Strong estimates for some coupled Navier-Stokes type systems

Chillán, agosto de 2012

Table of contents

Table of contents

Navier-Stokes 3D

$$(NS)\begin{cases} \partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}, \\ \nabla\cdot\boldsymbol{u} = 0 \qquad\qquad \text{in } \Omega\times(0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \boldsymbol{u}|_{\partial\Omega} = 0 \end{cases}$$

$(NS)_m$: Approximated Galerkin Pb. (eigenfunctions Stokes Pb. "special" basis of $\boldsymbol{V}$)

$$\boldsymbol{V} = \{\boldsymbol{u}\in\boldsymbol{H}_0^1 : \nabla\cdot\boldsymbol{u} = 0 \text{ in } \Omega\}$$

$A\boldsymbol{u}_m$ test function (A Stokes operator)

$$(\partial_t \boldsymbol{u}_m, A\boldsymbol{u}_m) - \nu(\Delta \boldsymbol{u}_m, A\boldsymbol{u}_m) + ((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, A\boldsymbol{u}_m) = (\boldsymbol{f}, A\boldsymbol{u}_m),$$

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m\|_1^2 + \nu\|\boldsymbol{u}_m\|_2^2 = -((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, A\boldsymbol{u}_m) + (\boldsymbol{f}, A\boldsymbol{u}_m).$$

$$|(\boldsymbol{f}, A\boldsymbol{u}_m)| \leq \varepsilon\nu\|\boldsymbol{u}_m\|_2^2 + C|\boldsymbol{f}|_2^2$$

Navier-Stokes

Au_m test function (A Stokes operator)

$$(\partial_t \boldsymbol{u}_m, A\boldsymbol{u}_m) - \nu(\Delta \boldsymbol{u}_m, A\boldsymbol{u}_m) + ((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, A\boldsymbol{u}_m) = (\boldsymbol{f}, A\boldsymbol{u}_m),$$

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m\|_1^2 + \nu\|\boldsymbol{u}_m\|_2^2 = -((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, A\boldsymbol{u}_m) + (\boldsymbol{f}, A\boldsymbol{u}_m).$$

$$|(\boldsymbol{f}, A\boldsymbol{u}_m)| \leq \varepsilon\nu\|\boldsymbol{u}_m\|_2^2 + C|\boldsymbol{f}|_2^2$$

Small data

$$|((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, A\boldsymbol{u}_m)| \leq |\boldsymbol{u}_m|_6 |\nabla \boldsymbol{u}_m|_3 |A\boldsymbol{u}_m|_2 \leq C\|\boldsymbol{u}_m\|_2^{3/2} \|\boldsymbol{u}_m\|_1^{3/2}$$
$$\leq \varepsilon\nu \|\boldsymbol{u}_m\|_2^2 + C\|\boldsymbol{u}_m\|_1^6$$

$$\Phi_m(t) = \|\boldsymbol{u}_m\|_1^2, \qquad \Psi_m(t) = \|\boldsymbol{u}_m\|_2^2.$$

$$\begin{cases} \Phi_m' + \nu\Psi_m \leq C_1\Phi_m^3 + C_2|\boldsymbol{f}|_2^2, \\ \Phi(0) = \Phi_{m0} \end{cases}$$

Small data

$$|((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, A\boldsymbol{u}_m)| \le |\boldsymbol{u}_m|_6 |\nabla \boldsymbol{u}_m|_3 |A\boldsymbol{u}_m|_2 \le C\|\boldsymbol{u}_m\|_2^{3/2}\|\boldsymbol{u}_m\|_1^{3/2}$$
$$\le \varepsilon\nu\|\boldsymbol{u}_m\|_2^2 + C\|\boldsymbol{u}_m\|_1^6$$

$$\Phi_m(t) = \|\boldsymbol{u}_m\|_1^2, \qquad \Psi_m(t) = \|\boldsymbol{u}_m\|_2^2.$$

$$\begin{cases} \Phi_m' + \nu\Psi_m \le C_1\Phi_m^3 + C_2|\boldsymbol{f}|_2^2, \\ \Phi(0) = \Phi_{m0} \end{cases}$$

$\Phi_m(0) \le \delta$, $\|\boldsymbol{f}\|_{L^2(L^2)} < \delta / C_2 \Rightarrow \Phi_m(t) < 2\delta$, $\forall t \in [0, T]$.
Indeed, (by contradiction) if there exists $T^* \in [0, T]$,

$$\Phi_m(T^*) = 2\delta \qquad \text{and} \qquad \Phi_m(s) < 2\delta \quad \forall s \in [0, T^*),$$

then (P Poincaré constant),

$$\Phi_m' + \nu P \Phi_m \le C_1 (2\delta)^2 \Phi_m + C_2 |\boldsymbol{f}|_2^2 \quad \text{in } [0, T^*].$$

$$\delta <<: \quad \Phi_m' + C\Phi_m \le C_2 |\boldsymbol{f}|_2^2 \quad \text{in } [0, T^*].$$

Integrating in $[0, T^*]$ with a Gronwall's technique,

$$\Phi_m(T^*) \le \Phi_m(0) e^{-CT^*} + C_2 \int_0^{T^*} |\boldsymbol{f}|_2^2 < 2\delta.!!$$

$$\boldsymbol{u}_m \in L^\infty(\boldsymbol{H}^1) \cap L^2(\boldsymbol{H}^2).$$

Large viscosity

$$|((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, A\boldsymbol{u}_m)| \leq |\boldsymbol{u}_m|_3 |\nabla \boldsymbol{u}_m|_6 |A\boldsymbol{u}_m|_2 \leq C\|\boldsymbol{u}_m\|_1 \|\boldsymbol{u}_m\|_2 \|\boldsymbol{u}_m\|_2$$

$$\leq \varepsilon\nu \|\boldsymbol{u}_m\|_2^2 + \frac{C}{\nu}\|\boldsymbol{u}_m\|_1^2 \|\boldsymbol{u}_m\|_2^2$$

$$\Phi_m(t) = \|\boldsymbol{u}_m\|_1^2, \qquad \Psi_m(t) = \|\boldsymbol{u}_m\|_2^2.$$

$$\begin{cases} \Phi_m' + \nu\Psi_m \leq \dfrac{C}{\nu}\Phi_m\Psi_m + C_2|\boldsymbol{f}|_2^2, \\ \Phi(0) = \Phi_{m0} \end{cases}$$

Large viscosity

$$|((\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, A\boldsymbol{u}_m)| \le |\boldsymbol{u}_m|_3 |\nabla \boldsymbol{u}_m|_6 |A\boldsymbol{u}_m|_2 \le C\|\boldsymbol{u}_m\|_1 \|\boldsymbol{u}_m\|_2 \|\boldsymbol{u}_m\|_2$$
$$\le \varepsilon\nu \|\boldsymbol{u}_m\|_2^2 + \frac{C}{\nu}\|\boldsymbol{u}_m\|_1^2 \|\boldsymbol{u}_m\|_2^2$$

$$\Phi_m(t) = \|\boldsymbol{u}_m\|_1^2, \qquad \Psi_m(t) = \|\boldsymbol{u}_m\|_2^2.$$

$$\begin{cases} \Phi_m' + \nu\Psi_m \le \dfrac{C}{\nu}\Phi_m\Psi_m + C_2|\boldsymbol{f}|_2^2, \\ \Phi(0) = \Phi_{m0} \end{cases}$$

$\Phi_m(t) \leq M$, $\forall t \in [0, T]$ where $M > \Phi(0) + C_2\|\boldsymbol{f}\|_2^2$.
Indeed, if there exists $T^* \in [0, T]$,

$$\Phi_m(T^*) = M \qquad \text{and} \qquad \Phi_m(s) < M \quad \forall s \in [0, T^*),$$

then,

$$\Phi'_m + (\nu - \frac{C}{\nu} M)\Psi_m \leq C_2|\boldsymbol{f}|_2^2 \quad \text{in } [0, T^*].$$

P Poincaré constant, $\nu >>$: $\quad \Phi'_m + P\Phi_m \leq C_2|\boldsymbol{f}|_2^2 \quad \text{in } [0, T^*]$.

Integrating in $[0, T^*]$ with a Gronwall's technique,

$$\Phi_m(T^*) \leq \Phi_m(0)e^{-PT^*} + C_2 \int_0^{T^*} |\boldsymbol{f}|_2^2 < M.!!$$

Table of contents

A Generalized Boussinesq Model

$$\begin{cases} \partial_t \boldsymbol{u} - \nabla \cdot (\nu(\theta)\nabla \boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \alpha \boldsymbol{g}\theta + \boldsymbol{f}, & \\ \nabla \cdot \boldsymbol{u} = 0, & \text{in } \Omega \times [0,\infty), \\ \partial_t \theta - \nabla \cdot (k(\theta)\nabla\theta) + (\boldsymbol{u}\cdot\nabla)\theta = 0, & \\ \boldsymbol{u} = 0, \qquad \partial_n \theta = 0 & \text{on } [0,\infty)\times\partial\Omega, \\ \boldsymbol{u}(0) = \boldsymbol{u}(T), \qquad \theta(0) = \theta(T) & \text{in } \Omega. \end{cases}$$

$$\Phi_m(t) = \int_\Omega (\nu(\theta_m)+1)|\nabla \boldsymbol{u}_m|^2 + \|\theta_m\|_2^2 + |\partial_t \theta_m|_2^2,$$

$$\Psi_m(t) = \|\boldsymbol{u}_m\|_2^2 + |\partial_t \boldsymbol{u}_m|_2^2 + \|\theta_m\|_3^2 + \|\partial_t\theta_m\|_1^2.$$

$\|\boldsymbol{f}\|_{L^2(L^2)} << \quad \Rightarrow \quad$ regular (and small) time-periodic solution.

A Generalized Boussinesq Model

$$\begin{cases} \partial_t \boldsymbol{u} - \nabla\cdot(\nu(\theta)\nabla\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \alpha\boldsymbol{g}\theta + \boldsymbol{f}, & \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega\times[0,\infty), \\ \partial_t\theta - \nabla\cdot(k(\theta)\nabla\theta) + (\boldsymbol{u}\cdot\nabla)\theta = 0, & \\ \boldsymbol{u} = 0, \qquad \partial_n\theta = 0 & \text{on } [0,\infty)\times\partial\Omega, \\ \boldsymbol{u}(0) = \boldsymbol{u}(T), \qquad \theta(0) = \theta(T) & \text{in } \Omega. \end{cases}$$

$$\Phi_m(t) = \int_\Omega (\nu(\theta_m)+1)|\nabla\boldsymbol{u}_m|^2 + \|\theta_m\|_2^2 + |\partial_t\theta_m|_2^2,$$

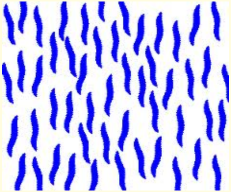
$$\Psi_m(t) = \|\boldsymbol{u}_m\|_2^2 + |\partial_t\boldsymbol{u}_m|_2^2 + \|\theta_m\|_3^2 + \|\partial_t\theta_m\|_1^2.$$

$\|\boldsymbol{f}\|_{L^2(L^2)} << \quad \Rightarrow$ regular (and small) time-periodic solution.

A Nematic Liquid Crystal Model

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = (\nabla\boldsymbol{d})^t\omega, & \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega\times[0,\infty), \\ \partial_t\boldsymbol{d} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{d} + \omega = 0 & \\ \boldsymbol{u}(x,t) = 0, \qquad \boldsymbol{d}(x,t) = \boldsymbol{h}(x,t) & \text{on } [0,\infty)\times\partial\Omega, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{d}(0) = \boldsymbol{d}_0 \quad \text{or} \quad \boldsymbol{u}(0) = \boldsymbol{u}(T),\ \boldsymbol{d}(0) = \boldsymbol{d}(T) \text{ in } \Omega. & \end{cases}$$

$$\boldsymbol{f}(\boldsymbol{d}) = \frac{1}{\varepsilon^2}(|\boldsymbol{d}|^2 - 1)\boldsymbol{d}, \qquad \omega = -\Delta\boldsymbol{d} + \boldsymbol{f}(\boldsymbol{d}),$$



A Nematic Liquid Crystal Model

$$
\begin{cases}
\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = (\nabla \boldsymbol{d})^t \omega, \\
\nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega\times[0,\infty), \\
\partial_t \boldsymbol{d} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{d} + \omega = 0 \\
\boldsymbol{u}(x,t) = 0, \qquad \boldsymbol{d}(x,t) = \boldsymbol{h}(x,t) & \text{on } [0,\infty)\times\partial\Omega, \\
\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{d}(0) = \boldsymbol{d}_0 \quad \text{or} \quad \boldsymbol{u}(0) = \boldsymbol{u}(T),\ \boldsymbol{d}(0) = \boldsymbol{d}(T) \text{ in } \Omega.
\end{cases}
$$

$$
\boldsymbol{f}(\boldsymbol{d}) = \frac{1}{\varepsilon^2}(|\boldsymbol{d}|^2 - 1)\boldsymbol{d}, \qquad \omega = -\Delta\boldsymbol{d} + \boldsymbol{f}(\boldsymbol{d}),
$$

$$
\Phi_m(t) = \|\boldsymbol{u}_m\|_1^2 + |\boldsymbol{\omega}_m + \partial_t \widetilde{\boldsymbol{d}}|_2^2,
$$

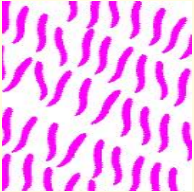
$$
\Psi_m^1(t) = \|\boldsymbol{u}_m\|_2^2, \quad \Psi_m^2(t) = |\nabla(\boldsymbol{\omega}_m + \partial_t \widetilde{\boldsymbol{d}})|_2^2
$$

$\nu >>$ $\Rightarrow$ Strong solution of (IVP) in $(0,+\infty)$ and regular time-periodic solution.

A Smectic-A Liquid Crystal Model

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} - \omega\nabla\varphi + \nabla p = 0 & \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } \Omega\times[0,\infty), \\ \partial_t\varphi + \boldsymbol{u}\cdot\nabla\varphi + \omega = 0, & \\ \boldsymbol{u}|_{\partial\Omega} = 0, \quad \varphi|_{\partial\Omega} = \varphi_1, \quad \partial_{\mathrm{n}}\varphi|_{\partial\Omega} = \varphi_2 & \text{on } [0,\infty)\times\partial\Omega, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0,\ \varphi(0) = \varphi_0 \quad \text{or} \quad \boldsymbol{u}(0) = \boldsymbol{u}(T),\ \varphi(0) = \varphi(T) \text{ in } \Omega. & \end{cases}$$

$$\omega = \Delta^2\varphi - \nabla\cdot\boldsymbol{f}(\nabla\varphi), \qquad \varphi_1 = \varphi_1(x,t),\ \varphi_2 = \varphi_2(x,t)$$



A Smectic-A Liquid Crystal Model

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu \Delta \boldsymbol{u} - \omega \nabla \varphi + \nabla p = 0 & \\ \nabla \cdot \boldsymbol{u} = 0, & \text{in } \Omega \times [0, \infty), \\ \partial_t \varphi + \boldsymbol{u} \cdot \nabla \varphi + \omega = 0, & \\ \boldsymbol{u}|_{\partial\Omega} = 0, \quad \varphi|_{\partial\Omega} = \varphi_1, \quad \partial_{\mathrm{n}} \varphi|_{\partial\Omega} = \varphi_2 & \text{on } [0, \infty) \times \partial\Omega, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \ \varphi(0) = \varphi_0 \quad \text{or} \quad \boldsymbol{u}(0) = \boldsymbol{u}(T), \ \varphi(0) = \varphi(T) \text{ in } \Omega. \end{cases}$$

$$\omega = \Delta^2 \varphi - \nabla \cdot \boldsymbol{f}(\nabla \varphi), \qquad \varphi_1 = \varphi_1(x, t), \ \varphi_2 = \varphi_2(x, t)$$

$$\Phi_m(t) = \|\boldsymbol{u}_m\|_1^2 + |\boldsymbol{\omega}_m - \partial_t \widetilde{\varphi}|_2^2,$$
$$\Psi_m^1(t) = \|\boldsymbol{u}_m\|_2^2, \quad \Psi_m^2(t) = \|\boldsymbol{\omega}_m - \partial_t \widetilde{\varphi}\|_2^2$$

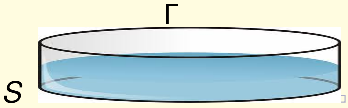
$\nu >>$ $\Rightarrow$ Strong solution of (IVP) in $(0, +\infty)$ and regular time-periodic solution.

A Bioconvective Model

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - 2\nabla \cdot (\nu(m) D(\boldsymbol{u})) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla q = -m\chi + \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} = 0, \qquad\qquad\qquad\qquad \text{in } (0,T) \times \Omega. \\ \dfrac{\partial m}{\partial t} - \theta \Delta m + \boldsymbol{u} \cdot \nabla m + U \dfrac{\partial m}{\partial x_3} = 0, \end{cases}$$

$$\begin{cases} \boldsymbol{u} = 0 \quad \text{on } (0,T) \times S, \qquad \boldsymbol{u} \cdot \mathbf{n} = 0 \quad \text{on } (0,T) \times \Gamma, \\ \nu(m)[D(\boldsymbol{u})\mathbf{n} - \mathbf{n} \cdot (D(\boldsymbol{u})\mathbf{n})\mathbf{n}] = 0 \quad \text{on } (0,T) \times \Gamma, \\ \theta \dfrac{\partial m}{\partial \mathbf{n}} - U m n_3 = 0 \quad \text{on } (0,T) \times \partial\Omega. \end{cases}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}(T), \quad m(0) = m(T), \quad \text{in } \Omega.$$



Some models

$$\Phi_n^1(t) = \int_\Omega (\nu(m^n) + 1)|D(\boldsymbol{u}^n - \boldsymbol{u}_\alpha)|_2$$

$$\Phi_n^2(t) = \|m^n - m_\alpha\|_2^2 + |\partial_t m^n|_2^2,$$

$$\Psi_n^1(t) = \|\boldsymbol{u}^n - \boldsymbol{u}_\alpha\|_2^2,$$

$$\Psi_n^2 = |\partial_t \boldsymbol{u}^n|_2^2 + \|m^n - m_\alpha\|_3^2 + \|\partial_t m^n\|_1^2$$

$\nu_{min} >>$ and $\boldsymbol{u}_0 - \boldsymbol{u}_\alpha,\ m_0 - m_\alpha <<$ $\Rightarrow$
Strong time-periodic solution.

Table of contents

Lemma

Let $\Phi \in L^1(0, +\infty)$ and $\Psi \in L^1_{loc}(0, +\infty)$ be two positive functions satisfying

$$\Phi'(t) + C\Psi(t) \leq A(\Phi(t)) + B(\Phi(t))\Psi(t)$$

$A(\Phi)$ will be an addition of powers (≥ 1) and $B(\Phi)$ an addition of powers (> 0) of Φ. Then,

$$\lim_{t \to +\infty} \Phi(t) = 0.$$

In particular, there exists $t^* \geq 0$ such that $\Phi \in C_b[t^*, +\infty)$.

Large time

A Double Penalized Smectic-A Model

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} - (\nabla\boldsymbol{n})^t\boldsymbol{\omega} + \nabla q = 0, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \partial_t\boldsymbol{n} + \boldsymbol{u}\cdot\nabla\boldsymbol{n} - \gamma\Delta\boldsymbol{\omega} = 0, \\ \mathcal{A}_{\varepsilon_2}(\boldsymbol{n}) + \boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}) - \boldsymbol{\omega} = 0, \quad \text{in } \Omega\times(0,+\infty) \end{cases}$$

$$\boldsymbol{u}|_{\partial\Omega} = 0, \quad \boldsymbol{n}|_{\partial\Omega} = \boldsymbol{n}_{\partial\Omega}, \quad \boldsymbol{\omega}|_{\partial\Omega} = 0$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \boldsymbol{n}(0) = \boldsymbol{n}_0 \quad \text{in } \Omega$$

$$\boldsymbol{f}_{\varepsilon_1}(\boldsymbol{n}) = \frac{1}{\varepsilon_1^2}(|\boldsymbol{n}|^2 - 1)\boldsymbol{n}$$

$$(\mathcal{A}_{\varepsilon_2}(\boldsymbol{n}), \overline{\boldsymbol{n}}) := (\nabla\boldsymbol{n}, \nabla\overline{\boldsymbol{n}}) + \frac{1}{\varepsilon_2^2}(\nabla\times\boldsymbol{n}, \nabla\times\overline{\boldsymbol{n}}).$$

Large time

$$\Phi_m(t) = |\nabla \boldsymbol{u}_m|_2^2 + |\nabla \boldsymbol{\omega}_m|_2^2,$$

$$\Psi_m(t) = \frac{1}{2}|A\boldsymbol{u}_m|_2^2 + K\|\partial_t \boldsymbol{n}_m\|_1^2$$

$$\Phi_m' + \Psi_m \leq C\Big(1 + \Phi_m^3\Big).$$

Strong solution of (PVI) in $(t^*, +\infty)$.