On an iterative method for the approximate solution of an initial and boundary-value problem for a generalized Boussinesq model

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Influence of temperature on hydrodynamics problem

Boussinesq approximation

The fluid is incompressible except insofar as the thermal expansion produces a buoyancy, represented by a term $\alpha g \varphi$, where

- g is the acceleration of gravity,
- α is the coefficient of thermal expansion,
- φ is the perturbation temperature.

Classical model

Continuity equation (Incompressibility)

$$\text{div}\,\mathbf{u} = 0 \qquad \text{in } (0,T)\times\Omega$$

Movement equations (NS)

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nabla\cdot(\nu\nabla\mathbf{u}) + \nabla p = \alpha\,\varphi\mathbf{g} + \mathbf{h} \qquad \text{in } (0,T)\times\Omega$$

Temperature equation

$$\frac{\partial \varphi}{\partial t} + \mathbf{u}\cdot\nabla\varphi - \nabla\cdot(k\nabla\varphi) = f \qquad \text{in } (0,T)\times\Omega,$$

Generalized model

Generalized Boussinesq model

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nabla\cdot(\nu(\varphi)\nabla\mathbf{u}) + \nabla p = \alpha\,\varphi\mathbf{g} + \mathbf{h}, \\ \text{div}\,\mathbf{u} = 0 \qquad\qquad \text{in } (0,T)\times\Omega, \\ \dfrac{\partial \varphi}{\partial t} + \mathbf{u}\cdot\nabla\varphi - \nabla\cdot(k(\varphi)\nabla\varphi) = f \end{cases}$$

$\nu : \mathbb{R} \to \mathbb{R}$ kinematic viscosity,
$k : \mathbb{R} \to \mathbb{R}$ thermal conductivity.

Generalized model

Generalized Boussinesq model

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot (\nu(\varphi) \nabla \mathbf{u}) + \nabla p = \alpha\, \varphi \mathbf{g} + \mathbf{h}, \\ \text{div}\ \mathbf{u} = 0 \qquad\qquad \text{in } (0,T) \times \Omega, \\ \dfrac{\partial \varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi - \nabla \cdot (k(\varphi) \nabla \varphi) = f \end{cases}$$

$$\mathbf{u}(x,t) = 0 \qquad \varphi(x,t) = 0 \qquad \text{on } (0,T) \times \partial\Omega,$$

$$\mathbf{u}(x,0) = \mathbf{u}_0(x) \qquad \varphi(x,0) = \varphi_0(x) \qquad \text{in } \Omega.$$

Previous result

S.A. Lorca, J.L. Boldrini, The initial value problem for a generalized Boussinesq model, Nonlinear Analysis (1999).

Goal

To show the existence of strong solution by using an iterative approach and also to give convergence-rates for the approximate solutions.

Notation

- $H = \{\boldsymbol{u} \in L^2;\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$,
  $V = \{\boldsymbol{u} \in H^1;\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{u} = 0 \text{ on } \partial\Omega\}$

- **Stokes operator**
  $P : L^2(\Omega) \mapsto H$ orthogonal projection (Helmholtz dec.).
  $A = -P\Delta$, $D(A) = H^2(\Omega) \cap V$.

Hypotheses

- Ω bounded domain in $\mathbb{R}^N (N = 2 \ \text{ou} \ 3)$, regular boundary.
- $\nu, \nu', \nu'', k, k', k''$ are bounded continuous functions.

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Iterative process

Let $\mathbf{u}^0(t) = \mathbf{u_0}$ and $\varphi^0(t) = \varphi_\mathbf{0}$ for all $t \in [0, T]$.
Step $n \geq 1$: First, given $\mathbf{u}^{n-1}$ and φ^{n-1} to find φ^n such that

$$\varphi_t^n - \operatorname{div}\left(k(\varphi^{n-1})\nabla\varphi^n\right) + \mathbf{u}^{n-1}\cdot\nabla\varphi^n = f,$$
$$\varphi^n(0) = \varphi_0 \quad \text{in } \Omega,$$
$$\varphi^n = 0 \quad \text{on } \partial\Omega.$$

Afterwards, given $\mathbf{u}^{n-1}$, φ^{n-1} and φ^n, to find $\mathbf{u}^n, p^n$ such that

$$\mathbf{u}_t^n - \operatorname{div}\left(\nu(\varphi^{n-1})\nabla\mathbf{u}^n\right) + \mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n + \nabla p^n = \mathbf{h} + \alpha\varphi^n\mathbf{g},$$
$$\operatorname{div}\mathbf{u}^n = 0,$$
$$\mathbf{u}^n(0) = \mathbf{u}_0 \quad \text{in } \Omega,$$
$$\mathbf{u}^n = 0 \quad \text{on } \partial\Omega.$$

Main results: approximate problems solution

Theorem

- $\mathbf{g} \in L^\infty(L^6)$; $f, \mathbf{h} \in L^\infty(L^2)$; $\mathbf{g}_t \in L^4(L^3)$, $f_t, \mathbf{h}_t \in L^2(L^2)$.
- f, $\mathbf{h}$, α, $\mathbf{u}_0$, φ_0 *small enough*

$\Rightarrow$ $\exists!$ $(\mathbf{u}^n, \varphi^n)$ *solution such that*

- $\mathbf{u}^n \in L^\infty(D(A))$, $\varphi^n \in L^\infty(H^2)$,
- $\mathbf{u}^n_t \in L^\infty(H) \cap L^2(V)$, $\varphi^n_t \in L^\infty(L^2) \cap L^2(H^1_0)$,

$$\begin{aligned} \sup_t \{|\mathbf{u}^n_t(t)|^2 + |\varphi^n_t(t)|^2\} &\leq M, \\ \int_0^t |\nabla \mathbf{u}^n_t(\tau)|^2 d\tau + \int_0^t |\nabla \varphi^n_t(\tau)|^2 d\tau &\leq M, \\ \sup_t \{|A\mathbf{u}^n(t)|^2 + |\Delta \varphi^n(t)|^2\} &\leq M, \end{aligned}$$

for all $t \in [0, T]$.

Theorem

Under the conditions of the previous theorem

$$(\mathbf{u}^n, \varphi^n) \to (\mathbf{u}, \varphi) \text{ in } L^2(D(A)) \times L^2(H^2(\Omega)).$$

$(\mathbf{u}, \varphi)$ *is the unique (strong) solution of the original problem.*

Main results: rate of convergence

Theorem

Rates of convergence:

$$\sup_{0\leq\tau\leq t}\{|\mathbf{u}^n(\tau)-\mathbf{u}(\tau)|^2+|\varphi^n(\tau)-\varphi(\tau)|^2\}\leq C\frac{(Dt)^n}{n!},$$

$$\sup_{0\leq\tau\leq t}\{\int_0^\tau|\nabla\mathbf{u}^n-\nabla\mathbf{u}|^2+\int_0^\tau|\nabla\varphi^n-\nabla\varphi|^2\}\leq C\frac{(Dt)^n}{n!},$$

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Proof of Theorem 1: Weak estimates

(eq. φ^n, φ^n) $+$ (eq. $\mathbf{u}^n, \mathbf{u}^n$) + Gronwall's lemma $\Rightarrow$

$$\mathbf{u}^n \text{ is bounded in } L^\infty(H) \cap L^2(V)$$

$$\varphi^n \text{ is bounded in } L^\infty(L^2) \cap L^2(H_0^1).$$

$$(\text{eq. } \varphi^n, -\Delta\varphi^n) + (\text{eq. } \mathbf{u}^n, A\mathbf{u}^n)$$
$$+(\text{eq. } \varphi^n, \varphi_t^n) + (\text{eq. } \mathbf{u}^n, \mathbf{u}_t^n)$$
$$+((\text{eq. } \varphi^n)_t, \varphi_t^n) + ((\text{eq. } \mathbf{u}^n)_t, \mathbf{u}_t^n),$$

$\Rightarrow$ differential inequality

Denoting

$$a = a(t) = C_2\alpha^4|g|_6^4 \in L^\infty(0,T)$$
$$b^n = b^n(t) = C_2(\alpha^2|g|_3^2|\nabla\varphi^n|^2 + \alpha^2|g_t|_3^2|\nabla\varphi^n|^2$$
$$+|f|^2 + |h|^2 + |f_t|^2 + |h_t|^2) \in L^1(0,T).$$

Integrating in $[0,t]$

$$
\begin{aligned}
&|\nabla \mathbf{u}^n|^2 + |\nabla \varphi^n|^2 + |\mathbf{u}_t^n|^2 + |\varphi_t^n|^2 \\
&+ \int_0^t (C - C_1(|\Delta \varphi^{n-1}| + |\nabla \mathbf{u}^{n-1}| + |\varphi_t^{n-1}|)(|\Delta \varphi^n|^2 + |A\mathbf{u}^n|^2)) \\
&+C \int_0^t (|\varphi_t^n|^2 + |\mathbf{u}_t^n|^2 + |\nabla \varphi_t^n|^2 + |\nabla \mathbf{u}_t^n|^2) \\
&\le |\nabla \mathbf{u}^n(0)|^2 + |\nabla \varphi^n(0)|^2 + |\mathbf{u}_t^n(0)|^2 + |\varphi_t^n(0)|^2 \\
&+C_1(\int_0^t |\nabla \mathbf{u}_t^{n-1}|^2 + |\nabla \varphi_t^{n-1}|^2)(|\Delta \varphi^n|^2 + |A\mathbf{u}^n|^2) \\
&+C_1 \int_0^t |\nabla \mathbf{u}^{n-1}|(|\mathbf{u}_t^n|^2 + |\varphi_t^n|^2) + \int_0^t (a|\varphi_t^n|^2 + b^n)
\end{aligned}
\tag{1}
$$

Proof of Theorem 1: Differential inequalities

$$(\text{eq. } \varphi^n, -\Delta\varphi^n) + (\text{eq. } \mathbf{u}^n, A\mathbf{u}^n) \Rightarrow$$

$$\begin{aligned} &(\frac{k_0}{2} - C_\varphi|\Delta\varphi^{n-1}|^2 - C_\varphi|\nabla\mathbf{u}^{n-1}|^2)|\Delta\varphi^n|^2 \\ &+(\frac{\nu_0}{2} - C_u|\Delta\varphi^{n-1}|^2 - C_u|\nabla\mathbf{u}^{n-1}|^2)|A\mathbf{u}^n|^2 \\ &\leq C(|\mathbf{u}_t^n|^2 + |\varphi_t^n|^2 + |g|_4^2|\nabla\varphi|^2) + C(|h|^2 + |f|^2) \end{aligned} \quad (2)$$

Proof of Theorem 1: Induction hypothesis (smallness)

$$\sup_{0\le t\le T} |\Delta\varphi^{n-1}|^2 < \delta^2 \qquad \sup_{0\le t\le T} |A\mathbf{u}^{n-1}|^2 < \delta^2$$

$$\sup_{0\le t\le T} |\nabla\varphi^{n-1}|^2 < \delta^2 \qquad \sup_{0\le t\le T} |\nabla\mathbf{u}^{n-1}|^2 < \delta^2$$

$$\sup_{0\le t\le T} |\varphi_t^{n-1}|^2 < \delta^2 \qquad \sup_{0\le t\le T} |\mathbf{u}_t^{n-1}|^2 < \delta^2$$

$$\sup_{0\le t\le T} \int_0^t |\nabla\varphi_t^{n-1}|^2 < \delta^2 \qquad \sup_{0\le t\le T} \int_0^t |\nabla\mathbf{u}_t^{n-1}|^2 < \delta^2$$

$\delta <<$

(1) + (2) + lemma's Gronwall $\Rightarrow$

$$\Phi_n(t) + \frac{C}{2}\int_0^t (|\Delta\varphi^n|^2 + |A\mathbf{u}^n|^2))$$
$$+C\int_0^t (|\varphi_t^n|^2 + |\mathbf{u}_t^n|^2 + |\nabla\varphi_t^n|^2 + |\nabla\mathbf{u}_t^n|^2)$$
$$\leq \left(\Phi_n(0) + C_2\int_0^t (|\nabla\mathbf{u}_t^{n-1}|^2 + |\nabla\varphi_t^{n-1}|^2)(|h|^2 + |f|^2) + \int_0^t b^n\right)$$
$$\cdot \exp\left(\int_0^t (|\nabla\mathbf{u}_t^{n-1}|^2 + |\nabla\varphi_t^{n-1}|^2) + C_1\delta T + \int_0^t a\right)$$

where $\Phi_n(t) = |\nabla\mathbf{u}^n|^2 + |\nabla\varphi^n|^2 + |\mathbf{u}_t^n|^2 + |\varphi_t^n|^2$.

Proof of Theorem 1

Induction hypothesis + hypothesis of small data $\Rightarrow$

$$|\nabla \mathbf{u}^n|^2 + |\nabla \varphi^n|^2 + |\mathbf{u}_t^n|^2 + |\varphi_t^n|^2 + \frac{C}{2}\int_0^t (|\Delta \varphi^n|^2 + |A\mathbf{u}^n|^2))$$
$$+C\int_0^t (|\varphi_t^n|^2 + |\mathbf{u}_t^n|^2 + |\nabla \varphi_t^n|^2 + |\nabla \mathbf{u}_t^n|^2) < \delta.$$

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Proof of Theorem 2: Equations

$$\mathbf{u}^{n,s}(t) = \mathbf{u}^{n+s}(t) - \mathbf{u}^{n}(t),$$
$$\varphi^{n,s}(t) = \varphi^{n+s}(t) - \varphi^{n}(t).$$

$$\mathbf{u}_t^{n,s} - \operatorname{div}\left(\nu(\varphi^{n+s-1})\nabla\mathbf{u}^{n,s}\right) + \nabla p^{n,s} = (\mathbf{u}^{n-1,s}\cdot\nabla)\mathbf{u}^{\mathbf{n}} + \alpha\mathbf{g}\varphi^{n,s}$$
$$+ \operatorname{div}\left((\nu(\varphi^{n+s-1}) - \nu(\varphi^{n-1}))\nabla\mathbf{u}^{n}\right) - (\mathbf{u}^{n+s-1}\cdot\nabla)\mathbf{u}^{n,s},$$

$$\varphi_t^{n,s} - \operatorname{div}\left(k(\varphi^{n+s-1})\nabla\varphi^{n,s}\right) = (\mathbf{u}^{n+s-1}\cdot\nabla)\varphi^{n,s}$$
$$+ \operatorname{div}\left((k(\varphi^{n+s-1}) - k(\varphi^{n-1}))\nabla\varphi^{n}\right) + (\mathbf{u}^{n-1,s}\cdot\nabla)\varphi^{n}.$$

Proof of Theorem 2: Differential inequalities

$$(\text{eq. } \varphi^{n,s}, -\Delta\varphi^{n,s}) + (\text{eq. } \mathbf{u}^{n,s}, A\mathbf{u}^{n,s}) \Longrightarrow$$

$$\begin{aligned}
& \frac{d}{dt}(|\nabla \mathbf{u}^{n,s}|^2 + |\nabla\varphi^{n,s}|^2) + \mu(|A\mathbf{u}^{n,s}|^2 + |\Delta\varphi^{n,s}|^2) \\
\leq\ & C(|\nabla \mathbf{u}^{n-1,s}|^2 + |\nabla\varphi^{n-1,s}|^2) + C(|\nabla \mathbf{u}^{n,s}|^2 + |\nabla\varphi^{n,s}|^2) \\
+\ & \delta|\Delta\varphi^{n-1,s}|^2.
\end{aligned}$$

Lemma

$a_n, b_n \in L^1(0,T),\ \ a_n(0) = 0$

$$a_n'(t) + b_n(t) \leq c_n(t)a_n(t) + d_n(t)a_{n-1}(t) + \epsilon b_{n-1}(t)$$

where c_n, d_n *positive and uniformly bounded in* $L^\infty(0,T)$ *and* $0 \leq \epsilon \leq 1/2$. *Then,*

$$||a_n||_{L^\infty(0,T)} \to 0 \quad \textit{and} \quad ||b_n||_{L^1(0,T)} \to 0$$

Proof of Theorem 2

$$\frac{d}{dt}\underbrace{(|\nabla \mathbf{u}^{n,s}|^2 + |\nabla \varphi^{n,s}|^2)}_{a_n(t)} + \mu\underbrace{(|A\mathbf{u}^{n,s}|^2 + |\Delta \varphi^{n,s}|^2)}_{b_n(t)}$$

$$\leq \underbrace{C}_{d_n(t)}(|\nabla \mathbf{u}^{n-1,s}|^2 + |\nabla \varphi^{n-1,s}|^2) + \underbrace{C}_{c_n(t)}(|\nabla \mathbf{u}^{n,s}|^2 + |\nabla \varphi^{n,s}|^2)$$

$$+ \quad \delta|\Delta \varphi^{n-1,s}|^2.$$

Proof of Theorem 2

$$a_n(t) = |\nabla \mathbf{u}^{n,s}(t)|^2 + |\nabla \varphi^{n,s}(t)|^2, \qquad \|a_n\|_\infty \to 0$$

$$b_n(t) = \mu(|A\mathbf{u}^{n,s}(t)|^2 + |\Delta \varphi^{n,s}(t)|^2) \qquad \|b_n\|_{L^1} \to 0$$

$$\mathbf{u}^n \to \mathbf{u} \quad \text{strongly in } L^\infty(V) \cap L^2(D(A)),$$

$$\mathbf{u}^n_t \to \mathbf{u}_t \quad \text{strongly in } L^2(H),$$

$$\varphi^n \to \varphi \quad \text{strongly in } L^\infty(H^1_0) \cap L^2(H^2),$$

$$\varphi^n_t \to \varphi_t \quad \text{strongly in } L^2(L^2).$$

$$a_n(t) = |\nabla \mathbf{u}^{n,s}(t)|^2 + |\nabla \varphi^{n,s}(t)|^2, \qquad \|a_n\|_\infty \to 0$$

$$b_n(t) = \mu(|A\mathbf{u}^{n,s}(t)|^2 + |\Delta \varphi^{n,s}(t)|^2) \qquad \|b_n\|_{L^1} \to 0$$

$$\mathbf{u}^n \to \mathbf{u} \quad \text{strongly in } L^\infty(V) \cap L^2(D(A)),$$

$$\mathbf{u}_t^n \to \mathbf{u}_t \quad \text{strongly in } L^2(H),$$

$$\varphi^n \to \varphi \quad \text{strongly in } L^\infty(H_0^1) \cap L^2(H^2),$$

$$\varphi_t^n \to \varphi_t \quad \text{strongly in } L^2(L^2).$$

$$\mathbf{v}^n = \mathbf{u}^n - \mathbf{u}$$
$$z^n = \varphi^n - \varphi$$
$$q^n = p^n - p$$

Subtracting the corresponding equations.

$$a_n(t) = |\mathbf{v}^n|^2(t) + C|z^n|^2(t)$$

$$b_n(t) = \nu_0|\nabla \mathbf{v}^n|^2(t) + C|\nabla z^n|^2(t)$$

$$a_n(0) = 0$$

Lemma

a_n, $b_n \in L^1(0,T)$ positive, $a_n(0) \le A_0 \in \mathbb{R}$

$$a_n'(t) + b_n(t) \le c_n(t) a_n(t) + d_n(t) a_{n-1}(t), \quad a.e.\ t \in (0,T),$$

where c_n, d_n positive, bounded in $L^1(0,T)$ and $L^\infty(0,T)$, respectively. Then,

$$a_n(t) + \int_0^t b_n(s)\,ds \le D A_0 e^{Dt} + |a_0|_\infty \frac{(D\,t)^n}{n!}.$$

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Open problems

- Strong convergence-rate bounds.
- Nonlinear iterative process.