.g. $(3) - (2)$ is a 0-boundary). Denote the groups of q-cycles and q-boundaries by Z_q and B_q respectively. Thanks to the nilpotence property of ∂, it is true that $Z_q \subseteq B_q$ for all $q \geq 0$. Define the *qth integer homology group* to be the quotient group Z_q/B_q, denoted by $\mathcal{H}_q(\mathcal{C})$. For each q, the integer qth homology group $\mathcal{H}_q(\mathcal{C})$ is a finitely generated abelian group isomorphic to $F_q \oplus T_q$, where F_q and T_q are the *free subgroup* and the *torsion subgroup* of $\mathcal{H}_q(\mathcal{C})$, respectively. We say that c is a *representative q-cycle* of the homology generator $c + B_q$ (denoted by $[c]$). The rank of F_q, denoted by β_q, is called the *qth Betti number* of $\mathcal{C}$. Intuitively, β_0 is the number of components of connected pieces, β_1 the number of independent holes and β_2 the number of cavities. Going back to the example, $\mathcal{H}_0(\mathcal{C}) \simeq \mathbf{Z}$ and $\mathcal{H}_1(\mathcal{C}) \simeq \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. Representative cycles of the homology generators of dim 1 are: e, $c + b + d$ and $a + b$. Hence, $\beta_0 = 1$ and $\beta_1 = 3$.

If $f, g : \mathcal{C} \to \mathcal{C}'$ are chain maps, then *a chain homotopy* $\phi : \mathcal{C} \to \mathcal{C}'$ of f to g is a family of homomorphisms $\{\phi_q : C_q \to C'_{q+1}\}$ such that $f_q - g_q = d'_{q+1}\phi_q + \phi_{q-1}d_q$. Then, f is called a *chain equivalence* and g its *chain homotopical inverse*.

Integral operators. Let $\mathcal{C} = (\{u^1, \ldots, u^n\}, \partial)$ be a chain complex. Let i and j be two integers such that $1 \leq i < j \leq n$, and $\dim u^j = 1 + \dim u^i$. An *integral operator* is a linear map $\phi : \mathcal{C} \to \mathcal{C}$ such that $\phi(u^i) = \lambda u^j$ for some $\lambda \in \mathbf{Z}$, $\lambda \neq 0$ and $\phi(u^k) = 0$ for any $1 \leq k \leq n$, $k \neq i$. An integral operator ϕ satisfies the *chain-homotopy property* if $\phi\partial\phi = \phi$. In this paper, we will deal with integral operators with the chain-homotopy property.

Proposition 1. *Consider the map* $\pi = id - \phi\partial - \partial\phi : \mathcal{C} \to im\pi$ *(where* $im\pi = (\{u^1, \ldots, \hat{u}^i, \ldots, \hat{u}^j, \ldots, u^n\}, \pi\partial\pi)$, *the hat means that the element is omitted) and the inclusion* $\iota : im\pi \to \mathcal{C}$. *If* ϕ *is an integral operator satisfying the chain-homotopy property, then* ϕ *is a chain homotopy of the identity map of* $\mathcal{C}$ *to* $\iota\pi$. *Therefore,* $\mathcal{C}$ *and* $im\pi$ *have isomorphic integer homology groups.*

Lemma 2. *Let* ϕ *be a linear map given by* $\phi(u^i) = \langle \partial u^j, u^i \rangle u^j$ *(where* $\langle \partial u^j, u^i \rangle$ *denotes the coefficient of* u^i *in the expression of* ∂u^j*). If* $\langle \partial u^j, u^i \rangle = \pm 1$ *then* ϕ *is an integral operator satisfying the chain-homotopy property.*

Geometric integral operators. *Collapse and face reduction.* There is a well-known process for thinning a simplicial complex using simplicial collapses [1]. It can be easily extended to chain complexes. Suppose $\mathcal{C} = (C, \partial)$ is a chain
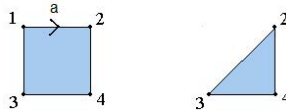
Fig. 2. A cell complex K (on the left) and the cell complex after applying the integral operator $\phi(1) = -a$ (on the right)

complex, $C = \{u^1, \ldots, u^n\}$. Consider two cells $u^i, u^j \in C$ such that u^i is a face of u^j. Then, define an integral operator involving these cells, $\phi(u^i) = \lambda u^j$: (1) if u^j is a maximal cell (that is, $u^j \notin im\partial$) and u^i is a free facet of u^j, (which means that $\langle \partial u^j, u^i \rangle \neq 0$ and $\langle \partial u^k, u^i \rangle = 0$ for any $k \neq j$) then, $\mathcal{C}$ *collapses* onto $\mathcal{C}' = (\{u^1, \ldots, \hat{u}^i, \ldots, \hat{u}^j, \ldots, u^n\}, \partial)$; (2) if u^i is a face of at least two cells, u^j and u^k, then $\mathcal{C}$ is reduced to $\mathcal{C}' = (C', \partial)$. C' coincides with C except for u^i and u^j which disappear. A chain equivalence between $\mathcal{C}$ and $\mathcal{C}'$ is given in [10]. The integral operator associated to this construction is given by $\phi(u^i) = \langle \partial u^j, u^i \rangle u^j$. In the particular case that ϕ satisfies the chain-homotopy property, we obtain the geometric collapse and face reduction. *Edge contractions.* Let K be a simplicial complex. Let τ be and edge of K and a, b two vertices of K such that $\partial(\tau) = \pm b \pm a$. An *edge contraction* is given by the vertex map $f^{vert} : K^{(0)} \to L^{(0)} = K^{(0)} \setminus \{b\}$ where $f^{vert}(b) = a$, and $f^{vert}(v) = v$ for all vertex $v \neq b$. It is known that if $Lk\,a \cap Lk\,b = Lk\,\tau$, then edge contraction preserves homotopy invariants. An edge contraction is obtained after applying the sequence of integral operators given by $\phi^{(\sigma,\mu)}(\sigma) = \langle \partial\mu, \sigma \rangle \mu$, for all σ having a facet in $Lk\tau$, σ being a facet of μ, b being a vertex of σ and a a vertex of μ.

Collapse of d Face reduction (f) Edge Contraction (d)
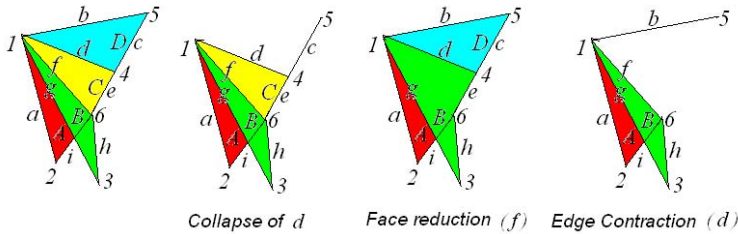
Fig. 3. A complex K, K after collapsing d, after reducing the face f and after contracting d

Algebraic integral operators. Given a chain complex $\mathcal{C} = (C, \partial)$, $C = \{u^1, \ldots, u^n\}$, consider the matrix corresponding to ∂. Some transformations on the matrix can lead to cases of integral operators.

Case 1. If $u^i, u^j \in C$ satisfy that $\langle \partial u^j, u^i \rangle = \pm 1$, consider the integral operator ϕ given by $\phi(u^i) = \langle \partial u^j, u^i \rangle u^j$. Then $im\pi = (\{u^1, \ldots, \hat{u}^i, \ldots, \hat{u}^k, \ldots, u^n\}, \pi\partial\pi)$. This is equivalent to the geometric operations of collapse or face reduction.

Case 2. If $u^i, u^j, u^k \in C$ satisfy that $\langle \partial u^k, u^i \rangle \cdot \langle \partial u^k, u^j \rangle \neq \pm 1$ and $\gcd(\langle \partial u^k, u^i \rangle, \langle \partial u^k, u^j \rangle) = 1$. then there exist two integers α and β such that $\alpha\langle \partial u^k, u^i \rangle + \beta\langle \partial u^k, u^j \rangle = 1$. Consider the 'integral operator' given by $\phi(\langle \partial u^k, u^i \rangle u^i + \langle \partial u^k, u^j \rangle$

$u^j) = u^k$ (which is, in fact, a linear combination of the integral operators $\phi^i(u^i) = \alpha u^k$ and $\phi^j(u^j) = \beta u^k$). Then $im\pi = (\{u^1, \ldots, \hat{u}^i, \ldots, \hat{u}^k, \ldots, u^n\}, \pi\partial\pi)$.

Case 3. If $u^i, u^j, u^k \in C$ satisfy that $\langle \partial u^j, u^i \rangle \cdot \langle \partial u^k, u^i \rangle \neq \pm 1$ and $\gcd(\langle \partial u^j, u^i \rangle, \partial u^k, u^i \rangle) = 1$; then there exist two integers α and β such that $\alpha \langle \partial u^j, u^i \rangle + \beta \langle \partial u^k, u^i \rangle = 1$. Consider the integral operator given by $\phi(u^i) = \alpha u^j + \beta u^k$. Then, $im\pi = (\{u^1, \ldots, \hat{u}^i, \ldots, \hat{u}^j, \ldots, u^n\}, \pi\partial\pi)$.

3 Integral Operators for Obtaining Small Chain Complexes

Consider a cell complex K of dimension d and the chain complex $\mathcal{C} = (K, \partial)$, associated to it. In this section, we establish a sequence of integral operators in order to obtain a small chain complex $\mathcal{C}' = (C', \partial')$ with the same homology than the former such that C' is a subset of K.

Geometric Part. From $i = d$ to $i = 0$, construct a graph G^i as follows: associate a node $v^{(i,u)}$ to each i-cell $u \in K$, except for the i-cells used in the step $i + 1$. Add an edge between two nodes $v^{(i,u)}$ and $v^{(i,u')}$ of the graph if u and u' share a not-used $(i - 1)$-cell. Construct a cover forest T^i of the graph G^i using, for example, breadth-first search algorithm with root in the center of the graph. Now, for each node $v^{(i,u)}$ from the leaves to the root of T^i, consider the integral operators:

- $\phi^{(i,u,u'')}(u'') := \langle \partial u, u'' \rangle u$ if $v^{(i,u)}$ is a leaf of T^i and there exists a free facet u'' of u in K (isolated nodes are considered leaves).
- $\phi^{(i,u,u'')}(u'') := \langle \partial u, u'' \rangle u$ if $v^{(i,u)}$ is a not-leaf node of T^i and u'' is the $(i - 1)$-cell of K corresponding to the edge of T^i connecting the nodes $v^{(i,u)}$ and $v^{(i,u')}$ (a child node of $v^{(i,u)}$).

Observe that we do not have to update the differential ∂ of the chain complex $\mathcal{C}$ each time we add an integral operator. On the contrary, we can update ∂ at the end of the process for each i. Let m be the number of the internal nodes of the cover forest T^i associated to the graph G^i. Then, in the worst case, we eliminate m i-cells and m $(i - 1)$-cells, for each $i = d, \ldots, 0$.

Algebraic Part. Now, initialize $\pi' = \pi$, $C' = C^\pi$ and $\partial' = \partial^\pi$. While π' changes do:

Step 1. While there exist ± 1 coefficients in the matrix of ∂' at any dimension do: let $u, u' \in C'$ such that $\langle \partial' u', u \rangle = \pm 1$. Then, consider the integral operator given by $\phi'(u) := \langle \partial' u', u \rangle u'$. Update $\pi' := id - \phi'\partial' - \partial'\phi'$, $C' := C' \setminus \{u, u'\}$ and $\partial' := \pi'\partial'\pi'$.

Step 2. While there exist coprime coefficients in a column of the matrix of ∂' at any dimension, do: let $u, u', u'' \in C'$ such that $\gcd(\langle \partial' u'', u \rangle, \langle \partial' u'', u' \rangle) = 1$. Consider the 'integral operator' given by $\phi'(\langle \partial' u'', u \rangle u + \langle \partial' u'', u' \rangle u') := u''$. Update $\pi' := id - \phi'\partial' - \partial'\phi'$, $C' := C' \setminus \{u, u''\}$ and $\partial' := \pi'\partial'\pi'$. If there is a

coefficient ± 1 in the matrix of ∂' at any dimension, go to Step 1. In other case, go to Step 3.

Step 3. While there exist coprime coefficients in a row of the matrix of ∂' at any dimension, do: let $u, u', u'' \in C'$ such that $\gcd(\langle\partial'u', u\rangle, \langle\partial'u'', u\rangle) = 1$. Let α and $\beta \in \mathbf{Z}$ such that $\alpha\langle\partial'u', u\rangle + \beta\langle\partial'u'', u\rangle = 1$. Then, consider the integral operator $\phi'(u) := \alpha u' + \beta u''$. Update $\pi' := id - \phi'\partial' - \partial'\phi'$, $C' := C' \setminus \{u, u''\}$ and $\partial' := \pi'\partial'\pi'$. Go to Step 1.

Observe that the final chain complex C' has the property that not any element of its differential is ± 1 and not any column nor row contains coprime coefficients. Moreover, this small chain complex contains the same torsion information as the original one.

Example 1. Consider the simplicial complex K of dim 2 shown in Figure 4 a). The cover tree obtained for dim 2 is shown in b). The geometric part for $i = 2$ gives place to the following integral operators: $\phi_1(d) = -B$, $\phi_2(b) = -C$ and $\phi_3(c) = A$. The resulting complex is c). For $i = 1$, we obtain: $\phi_4(\langle 0\rangle) = -a$, $\phi_5(\langle 2\rangle) = e$, $\phi_6(\langle 3\rangle) = f$. The cover tree for dim 1 is shown in d) and the resulting complex is $\mathcal{C}^\pi = (C^\pi, \partial^\pi)$ (see e)) where

∂^π	$\langle 1\rangle$	$\langle 4\rangle$	h	g
$\langle 1\rangle$	0	0	-1	-1
$\langle 4\rangle$	0	0	1	1

Now, applying the algebraic part to $\mathcal{C}^\pi$, we obtain that $\phi_7(\langle 1\rangle) = -g$. Then, $\mathcal{C}' = (C', \partial')$ is given by $C' = \{\langle 4\rangle, h\}$ and ∂' is null (see Figure 4 f)). Then $C' \simeq H(K)$. Therefore, $\mathcal{H}_0(K) \simeq \mathbf{Z}$ and $\mathcal{H}_1(K) \simeq \mathbf{Z}$.

Example 2. Consider a digital bone volume image obtained from a micro magnetic resonance of a trabecular bone (it has $85 \times 85 \times 20 = 144500$ voxels). Create a simplicial complex K which is topologically equivalent to the bone using the
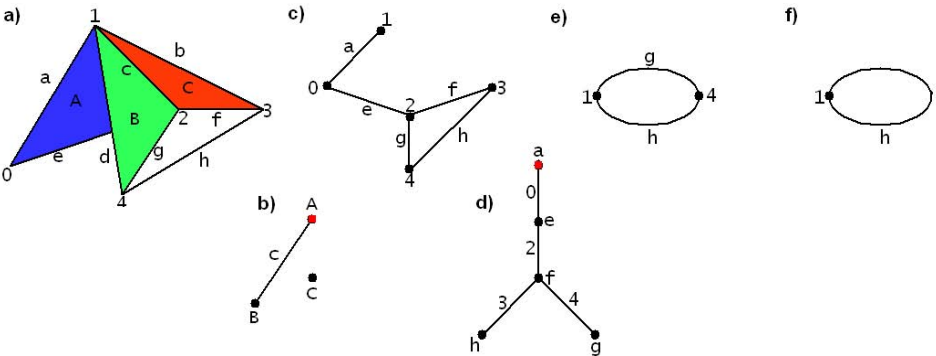

Fig. 4. a) Simplicial complex K; b) cover forest corresponding to dim 2; c) resulting complex after applying the geometric part for $i = 2$; d) cover forest corresponding to dim 1; e) resulting complex after applying the geometric part for $i = 1$; f) the cell complex resulting after applying the algebraic part of the method
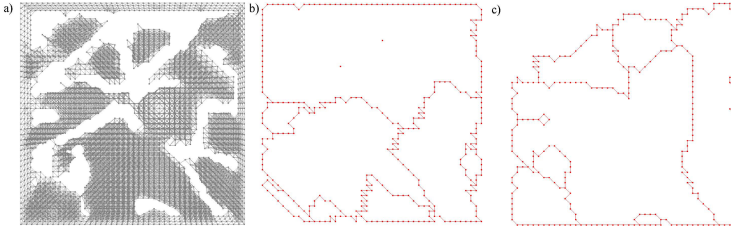
Fig. 5. a) Simplicialization of the foreground of a binary bone volume; preserving-homology thinning of the foreground (b) and the background (c) obtained using integral operators
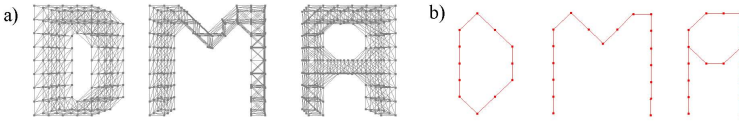

Fig. 6. a) A simplicial complex; b) a thinned one preserving end points obtained using integral operators

(26, 6)-adjacency (26 for the trabecular bone and 6 for the bone marrow). The complete volume's simplicialization of the foreground has 90530 voxels, 543327 edges, 826320 triangles and 376434 tetrahedrons. The number of connected components, holes and cavities of the foreground is 1, 3718 and 806 respectively. In Figure 5, only 5 frames (with 74835 simplices) are shown, in order to get a better visualization.

4 Conclusions

In this paper, we tackle the problem of computing integer homology of a chain complex associated to a cell complex. The existing methods, such as incremental algorithms or the performance of the SNF matrix, accomplish the computation from the initial chain complex. We have developed here a powerful technique that allows, in each step, to move on a smaller chain complex with the same homology than the original one. This is done via the construction of integral operators (elementary chain homotopies) which act as some kind of local inverse of the differential of the chain complex. Moreover, this method acts, in some sense, as a parallel algorithm, since the integral operators are defined only once for each pair of elements considered and independently of the rest of the given values. The method developed here find direct application in the work of Peltier et al. [15] since the authors use simplicial collapse, face reductions and edge contractions in order to compute homology generators of images via irregular graph pyramids. One initial work done in this direction is [9]. Another advantage of this method is that it is possible to reuse the output of this technique for obtaining finer

algebraic topological invariants (such as, cohomology ring, cohomology operators and homotopy groups) of the object.

References

1. Björner, A.: Topological Methods. Handbook on Combinatorics 2, 1819–1872 (1995)
2. Dahmen, W., Micchelli, C.A.: On the Linear Independence of Multivariate b-Splines. Triangulation of Simploids. SIAM J. Numer. Anal. 19 (1982)
3. Delfinado, C.J.A., Edelsbrunner, H.: An Incremental Algorithm for Betti Numbers of Simplicial Complexes on the 3–Sphere. Comput. Aided Geom. Design 12(7), 771–784 (1995)
4. Dumas, J.G., Saunders, B.D., Villad, G.: On Efficient Sparse Integer Matrix SNF Computations. J. Symb. Comput. 32(1), 71–99 (2001)
5. Gonzalez-Diaz, R., Real, P.: Computation of Cohomology Operations on Finite Simplicial Complexes. HHA 5(2), 83–93 (2001)
6. Gonzalez-Diaz, R., Real, P.: On the Cohomology of 3D Digital Images. Discrete Appl. Math. 147(2-3), 245–263 (2005)
7. Gonzalez-Diaz, R., Medrano, B., Real, P., Sanchez-Pelaez, J.: Reusing Integer Homology Information of Digital Images. In: Kuba, A., Nyúl, L.G., Palágyi, K. (eds.) DGCI 2006. LNCS, vol. 4245, pp. 199–210. Springer, Heidelberg (2006)
8. Gonzalez-Diaz, R., Jimenez, M.J., Medrano, B., Real, P.: Chain Homotopies for Object Topological Representations. Discrete Appl. Math. (to appear)
9. Gonzalez-Diaz, R., Iglesias-Ham, M., Ion, A., Kropatsch, W.G.: Irregular Graph Pyramids, Integral Operators and Representative Cocycles of Cohomology Generators. In: Workshop on Computational Topology in Image Context, Poitiers, France (2008)
10. Kaczynski, T., Mischaikow, K., Mrozek, M.: Computational Homology. In: AMS, Providence, Rhode Island, vol. 157 (2004)
11. Massey, W.M.: A Basic Course in Algebraic Topology. Graduate Texts in Math., vol. 127. Springer, New York (1991)
12. MacLane, S.: Homology. Classics in Mathematics. Springer, Berlin (1995)
13. Munkres, J.R.: Elements of Algebraic Topology. Addison-Wesley, Canada (1993)
14. Peltier, S., Alayrangues, S., Fuchs, L., Lachaud, J.: Computation of Homology Groups and Generators. In: Andrès, É., Damiand, G., Lienhardt, P. (eds.) DGCI 2005. LNCS, vol. 3429, pp. 195–205. Springer, Heidelberg (2005)
15. Peltier, S., Ion, A., Haxhimusa, Y., Kropatsch, W.G., Damiand, G.: Computing Homology Generators Using Irregular Graph Pyramids. In: Escolano, F., Vento, M. (eds.) GbRPR 2007. LNCS, vol. 4538, pp. 283–294. Springer, Heidelberg (2007)