


# A Graph-with-Loop Structure for a Topological Representation of 3D Objects\*

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**Abstract.** Given a cell complex  $K$  whose geometric realization  $|K|$  is embedded in  $\mathbf{R}^3$  and a continuous function  $h : |K| \rightarrow \mathbf{R}$  (called the *height function*), we construct a graph  $G_h(K)$  which is an extension of the Reeb graph  $R_h(|K|)$ . More concretely, the graph  $G_h(K)$  without loops is a subdivision of  $R_h(|K|)$ . The most important difference between the graphs  $G_h(K)$  and  $R_h(|K|)$  is that  $G_h(K)$  preserves not only the number of connected components but also the number of “tunnels” (the homology generators of dimension 1) of  $K$ . The latter is not true in general for  $R_h(|K|)$ . Moreover, we construct a map  $\psi : G_h(K) \rightarrow K$  identifying representative cycles of the tunnels in  $K$  with the ones in  $G_h(K)$  in the way that if  $e$  is a loop in  $G_h(K)$ , then  $\psi(e)$  is a cycle in  $K$  such that all the points in  $|\psi(e)|$  belong to the same level set in  $|K|$ .

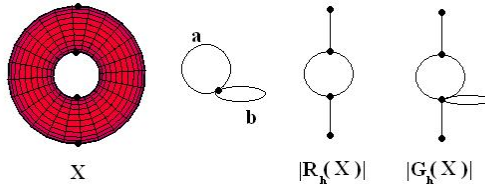
## 1 Reeb Graphs and Tunnels

We are interested in analyzing and visualizing intrinsic properties of geometric models and scientific data. Specifically, Reeb graphs [13], which express the connectivity of level sets, have been used in the past to construct data structures and user-interfaces for modeling and visualization applications [5].

Let  $X$  be a topological space and  $h : X \rightarrow \mathbf{R}$  a continuous map. A *level set* is the preimage of a constant value,  $h^{-1}(t)$ . Call a connected component of a level set a *contour*. Two points  $x, y \in X$  are equivalent,  $x \sim y$ , if they belong to the same contour, that is, if  $h(x) = h(y)$  and  $x$  and  $y$  are connected by a path on  $X$ . The *Reeb graph* of  $h$ ,  $R_h(X)$ , is the quotient space defined by this equivalence relation. Observe that, by construction, the Reeb graph has a point for each contour and the connection is provided by  $\psi : X \rightarrow R_h(X)$  that maps each point  $x$  to its equivalence class. Even though the Reeb graph loses a lot of the original topological structure, some things can be said: a tunnel in  $X$  that maps (by  $\psi$ ) to a tunnel in  $R_h(X)$  cannot be continuously deformed to a single point, and two tunnels in  $X$  that map to different tunnels in  $R_h(X)$ . The number of connected components of  $X$ ,  $\beta_0(X)$ , is preserved and the number of tunnels of  $X$ ,  $\beta_1(X)$ , cannot increase, i.e.  $\beta_0(R_h(X)) = \beta_0(X)$  and  $\beta_1(R_h(X)) \leq \beta_1(X)$ .

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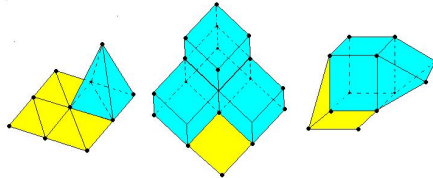


**Fig. 1.** From left to right, the torus  $X$ ; representative cycles of the two tunnels of  $X$  (the tunnel  $a$  is obvious and the other one,  $b$ , is not); a geometric realization of the Reeb Graph  $R_h(X)$  that  $h$  associates to each point on  $X$  its elevation; and a geometric realization of the graph with loops  $G_h(X)$

In [2] the authors adapt concepts developed for smooth manifolds to discrete surface models, introducing an extended Reeb graph representation for a generic polyhedral surface. Their approach is based on the computation of a sufficiently dense number of contour lines and the definition of the Reeb graph from the contour set. However, such a construction is actually not an extension of the Reeb graph itself, but rather an application of its definition in the discrete domain. In [4], tight upper and lower bounds of the number of tunnels in the Reeb graph that depend on the genus, the number of boundary components and whether or not the 2-manifold is orientable, is given.

In this paper, we focus on objects embedded in  $\mathbf{R}^3$ . Several combinatorial structures may represent a cellular subdivision which models an object such as simplicial, cubical and simploidal complexes. Roughly speaking, the *cells* of a given simplicial complex are simplices (vertices, edges, triangles and tetrahedra); vertices, edges, squares and cubes constitute the collection of cells of a cubical complex; in the case of simploidal complexes, which generalize both simplicial and cubical complexes (see [3]), cells are cartesian products of simplices. In all the cases they fit together in a natural way to form the object (see [11,3]).

From now on, a graph  $G = (V, E)$ , where  $V$  is a set of vertices and  $E$  a set of edges, is considered as a particular complex only with vertices and edges. Given



**Fig. 2.** A simplicial, cubical and simploidal complex

a complex  $K$  (simplicial, cubical or simploidal), a geometric realization of it (e.g. a 3D triangulated surface)  $|K|$ , and a continuous function  $h : |K| \rightarrow \mathbf{R}$ , our aim is the computation of a graph  $G_h(K)$  and a function  $\Psi : G_h(K) \rightarrow K$  with the following properties:

- $G_h(K)$  has the same number of connected components and tunnels than  $K$ .
- Each loop (an edge such that its endvertices are the same) of  $G_h(K)$  maps to a non-contractible cycle  $c$  in  $K$  such that  $|c|$  (the geometric realization of all the cells in  $c$ ) lies in a contour of  $|K|$ , two loops in  $G_h(K)$  map to non-homologous cycles in  $K$ , and each edge in  $G_h(K)$  map to a path in  $K$ .
- If we do not consider the loops in  $G_h(K)$ , then  $G_h(K)$  is a subdivision of the Reeb graph  $R_h(K)$ .

Therefore,  $G_h(K)$  can be seen as an extension of  $R_h(K)$  such that not only the number of connected components and tunnels of  $G_h(K)$  and  $K$  coincide (this is not true, in general, for  $R_h(K)$ ), but also there exists a one-by-one identification of tunnels in  $G_h(K)$  with tunnels in  $K$  in the way that if  $e$  is a loop in  $G_h(K)$ , then  $\psi(e)$  is a cycle in  $K$  such that all the points in  $|\psi(e)|$  have the same height.

## 2 Algebraic-Topological Models for 3D Objects

This section introduces the algebraic topology background needed to understand the rest of the paper, which is essentially extracted from Munkres’ book [12]. The concept of AT-model established in [8,9] for cohomology computation of 3D digital images is adapted here to solve the problem of computing the graph  $G_h(K)$  and the function  $\Psi : G_h(K) \rightarrow K$ . Without lost of generality, we will consider that the ground ring is  $\mathbf{Z}/2$ .

A *chain complex*  $\mathcal{C}$  is a sequence  $\{C_q, d_q\}$  of abelian groups  $C_q$  and homomorphisms  $d_q : C_{q+1} \rightarrow C_q$ , such that, for all  $q$ ,  $d_q d_{q+1} = 0$ . The set  $\{d_q\}_{q \geq 0}$  is called the *differential* of  $\mathcal{C}$ . The chain complex  $\mathcal{C}$  is *free* if  $C_q$  is a free abelian group for each  $q$ ; it is *finite* if there exists an integer  $n > 0$  such that  $C_q = 0$  for  $q > n$  and each abelian group  $C_q$  is finitely generated. All chain complexes considered here are finite and free. A chain  $c$  in  $\mathcal{C}$  is a *q-cycle* if  $c \in \text{Ker } d_q$ . If  $c \in \text{Im } d_{q+1}$  then  $a$  is called a *q-boundary*. Denote the groups of  $q$ -cycles and  $q$ -boundaries by  $Z_q$  and  $B_q$  respectively. Define the integer *qth homology group* to be the quotient group  $Z_q/B_q$ , denoted by  $H_q(\mathcal{C})$ . We say that  $c$  is a *representative q-cycle* of the homology generator  $c + B_q$  (denoted by  $[c]$ ). For each  $q$ , the *qth homology group*  $H_q(\mathcal{C})$  is a finitely generated free abelian group. The rank of  $H_q$ , denoted by  $\beta_q$ , is called the *qth Betti number* of  $\mathcal{C}$ . Homology is a powerful topological invariant, which characterizes an object by its  $q$ -dimensional “holes” (connected components, tunnels and cavities).

Let  $K$  be a complex (simplicial, cubical or simploidal). A  $q$ -chain  $c$  is a formal sum of  $q$ -cells (where  $q$  is the dimension of the cell) in  $K$ . Let  $\{\sigma_1^q, \dots, \sigma_{m_i}^q\}$  be the set of  $q$ -cells in  $K$ , then  $c = \sum_{i=1}^{m_i} \lambda_i \sigma_i^q$ , where  $\lambda_i \in \{0, 1\}$ . Alternatively, we can think of  $c$  as the set  $\{\sigma_i^q, \text{ such that } \lambda_i = 1\}$ , and the sum of two  $q$ -chains as their symmetric difference. The  $q$ -chains together with the addition operation form the group of  $q$ -chains denoted as  $C_q(K)$ . The differential of a  $q$ -cell  $\sigma$  in  $K$ ,  $d_q(\sigma)$ , is the sum of the  $(q - 1)$ -cells in  $K$  that belong to the boundary of  $\sigma$ . By linearity, the differential can be extended to  $q$ -chains. The *chain complex*  $C(K)$  is the sequence of chain groups  $C_q(K)$  connected by the homomorphisms  $d_q$ . The homology of  $K$  is defined as the homology of  $C(K)$ . Since we work with

objects embedded in  $\mathbf{R}^3$ , the homology groups are torsion-free (see [1, ch.10]). Moreover, Theorem of Universal Coefficient [12] ensures that all the homology information can be computed working with coefficients in  $\mathbf{Z}/2$ .

An *AT-model* for  $K$  is established in [8,9] and used to obtain the homology and representative cycles of homology generators of  $K$ . An AT-model can be computed starting from an ordering of the cells in  $K$  [8,9]. We deal here with a particular ordering based on a *cover forest*  $T$  of  $K$  (any two vertices are connected by exactly one path in  $T$  if and only if they are connected in  $K$ ). Let  $T = (V, E)$  be a cover forest of  $K$  where  $V$  is the set of all the vertices of  $K$  and  $E$  a subset of edges of  $K$ .  $S = (\sigma_0, \dots, \sigma_m)$  is a *T-filter* if it is an ordering of all the cells in  $K$  such that:

- for each  $j$  (where  $0 \leq j \leq m$ ),  $\{\sigma_0, \dots, \sigma_j\}$  is a subcomplex of  $K$ ;
- if  $i < j$ , the dimension of  $\sigma_i$  is less or equal than the dimension of  $\sigma_j$ ;
- if  $i < j$ ,  $\sigma_i$  and  $\sigma_j$  are two edges and  $\sigma_j \in T$ , then  $\sigma_i \in T$  That is, the edges of the cover forest are in first positions in  $S$ ).

Observe that  $\sigma_0$  is always a vertex of  $K$ . An AT-model for  $K$  is then defined as the output of the following algorithm, having as the input a complex  $K$  and a *T-filter*  $S$  of  $K$ .

**Algorithm 1.** [8,9] *AT-model Algorithm.*

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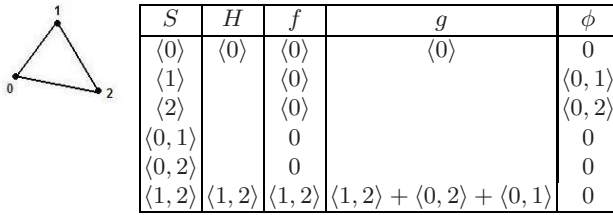
INPUT: a T-filter  $S = (\sigma_0, \dots, \sigma_m)$  of  $K$ ,
 $H := \{\sigma_0\}$ ,  $f(\sigma_0) := \sigma_0$ ,  $g(\sigma_0) := \sigma_0$ ,  $\phi(\sigma_0) := 0$ .
For  $i = 1$  to  $m$  do
  If  $fd(\sigma_i) = 0$ , then
     $H := H \cup \{\sigma_i\}$ ,  $f(\sigma_i) := \sigma_i$ ,  $\phi(\sigma_i) := 0$ ,  $g(\sigma_i) := \sigma_i + \phi d(\sigma_i)$ .
  If  $fd(\sigma_i) \neq 0$ , then:
     $k := \max \{j \text{ such that } \sigma_j \in fd(\sigma_i), j = 1, \dots, i - 1\}$ ,
     $H := H \setminus \{\sigma_k\}$ ,  $f(\sigma_i) := 0$ ,  $\phi(\sigma_i) := 0$ .
    For  $j = 1$  to  $i - 1$  do if  $\sigma_k \in f(\sigma_j)$ ,
       $f(\sigma_j) := f(\sigma_j) + fd(\sigma_i)$ ,  $\phi(\sigma_j) := \phi(\sigma_j) + \sigma_i + \phi d(\sigma_i)$ .
OUTPUT: the set  $(S, H, f, g, \phi)$ .
    
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Notice that in the  $i$ th step of the algorithm ( $i = 1, \dots, m$ ), exactly one homology generator is created or destroyed. The algorithm runs in time at most  $\mathcal{O}(m^3)$ .

**Proposition 1.** *Let  $K$  be a complex (simplicial, cubical or simploidal), let  $T = (V, E)$  be a cover forest of  $K$  and  $S$  a T-filter of  $K$ . The output of Algorithm 1,  $(S, H, f, g, \phi)$ , satisfies that:*

- $H$  is a subset of  $S$  such that no edge in  $T$  is an edge in  $H$ .  $H$  generates a chain complex denoted by  $\mathcal{H}$  with null differential;
- The number of vertices, edges and triangles in  $H$  equals the number of connected components, tunnels and cavities in  $|K|$ , respectively. In other words, the homology of  $K$  is isomorphic to  $\mathcal{H}$ .
- $f : C(K) \rightarrow \mathcal{H}$  satisfies that if  $c$  and  $c'$  are two cycles in  $K$  such that  $f(c) = f(c')$  then  $c$  and  $c'$  are homologous.

- $g : \mathcal{H} \rightarrow C(K)$  satisfies that  $\{[g(h)] : h \in H\}$  is a set of homology generators of  $K$ . If  $h, h' \in H, h \neq h'$ , then  $g(h)$  and  $g(h')$  are not homologous. Moreover, if  $a$  is an edge in  $H$ , then  $g(a)$  is a simple cycle in  $K$  and all the edges in  $g(a) \setminus \{a\}$  are edges in  $T$ . In fact,  $g(a) \setminus \{a\}$  is the simple path in  $T$  connecting the endvertices of  $a$ .
- $\phi : C(K) \rightarrow C(K)$  satisfies that if  $x \in H$  then  $\phi(x) = 0$  and there is no  $y \in S$  such that  $\phi(y) = x$ . Moreover, if  $v$  is a vertex in  $K$ , then  $\phi(v)$  is the simple path in  $T$  connecting  $v$  with the vertex in  $H$  that belongs to the same connected component in  $K$  than  $v$ .



**Fig. 3.** A filter  $S$  of a simplicial complex  $K$  and the result of applying Algorithm 1 to  $S$  (an AT-model for  $K$ )

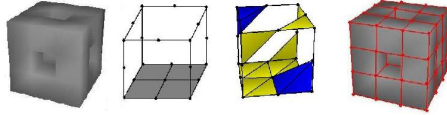
### 3 Computing a Graph-with-Loop Representation of a 3D Object

Let  $K$  be a complex (simplicial, cubical or simploidal);  $|K|$  its geometric realization in  $\mathbf{R}^3$ ; and  $h : |K| \rightarrow \mathbf{R}$  a continuous function. Let  $e_{xy}$  denote an edge with endvertices  $x$  and  $y$ . We say that the *height of a point*  $p \in |K|$  is  $t$  if  $h(p) = t$  and the *height of a cell*  $\sigma \in K$  is the minimum of the heights of all the points on  $|\sigma|$ . We say that  $K$  is an  *$h$ -complex* if:

- the set of the vertices of  $K$  can be partitioned into a finite number of subsets in terms of their height,  $V = \bigcup_{i=1}^r V_i$ , where  $V_i = \{v \in V : h(v) = t_i \text{ and } t_1 < \dots < t_r\}$ .
- if  $e_{vw}$  is an edge in  $K$  then  $v$  and  $w$  belong to  $V_i$  for some  $i = 1, \dots, r$  or  $v \in V_{i-1}$  and  $w \in V_i$  for some  $i = 2, \dots, r$ .

$h$ -Complexes appear in a natural way when they are defined by the neighborhood relations of voxels of a 3D digital image and  $h$  is the real function that associates to each point on  $|K|$  its elevation.

Let  $K$  be an  $h$ -complex and  $\sigma$  a cell in  $K$ . We say that  $\sigma$  is *horizontal* if the heights of all the points on  $|\sigma|$  coincide; otherwise, it is *vertical*. For  $i = 1, \dots, r$ , let  $K_i$  be the collection of all the horizontal cells in  $K$  with the same height  $t_i$ ,  $i = 0, 1, \dots, r$ .  $K_i$  is a subcomplex of  $K$  and if a cell  $\sigma$  is not in  $K_i$ , then  $\sigma$  is vertical. Let  $T_i = (V_i, E_i)$  be a cover forest of  $K_i$  and  $S_i$  a  $T_i$ -filter of  $K_i$ . Denote by  $(S_i, H_i, f_i, g_i, \phi_i)$ ,  $i = 1, \dots, r$ , the AT-models obtained using Algorithm 1. Let  $V$  be the set of vertices in  $K$ ,  $T = (V, E)$  a cover forest of  $K$  (obtained



**Fig. 4.** From left to right: a digital image and 3  $h$ -complexes associated to it considering the 6, 14 and 26-adjacency, respectively

after adding vertical edges in  $K$  in increasing ordering in height to the graph  $(V; \bigcup_{i=1}^r E_i)$ , and  $S$  a  $T$ -filter of  $K$ . Denote by  $(S, H, f, g, \phi)$  the AT-model obtained using Algorithm 1.

**Proposition 2.** *The AT-model  $(S, H, f, g, \phi)$  satisfies that:*

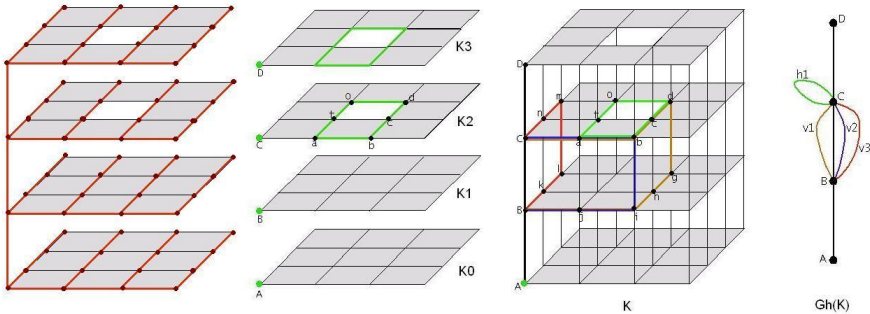
- If  $a$  is a horizontal edge in  $H$ , then  $a \in H_i$  for some  $i$  and  $g(a) = g_i(a)$  is a simple cycle such that its edges are in  $K_i$ .
- If  $a$  is a vertical edge in  $H$ , then for each level  $i, i = 1, \dots, r, g(a)$  has an even number of vertical edges of height  $t_i$ .

Now, let us explain how to construct the graph  $G_h(K)$  and the function  $\Psi : G_h(K) \rightarrow K$  using the AT-models  $(S_i, H_i, f_i, g_i, \phi_i), i = 1, \dots, r$  and  $(S, H, f, g, \phi)$  computed before. First, the vertices in  $G_h(K)$  in each level  $i$  are the vertices in  $H_i, i = 1, \dots, r$ . If  $v$  is a vertex in  $G_h(K)$ , then  $\Psi(v) = v$ . Second, for each level  $i$  and (horizontal) edge  $a$  in  $H_i \cap H$ , we add a loop  $\alpha$  in  $G_h(K)$  such that its endvertex is the vertex in the level  $i$  of  $G_h(K)$  which belongs to the same connected component than  $|a|$  in  $|K_i|$ . Define  $\Psi(\alpha) = g_i(a) = g(a)$ . Third, we add an edge  $e_{xy}$  between two vertices  $x$  and  $y$  in  $G_h(K)$  if  $x \in H_i$  and  $y \in H_{i+1}$  for some  $i$  and  $f(x) = f(y) = z \in H$  (i.e.  $x$  and  $y$  belong to the same connected component in  $K$ ). Define  $\Psi(e_{xy}) = \phi(x) + \phi(y)$  (the simple path in  $T$  connecting the vertices  $x$  and  $y$ ). Finally, for each vertical edge  $e_{vw}$  in  $H$ , an edge  $b$  is added to  $G_h(K)$ . Since  $e_{vw}$  is vertical, then  $v \in H_i$  and  $w \in H_{i+1}$  for some  $i$ . The endvertices of  $b$  are the vertices in  $G_h(K)$  which belong to the same connected component than  $v$  in  $K_i$  and  $w$  in  $K_{i+1}$ , respectively. Define  $\Psi(b) = e_{vw} + \phi_i(v) + \phi_{i+1}(w)$ .

**Theorem 2.** *Given a complex  $K$  and a continuous function  $h : |K| \rightarrow \mathbf{R}$ . If  $K$  is an  $h$ -complex, then:*

1. The graph  $G_h(K)$  and the complex  $K$  has the same number of tunnels and connected components.
2. For each loop  $\alpha \in G_h(K), \Psi(\alpha)$  is a simple cycle representative of a homology generator of  $K$ . If  $\alpha_1$  and  $\alpha_2$  are two different loops in  $G_h(K)$ , then  $\Psi(\alpha_1)$  and  $\Psi(\alpha_2)$  are two representative cycles of two non-equivalent generators of homology.
3. For each edge  $e_{xy}$  in  $G_h(K)$  that comes from a vertical edge  $e_{vw} \in H$ , then  $\Psi(e_{xy}) + \phi(x) + \phi(y) = g(e_{vw})$  is a representative cycle of a homology generator of  $K$ .
4. The graph  $G_h(K)$  without loops is a subdivision of the Reeb graph  $R_h(K)$ .
5. The graph  $G_h(K)$  and the function  $\Psi$  can be computed in  $\mathcal{O}(m^3)$ , where  $m$  is the number of cells in  $K$ .

*Proof.* The number of tunnels of  $K$  is the number of edges in  $H$ . By construction, each horizontal edge in  $H$  produces a loop in  $G_h(K)$  (i.e. a tunnel in  $G_h(K)$ ). Each vertical edge  $e_{vw}$  in  $H$  produces a vertical edge  $\beta$  in  $G_h(K)$ . Let  $v$  in  $K_i$  and  $w$  in  $K_{i+1}$ . Let  $V$  and  $W$  be the two vertices in  $G_h(K)$  that belong to the same connected component than  $v$  and  $w$ , respectively. Since  $e_{vw} \in H$ , then  $e_{vw}$  created a cycle when it was added. Therefore,  $v$  and  $w$  belong to the same connected component in  $K$  and so, there exists a path  $p$  between  $V$  and  $W$  in  $G_h(K)$  apart from the edge  $\beta$  that produces  $e_{vw}$ , by construction. Then,  $p + \beta$  is a cycle in  $G_h(K)$ . Moreover,  $\Psi(p + \beta) = g(b)$  is a representative cycle of the homology generators of dimension 1 of  $K$ . Since representative cycles of a homology generator of dimension 1 of  $K$  map by  $\psi$  to a cycle in  $R_h(K)$ , then  $\psi(g(b))$  is a cycle in  $R_h(K)$ . Since  $K$  is an  $h$ -complex, then a vertex in  $R_h(K)$  corresponds to a contour in a level  $t_i$ ,  $i = 1, \dots, r$ . Therefore, a vertex in  $R_h(K)$  is a vertex in  $G_h(K)$ .  $\square$



**Fig. 5.** From left to right: a cover forest  $T$  of the cubical complex  $K$  showed on the right of Figure 4; the complexes  $K_0, K_1, K_2$  and  $K_3$ ; a set of representative cycles of the generators  $H_1(K)$ ; and the graph  $G_h(K)$

*Example 1.* Let  $K$  be the cubical complex  $K$  on the right in Figure 4. A cover forest  $T$  of  $K$ ; the complexes  $K_0, K_1, K_2$  and  $K_3$ ; a set of representative cycles of the generators  $H_1(K)$ ; and the graph  $G_h(K)$  The non-trivial identification of the edges and loops in  $G_h(K)$  and  $K$  by  $\Psi$  are:

$G_h(K)$	$\Psi$
$h_1$	$e_{ab} + e_{bc} + e_{cd} + e_{do} + e_{of} + e_{af}$
$v_1$	$e_{Bj} + e_{ij} + e_{hi} + e_{gh} + e_{dg} + e_{cd} + e_{ac} + e_{ab} + e_{bc}$
$v_2$	$e_{Bj} + e_{ij} + e_{bi} + e_{ab} + e_{aC}$
$v_3$	$e_{Bk} + e_{lk} + e_{ml} + e_{mn} + e_{nC}$

The representative cycles of generators of  $H_1(K)$  are:

$$\alpha_0 = \Psi(h_1) = e_{ab} + e_{bc} + e_{cd} + e_{do} + e_{of} + e_{af}$$

$$\alpha_1 = \Psi(v_1) + e_{BC} = e_{Bj} + e_{ij} + e_{hi} + e_{gh} + e_{dg} + e_{cd} + e_{ac} + e_{ab} + e_{bc} + e_{BC}$$

$$\alpha_2 = \Psi(v_2) + e_{BC} = e_{Bj} + e_{ij} + e_{bi} + e_{ab} + e_{aC} + e_{BC}$$

$$\alpha_3 = \Psi(v_3) + e_{BC} = e_{Bk} + e_{lk} + e_{ml} + e_{mn} + e_{nC} + e_{BC}$$

## 4 Conclusions and Future Work

It is possible to obtain representative cycles on the boundary of the given complex  $K$  if we compute a cover forest of  $K$  first adding the edges on the boundary. Another task is the generalization of the method to any dimension. The problem is that the homology of a complex of a dimension higher than 3 can have torsion groups. In order to capture the torsion part of the homology we could use the concept of  $\lambda$ -AT-model developed in [10].

A possible extension of this work is the construction of a discrete Morse complex  $M_h(K)$  associated to a cell complex  $K$ , such that there is not only a one-by-one identification of all the homology generators of  $M_h(K)$  with that of  $K$ , but also an isomorphism between cohomology rings.  $M_h(K)$  can be constructed using a gradient vector field  $\mathcal{V}_K$  associated to a discrete Morse function (see [6,7]) that can be obtained from an AT-model for  $K$ .

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