Some properties of a class of abstract stationary equations

E.J. Villamizar-Roa [a,1], M.A. Rodríguez-Bellido [b,2] and M.A. Rojas-Medar [a,3]

[a] *IMECC-UNICAMP, CP 6065, 13083-970, Campinas-SP, Brazil*

[b] *Departamento de Ecuaciones diferenciales y Análisis Numérico, Universidad de Sevilla, Facultad de Matemáticas, 41012 Sevilla, Spain*

Abstract

We study a class of abstract nonlinear equations in a separable Hilbert space for which we prove properties of the set of solutions. The results apply, in particular, in several models of hydrodynamics, such as magneto-micropolar equations, micropolar fluid equations, Boussinesq and Navier-Stokes equations.

Key words: Abstract stationary equations, Sard's Theorem, Navier-Stokes equations.

1 Introduction

We are concerned in the study of properties of the following class of abstract stationary nonlinear equations in a separable Hilbert space X

$$A\mathbf{u} + B(\mathbf{u},\mathbf{u}) + B_1\mathbf{u} + B_2\mathbf{u} = \mathbf{f}. \tag{1}$$

⋆ The Second and third authors has been partially supported by D.G.E.S. and M.C. y T. (Spain), Projet BFM2003-06446.

[1] E.J. Villamizar-Roa is partially supported by COLCIENCIAS - Colombia, Project COLCIENCIAS-BID III etapa. E-mail: evillami@ime.unicamp.br

[2] M.A. Rodríguez-Bellido apologizes the FAPESP for the financial support for her stay in the UNICAMP, process 04/02889-6. E-mail: angeles@us.es

[3] M.A. Rojas-Medar is partially supported by CNPq-Brazil, grant No 301354/03-0. E-mail: marko@ime.unicamp.br

In (1) A is a self-adjoint, strictly positive operator in X with domain $D(A)$ and inverse compact. Thus there exists an ortonormal basis of X, $\{\mathbf{w}_j\}_{j\in\mathbb{N}}$ such that

$$A\mathbf{w}_j = \lambda_j \mathbf{w}_j,\ j = 1, 2, ... \quad 0 < \lambda_1 \leq \lambda_2 \leq ... \longrightarrow \infty.$$

The scalar product and the norm in X are denoted by $(\cdot,\cdot)$ and $|\cdot|$, respectively. As $\{\mathbf{w}_j\}_{j\in\mathbb{N}}$ is an ortonormal basis in X, for all $\mathbf{u} \in X$ we have:

$$\mathbf{u} = \sum_{j=1}^{\infty} c_j \mathbf{w}_j,\ c_j = (\mathbf{u}, \mathbf{w}_j).$$

The domain of operator A is characterized by

$$D(A) = \{\mathbf{u} = \sum_{j=1}^{\infty} c_j \mathbf{w}_j : \sum_{j=1}^{\infty} \lambda_j^2 c_j^2 < \infty\}.$$

For $\mathbf{u} \in D(A)$ we have $A\mathbf{u} = \sum_{j=1}^{\infty} \lambda_j c_j \mathbf{w}_j$. We can define the powers $A^\alpha : D(A^\alpha) \longrightarrow X,\ \alpha \in \mathbb{R},\ 0 \leq \alpha \leq 1$ with domain

$$D(A^\alpha) = \{\mathbf{u} = \sum_{j=1}^{\infty} c_j \mathbf{w}_j : \sum_{j=1}^{\infty} \lambda_j^{2\alpha} c_j^2 < \infty\}$$

defines by $A^\alpha \mathbf{u} = \sum_{j=1}^{\infty} \lambda_j^\alpha c_j \mathbf{w}_j$ and $c_j = (\mathbf{u}, \mathbf{w}_j)$. Observe that $X^\alpha = D(A^{\alpha/2})$ is a Hilbert space with the product

$$(\mathbf{u}, \mathbf{v})_\alpha = (A^{\alpha/2}\mathbf{u}, A^{\alpha/2}\mathbf{v}).$$

The associated norm is denoted by $|\cdot|_\alpha$. We also denoted $((\mathbf{u}, \mathbf{v})) = (\mathbf{u}, \mathbf{v})_1$ and $|\cdot|_1 = \|\cdot\|$. The dual space of X^α is denoted by $X^{-\alpha}$. Thus, identifying X with its dual space we have

$$X^\alpha \hookrightarrow X \hookrightarrow X^{-\alpha}$$

where the injections are continuous and with dense image. In (1) B_1 and B_2 are linear continuous operators and B is a bilinear continuous operator in X satisfying the following assumptions: $B : D(A^{1/2}) \times D(A^\beta) \longrightarrow X,\ B_i : D(A^\beta) \longrightarrow X$ with $\beta \in (3/4, 1)$ such that:

$$(B(\mathbf{u}, \mathbf{v}), \mathbf{v}) = 0, \quad \forall \mathbf{u}, \mathbf{v} \in D(A^{1/2}), \tag{2}$$

$$(B_1\mathbf{u}, \mathbf{v}) + (B_1\mathbf{v}, \mathbf{u}) = 0, \quad \forall \mathbf{u}, \mathbf{v} \in D(A^{1/2}). \tag{3}$$

We assume such that B, B_i have continuous extensions $B : X \times X^1 \to X^{-2\beta}$, $B : X^1 \times X^1 \to X^{-2\beta+1}$, $B : X^1 \times X \to X^{-2\beta}$, $B_i : X \to X^{-2\beta}$ and $B_i : X^1 \to X^{1-2\beta}$. Note that by interpolation inequalities, (2),(3) implies that

$$|(B(\mathbf{u},\mathbf{v}),\mathbf{w})| \leq c_1|A^{1/2}\mathbf{u}||A^{1/2}\mathbf{v}|^{2-2\beta}|A\mathbf{v}|^{2\beta-1}|\mathbf{w}|, \text{ for } \mathbf{v} \in D(A)$$

and

$$|(B_i\mathbf{u},\mathbf{v})| \leq c_{2i}|A^{1/2}\mathbf{u}|^{2-2\beta}|A\mathbf{u}|^{2\beta-1}|\mathbf{v}|, \text{ for } \mathbf{u} \in D(A).$$

The norms of B and B_i with values in $X^{-\alpha}$ are denoted by $\|B\|_\alpha$ and $\|B_i\|_\alpha$. $\mathbf{f}$ is assumed in X while $\mathbf{u}$ is unknown.

Equations (1) cover several models of hydrodynamics, such as magneto-micropolar equations, micropolar fluid equations, classical hydrodynamics, Boussinesq and Navier-stokes equations. Many authors have studied this models separately and many results relatives to existence, uniqueness, regularity of solutions, for instance, were encountred. The structure of the set of stationary solutions of the Navier-Stokes equations was considered in [3], [4],[8] and references therein. We study some of those properties for the set of solutions of (1) and the technique of analysis is closely related to those in the above cite papers. We will prove the following theorems:

Theorem 1.1 *We consider the equation (1) and the assumptions (2)-(3) hold. Then there exists a dense open set $\mathcal{O} \subset X$ such that for every $\boldsymbol{f} \in \mathcal{O}$, the set of solutions of (1) is finite and odd number. Moreover, the number of solutions $\mathcal{R}(\boldsymbol{f})$ of (1) on each connected component of $\mathcal{O}$ is constant.*

Theorem 1.2 *(Continuous Dependence) Let $\boldsymbol{f}_0 \in \mathcal{O}$. Then there exists a neighborhood of $\boldsymbol{f}_0$ such that for all $\boldsymbol{f}$ in it, Card $\mathcal{R}(\boldsymbol{f}) =$ Card $\mathcal{R}(\boldsymbol{f}_0) < \infty$. Moreover, if $\boldsymbol{f}$ converges to $\boldsymbol{f}_0$ in X, $\mathcal{R}(\boldsymbol{f})$ converges to $\mathcal{R}(\boldsymbol{f}_0)$ in the Hausdorff metric over X.*

2 Preliminaries.

We recall first some definitions and facts from the linear theory of *Fredholm operators* (see [5], and the references therein for details). If E_1 and E_2 are two real Banach spaces, a linear continuous operator $L : E_1 \longrightarrow E_2$ is called a Fredholm operator if

(1) Ker L has finite dimension
(2) Range L is closed
(3) Coker $L = E_2/$Range L, has finite dimension

If L is Fredholm, then its *index* is the integer ind $L =$ dim Ker$L$-dim Coker L. If $L = L_1 + L_2$ where L_1 is compact from E_1 into E_2 and L_2 is an isomorphism (respectively is surjective and dim Ker $L_2 = q$), then L is Fredholm operator of index 0 (respectively q). Let $M \subset E_1$ a differentiable manifold. A *Fredholm* *map* is a C^1 map $\mathcal{F} : M \longrightarrow E_2$ such that for each $u \in M$, the differential $D\mathcal{F}(u)$ is a Fredholm operator. In this case it follows from the properties of Fredholm operators that the index doesn't depend on u [5]; in this case the index of $\mathcal{F}$ is defined as index of $D\mathcal{F}(u)$. Let $\mathcal{F}$ be a C^1 map from a differentiable manifold M of E_1 into E_2. We recall that $u \in M$ is called a *regular point* of $\mathcal{F}$ if $D\mathcal{F}(u)$ is surjective and is *singular* if its not regular. The images of the singular points under $\mathcal{F}$ are called the singular values or critical values. Its complement in E_2 constitutes the set of regular values of $\mathcal{F}$. Thus a regular value of $\mathcal{F}$ is a point $f \in E_2$ which does not belong to the image $\mathcal{F}(M)$, or such that $D\mathcal{F}(u)$ is onto at every point u in the pre image $\mathcal{F}^{-1}(f)$. To prove the Theorem 1.1, we going to make use of the following infinite dimensional version of Sard's Theorem [5].

Theorem 2.1 *(Smale). Let E_1 and E_2 be two Banach spaces, $M \subset E_1$ a connected open and $\mathcal{F} : M \longrightarrow E_2$ a proper C^k Fredholm map with $k > max(ind\ \mathcal{F}, 0)$. Then the regular values of $\mathcal{F}$ form a dense set of E_2. If ind $\mathcal{F} = 0$ and f is a regular value of $\mathcal{F}$, then $\mathcal{F}^{-1}(f)$ is a discrete set. If ind $\mathcal{F} > 0$, and f is a regular value of $\mathcal{F}$, then $F^{-1}(f)$ is either empty or a manifold in M of class C^k and dimension ind $\mathcal{F}$.*

3 Proof of Theorems.

With respect to Equations (1) we have the following results

Theorem 3.1 *If $\|B_2\|_1 < 1$, then for all $\boldsymbol{f} \in X^{-1}$ the equation (1) have at least one weak solution $\boldsymbol{u} \in X^1$ in the following sense: for all $\boldsymbol{v} \in X^1$, $\boldsymbol{u}$ verifies*

$$((\boldsymbol{u}, \boldsymbol{v})) + (B(\boldsymbol{u}, \boldsymbol{u}), \boldsymbol{v}) + (B_1\boldsymbol{u}, \boldsymbol{v}) + (B_2\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}). \tag{4}$$

If B_2 satisfies

$$1 - \|B_2\|_1 > (\|B\|_1\|\boldsymbol{f}\|_{-1}^2)^{1/2}, \tag{5}$$

then the equations (1) has a unique solution.

To demonstrate the Theorem 3.1 (existence) we considered the Galerkin approximations and used the following result which is an consequence of Brouwer's Theorem (see [2]):

Lemma 3.2 *Let H a Hilbert space of finite dimension, with inner product $[\cdot,\cdot]$ and norm $[\cdot]$. If the operator $P : H \longrightarrow H$ is continuous and if there* *exists* $a > 0$ *with* $[y] = a : [P(y), y] > 0$, *then there exists* $y \in H$ *with* $[y] \leq a$ *such that* $P(y) = 0$.

The Galerkin approximation is given by the expression

$$\mathbf{u}^k = \sum_{j=1}^{k} c_{jk} \mathbf{w}_j, \ \ k \in \mathbb{N},$$

where the coefficients c_{jk} are real numbers which determined such that $\mathbf{u}^k$ is solution of the following problem

$$A\mathbf{u}^k + P_k(B(\mathbf{u}^k) + B_1\mathbf{u}^k + B_2\mathbf{u}^k) = P_k\mathbf{f}. \tag{6}$$

Here P_k represent the orthogonal projection associated to vectorial closed space $X_k = span\{\mathbf{w}_1, ..., \mathbf{w}_k\}$ or equivalently for $j = 1, ..., k$,

$$((\mathbf{u}^k, \mathbf{w}_j)) + (B(\mathbf{u}^k, \mathbf{u}^k), \mathbf{w}_j) + (B_1\mathbf{u}^k, \mathbf{w}_j) + (B_2\mathbf{u}^k, \mathbf{w}_j) = (\mathbf{f}, \mathbf{w}_j). \tag{7}$$

Taking $\mathbf{v} = \mathbf{u}$ ($\mathbf{u}$ solution of (1) gives by Theorem 3.1) in (4) and using (2), (3) is easy obtain the following:

Theorem 3.3 *The solutions of (1) belong to the ball* $\{\boldsymbol{u} \in X^1 : \|\boldsymbol{u}\|^2 \leq r\}$, *where*

$$r = \frac{\|\boldsymbol{f}\|_{-1}^2}{1 - \|B_2\|_1}.$$

Now, with $\mathbf{f} \in X$ we show an estimate of $\mathbf{u}$ in the norm of X^2. From (7), for all $\phi \in X_k$, we have

$$(A^{1/2}\mathbf{u}^k, A^{1/2}\phi) + (B(\mathbf{u}^k, \mathbf{u}^k), \phi) + (B_1\mathbf{u}^k, \phi) + (B_2\mathbf{u}^k, \phi) = (\mathbf{f}, \phi), \ \ k = 1, 2, ..., k.$$

Considering $\phi = A\mathbf{u}^k \in X_k$, using (2),(3), Young's inequality and interpolation inequalities, we obtain:

$$\begin{aligned} |A\mathbf{u}^k|^2 &\leq |\mathbf{f}||A\mathbf{u}^k| + |(B(\mathbf{u}^k, \mathbf{u}^k), A\mathbf{u}^k)| + |(B_1\mathbf{u}^k, A\mathbf{u}^k)| + |(B_2\mathbf{u}^k, A\mathbf{u}^k)| \\ &\leq |f||A\mathbf{u}^k| + c_1|A^{1/2}\mathbf{u}^k|^{3-2\beta}|A\mathbf{u}^k|^{2\beta} + c_2|A^{1/2}\mathbf{u}^k|^{2-2\beta}|A\mathbf{u}^k|^{2\beta} \\ &\leq c_\epsilon|\mathbf{f}|^2 + \epsilon|A\mathbf{u}^k|^2 + c_\epsilon|A^{1/2}\mathbf{u}^k|^\alpha + \epsilon|A\mathbf{u}^k|^2 + c_\epsilon|A^{1/2}\mathbf{u}^k|^2 + \epsilon|A\mathbf{u}^k|^2, \end{aligned}$$

with $\alpha = (3 - 2\beta)/(1 - \beta)$. Here, we mean by c_ϵ a generic positive constant which depend of ϵ. Taking ϵ sufficiently small and using that $|A^{1/2}\mathbf{u}^k| \leq r$, we obtain $|A\mathbf{u}^k|^2 \leq C$, where C does not depend of k. This estimate implies that the solutions u in X^1 of (1) belong $D(A)$. From (1) we obtain

$$
\begin{aligned}
|A\mathbf{u}| &\leq |\mathbf{f}| + |B(\mathbf{u},\mathbf{u})| + |B_1\mathbf{u}| + |B_2\mathbf{u}| \\
&\leq |\mathbf{f}| + c_1|A^{1/2}\mathbf{u}|^{3-2\beta}|A\mathbf{u}|^{2\beta-1} + c_2|A^{1/2}\mathbf{u}|^{2-2\beta}|A\mathbf{u}|^{2\beta-1} \\
&\leq |\mathbf{f}| + c_1|A^{1/2}\mathbf{u}|^{(3-2\beta)/(1-\beta)} + c_2|A^{1/2}\mathbf{u}|^2.
\end{aligned}
$$

Therefore using that $|A^{1/2}\mathbf{u}| \leq r$ (which depend of $|\mathbf{f}|$) we have

$$
|A\mathbf{u}| \leq c(|\mathbf{f}|) \tag{8}
$$

for some constant $c > 0$ independent of $\mathbf{u}$. Now we are going to apply the Theorem 2.1 to show the Theorem 1.1. For this, we considered $M = E_1 = D(A),\ E_2 = X$ and $\mathcal{F}(\mathbf{u}) = A\mathbf{u} + B(\mathbf{u},\mathbf{u}) + B_1\mathbf{u} + B_2\mathbf{u}$. From assumptions of $B, B_i, i = 1, 2$ we have that $\mathcal{F}$ makes since as a mapping from $D(A)$ into X. Let us remark that for any $\mathbf{f} \in X : \mathcal{R}(\mathbf{f}) := \mathcal{F}^{-1}(\mathbf{f})$.

Lemma 3.4 *The application $\mathcal{F} : D(A) \longrightarrow X$ is proper.*

PROOF. Let K a compact set of X. It follows of (8) that $\mathcal{F}^{-1}(K)$ is bounded in $D(A)$ because K is bounded in X. Thus $\mathcal{F}^{-1}(K)$ is compact in $D(A^\beta)$ since $D(A) = X^2 \hookrightarrow X^{2\beta} = D(A^\beta)$ compactly. As $B(\mathbf{u},\mathbf{u}), B_i, i = 1, 2$ are continuous from $D(A^\beta) \longrightarrow X$, we have that $B(\mathcal{F}^{-1}(K)), B_i(\mathcal{F}^{-1}(K))$ are compact in X. As the set $A^{-1}(K - B(\mathcal{F}^{-1}(K)) - \sum_{i=1}^{2} B_i(\mathcal{F}^{-1}(K)))$ is relatively compact in $D(A)$ and contain $\mathcal{F}^{-1}(K)$, we conclude that $\mathcal{F}^{-1}(K)$ is compact in $D(A)$.

Remark 3.5 *The Lemma 3.4 implies that the set $\mathcal{R}(\boldsymbol{f}) = \mathcal{F}^{-1}(\boldsymbol{f})$ is a compact subset of $D(A)$ for any $\boldsymbol{f} \in X$.*

Note that $\mathcal{F}$ is a C^∞ mapping from $D(A)$ into X and

$$
D\mathcal{F}(\mathbf{u}).\mathbf{v} = A\mathbf{v} + B(\mathbf{u},\mathbf{v}) + B(\mathbf{v},\mathbf{u}) + B_1\mathbf{v} + B_2\mathbf{v},\ \forall \mathbf{u},\mathbf{v} \in D(A).
$$

We also have that for each $\mathbf{u} \in D(A)$ the linear applications $\mathbf{v} \mapsto B(\mathbf{u},\mathbf{v}),\ \mathbf{v} \mapsto B(\mathbf{v},\mathbf{u}),\ \mathbf{v} \mapsto B_i\mathbf{v},\ i = 1, 2$ are continuous from $X^{2\beta}$ into X; therefore, if $(\mathbf{v}_n)$ is a bounded sequence in $D(A)$, then as $D(A) \hookrightarrow X^{2\beta}$ compactly, there exists a subsequence $\mathbf{v}_{nk}$ that converges strongly in $X^{2\beta}$ to some $\mathbf{v} \in X^{2\beta}$. Consequently, $B(\mathbf{u},\mathbf{v}_{nk}) \to B(\mathbf{u},\mathbf{v}),\ B(\mathbf{v}_{nk},\mathbf{u}) \to B(\mathbf{v},\mathbf{u})$, and $B_i\mathbf{v}_{nk} \to B_i\mathbf{v}$ in X, i.e, $\mathbf{v} \mapsto B(\mathbf{u},\mathbf{v}),\ \mathbf{v} \mapsto B(\mathbf{v},\mathbf{u}),\ \mathbf{v} \mapsto B_i\mathbf{v},\ i = 1, 2$ are compact from $D(A)$ into X. Since A is an isomorphism from $D(A)$ into X, then using the property of Fredholm commented in the begin of this section, we concluded that $D\mathcal{F}(\mathbf{u})$ is a Fredholm operator of index 0. Applying the Theorem 2.1, we have that $\mathcal{F}^{-1}(\mathbf{f})$ is a discrete set for each $\mathbf{f}$ regular value of $\mathcal{F}$. Since $\mathcal{F}$ is proper $\mathcal{F}^{-1}(\mathbf{f})$ is a finite set; consequently setting $\mathcal{O} = \{$the set of regular values of $\mathcal{F}\}$ which is dense and open by Smale's Theorem, we concluded a part of demonstration of Theorem 1.1. Now, we going to show that the number of solutions of (1) an each connected component of $\mathcal{O}$ is constant and every solution is a C^∞ function of $\mathbf{f}$. Let $\mathcal{O}_j,\ j \in \Lambda$ be the connected components of $\mathcal{O}$ (which are open) and let $\mathbf{f}_0, \mathbf{f}_1$ be two points of $\mathcal{O}_j$, for some j. Take $\mathbf{u}_0 \in \mathcal{F}^{-1}(\mathbf{f}_0)$. Then exists a continuous curve $t \in [0,1] \mapsto \mathbf{f}(t) \in \mathcal{O}_j,\ \mathbf{f}(0) = \mathbf{f}_0, \mathbf{f}(1) = \mathbf{f}_1$. We extend the above application as $\mathbf{f}(t+z) = \mathbf{f}(t),\ z \in \mathbb{Z}$ and we define the map $T : \mathbb{R} \times D(A) \longrightarrow X$ such that

$$(s, \mathbf{u}) \longmapsto T(s, \mathbf{u}) = \mathcal{F}(\mathbf{u}) - \mathbf{f}(s).$$

Directly from the assumptions, it follows that $T(0, \mathbf{u}_0) = 0$, the mapping $T(s, \cdot)$ is of class C^∞ and moreover for $(s, \mathbf{u}) \in \mathbb{R} \times D(A) : D_{\mathbf{u}} T(s, \cdot) = D\mathcal{F}(\mathbf{u})$. $D_{\mathbf{u}_0} T(0, \cdot)$ is an isomorphism of $D(A)$ in X since $D\mathcal{F}(\mathbf{u})$ is a Fredholm operator of Index 0, hence, by the Implicit Function Theorem there exists neighborhoods $\mathcal{U}_0, \mathcal{U}_{\mathbf{u}_0}$ of 0 and $\mathbf{u}_0$, respectively, and a unique continuous function $s \mapsto \mathbf{u}(s),\ s \in \mathcal{U}_0$, with $\mathcal{F}(\mathbf{u}(s)) = \mathbf{f}(s),\ \mathbf{u}(0) = \mathbf{u}_0$. Since $\mathbf{f}(s)$ is regular value of $\mathcal{F}$, for all $s \in [0,1],\ \mathbf{u}(s)$ is defined for every $s,\ 0 \le s \le 1$, and therefore $\mathbf{u}(1) \in \mathcal{F}^{-1}(\mathbf{f}_1)$. Now, $Card\ \mathcal{F}^{-1}(\mathbf{f}_0) \le Card\ \mathcal{F}^{-1}(\mathbf{f}_1)$ because the uniqueness of $\mathbf{u}(s)$. By symmetry, the number of points is the same. It remain to show that the number of solutions of (1) is odd. For fixed $\mathbf{f} \in \mathcal{O}$, we rewrite (1) as

$$F(\mathbf{u}) = \mathbf{u} + A^{-1}\Big(B(\mathbf{u}, \mathbf{u}) + \sum_{j=1}^{j=2} B_j \mathbf{u}\Big) = A^{-1}(\mathbf{f}) = \mathbf{g},\ \mathbf{u}, \mathbf{g} \in D(A).$$

Note that $F = (I - F_0)$, with $-F_0 := A^{-1}(B + \sum_{j=1}^{j=2} B_j)$ which is compact. If $\mathbf{u}_\lambda$ is solution of $F(\mathbf{u}_\lambda) = \lambda \mathbf{g},\ 0 \le \lambda \le 1$, then by (8)

$$|A\mathbf{u}_\lambda| \le c(|\mathbf{f}| + |\mathbf{f}|^3) < R := 1 + c(|\mathbf{f}| + |\mathbf{f}|^3).$$

Therefore the Leray-Schauder degree $D(F, B_R, \lambda\mathbf{g})$ (see [1]) where B_R is the open ball of $D(A)$ of radius R, is well defined. If $\lambda\mathbf{f}$ is a regular value of $\mathcal{F}$, then $\lambda\mathbf{g}$ is a regular value of F. Hence, if $F^{-1}(\lambda\mathbf{g}) = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ (finite set), we have that

$$D(F, B_R, \lambda\mathbf{g}) = \sum_{i=1}^{k} ind(\mathbf{u}_j).$$

There exist $\lambda_0 \in [0,1]$, such that B_2 satisfies $1 - \|B_2\|_1 > (\sqrt{\|B\|_1 \|\lambda\mathbf{f}\|_{-1}^2})^{1/2}$ for all $0 \le \lambda \le \lambda_0$ implying that $\mathcal{F}^{-1}(\lambda\mathbf{f})$ contains only one point $\mathbf{u}_\lambda$. For this values of λ, using arguments similar to that used before, we can show that $D\mathcal{F}(\mathbf{u}_\lambda)$ is an isomorphism and consequently $D(F, B_R, \lambda\mathbf{g}) = \pm 1$. As $D(F, B_R, \lambda\mathbf{g})$ is invariant by homotopy, we have that $D(F, B_R, \mathbf{g}) = \pm 1$. This implies that k is an odd number.

To prove the Theorem 1.2 we need the following lemma:

Lemma 3.6 *Let $\boldsymbol{f}_0 \in \mathcal{O},\ \boldsymbol{u}_0 \in \mathcal{R}(\boldsymbol{f}_0)$. There exists a neighborhoods $\mathcal{U}_{\boldsymbol{f}_0}, \mathcal{U}_{\boldsymbol{u}_0}$*

of $\boldsymbol{f}_0$ and $\boldsymbol{u}_0$, respectively, such that Card $(\mathcal{U}_{\boldsymbol{u}_0} \cap \mathcal{R}(\boldsymbol{f})) = 1$ for all $\boldsymbol{f} \in \mathcal{U}_{\boldsymbol{f}_0}$. Moreover the application $\mathcal{U}_{\boldsymbol{f}_0} \ni \boldsymbol{f} \longmapsto \boldsymbol{u}_{\boldsymbol{f}} \in \mathcal{U}_{\boldsymbol{u}_0}$ is of class C^∞, where $\{\boldsymbol{u}_{\boldsymbol{f}}\} := \mathcal{U}_{\boldsymbol{u}_0} \cap \mathcal{R}(\boldsymbol{f})$.

PROOF. . we consider the map $T : X \times D(A) \longrightarrow X$ such that $T(\mathbf{f}, \mathbf{u}) = \mathcal{F}(\mathbf{u}) - \mathbf{f}$. Note that $T(\mathbf{f}_0, \mathbf{u}_0) = 0$ and T is of class C^∞. As $D\mathcal{F}(\mathbf{u})$ is a Fredholm operator of index 0, we have that $D_{\mathbf{u}_0}T(\mathbf{f}_0, \cdot)$ is an isomorphism of $D(A)$ to X. Hence, from the Implicit Function Theorem, it follows that there exists neighborhoods $\mathcal{U}_{\mathbf{f}_0}, \mathcal{U}_{\mathbf{u}_0}$ of $\mathbf{f}_0$ and $\mathbf{u}_0$, respectively, and a $C^\infty-$ function $\xi : \mathcal{U}_{\mathbf{f}_0} \longrightarrow \mathcal{U}_{\mathbf{u}_0}$ such that $T(\mathbf{f}, \xi(\mathbf{f})) = 0$, for all $\mathbf{f} \in \mathcal{U}_{\mathbf{f}_0}$. In particular for any $\mathbf{f} \in \mathcal{U}_{\mathbf{f}_0}$, there exists an unique $\mathbf{u}_{\mathbf{f}} \in \mathcal{U}_{\mathbf{u}_0}$ with $\xi(\mathbf{f}) = \mathbf{u}_{\mathbf{f}}$, i.e, $T(\mathbf{f}, \mathbf{u}_{\mathbf{f}}) = 0$, or equivalently $\mathbf{u}_{\mathbf{f}} \in \mathcal{U}_{\mathbf{u}_0} \cap \mathcal{R}(\mathbf{f})$. Moreover the application $\mathcal{U}_{\mathbf{f}_0} \ni \mathbf{f} \longmapsto \mathbf{u}_{\mathbf{f}} \in \mathcal{U}_{\mathbf{u}_0}$ is of class C^∞.

Proof of Theorem 1.2.

From Theorem 1.1, the set $\mathcal{R}(\mathbf{f}_0) < \infty$. Let $\mathbf{u} \in \mathcal{R}(\mathbf{f}_0)$. By the Lemma 3.6 there exists a neighborhoods $\mathcal{U}_{\mathbf{f}_0}$, $\mathcal{U}_{\mathbf{u}}$ of $\mathbf{f}_0$ and $\mathbf{u}$, respectively, such that $Card(\mathcal{U}_{\mathbf{u}} \cap \mathcal{R}(\mathbf{f}_0)) = 1$ for all $\mathbf{f} \in \mathcal{U}_{\mathbf{f}_0}$. Moreover the application

$$\mathcal{U}_{\mathbf{f}_0} \ni \mathbf{f} \longmapsto \mathbf{u}_{\mathbf{f}} \in \mathcal{U}_{\mathbf{u}} \tag{9}$$

is of class C^∞, where $\{\mathbf{u}_{\mathbf{f}}\} = \mathcal{U}_{\mathbf{u}} \cap \mathcal{R}(\mathbf{f})$. As $\mathcal{R}(\mathbf{f}_0) < \infty$, there exists $\delta > 0$ so small that the ball $\{B(\mathbf{u}, \delta) : \mathbf{u} \in \mathcal{R}(\mathbf{f}_0)\}$ are pairwise disjoint and for all $\mathbf{u} \in \mathcal{R}(\mathbf{f}_0) : \ B(\mathbf{u}, \delta) \subset \mathcal{U}_{\mathbf{u}}$. By the continuity of the function (9), there exists a neighborhood $\mathcal{U} \subset \mathcal{U}_{\mathbf{f}_0}$ such that $\mathbf{u}_{\mathbf{f}} \in B(\mathbf{u}, \delta)$, for $\mathbf{f} \in \mathcal{U}$. Making $U = \cap\{\mathcal{U} : \ \mathbf{u} \in \mathcal{R}(\mathbf{f}_0)\}$, we have that if $\mathbf{f} \in U$, then the function

$$\mathcal{R}(\mathbf{f}_0) \ni \mathbf{u} \longmapsto \mathbf{u}_{\mathbf{f}} \in \mathcal{U}_{\mathbf{u}_0} \cap \mathcal{R}(\mathbf{f})$$

is an injection and as a consequence, $Card\ \mathcal{R}(\mathbf{f}_0) \leq Card\ \mathcal{R}(\mathbf{f})$. To see the equality assume contrary to our claim, that the set $\{\mathbf{f} \in U : \ Card\ \mathcal{R}(\mathbf{f}_0) < Card\ \mathcal{R}(\mathbf{f})\}$ is infinite. Then, there exists a sequence $(\mathbf{f}_k), \mathbf{f}_k \in U$ convergent to $\mathbf{f}_0$ in X such that $Card\ \mathcal{R}(\mathbf{f}_0) < Card\ \mathcal{R}(\mathbf{f}_k), \ k \in \mathbb{N}$. The subset $\{\mathbf{u}_{\mathbf{f}_k} : \mathbf{u} \in \mathcal{R}(\mathbf{f}_0)\}$ has exactly $Card\ \mathcal{R}(\mathbf{f}_0)$ elements because $\mathbf{u}_{\mathbf{f}_k}$ is the unique element of the set $\mathcal{R}(\mathbf{f}_k)$ which is in a suitable neighborhood of the solution $\mathbf{u} \in \mathcal{R}(\mathbf{f}_0)$, therefore it is not the whole $\mathcal{R}(\mathbf{f}_k)$. We select a sequence $a_k \in \mathcal{R}(\mathbf{f}_k) \setminus \{\mathbf{u}_{\mathbf{f}_k} : \mathbf{u} \in \mathcal{R}(\mathbf{f}_0)\}$. Note that a_k is not belong to set $\cup\{\mathcal{U}_{\mathbf{u}} : \mathbf{u} \in \mathcal{R}(\mathbf{f}_0)\}$. By (8), the sequence $(a_k)_{k \in \mathbb{N}}$ is bounded in $D(A)$, thus in virtue of the Banach- Alauglu theorem there exists $a \in D(A)$, and an infinite subset $\mathcal{N} \subset \mathbb{N}$ such that a_k converges weakly to a in $D(A)$ as $\mathcal{N} \ni k \longrightarrow \infty$. As $D(A) \hookrightarrow\hookrightarrow X, \ a_k \longrightarrow a$ strong in X and this is sufficiently for the pass the limit. Consequently, the limit $a \in \mathcal{R}(\mathbf{f}_0)$. This leads to a contradiction

$$\cup\{\mathcal{U}_{\mathbf{u}} : \mathbf{u} \in \mathcal{R}(\mathbf{f}_0)\}^C \ni a \in \mathcal{R}(\mathbf{f}_0) \subset \cup\{\mathcal{U}_{\mathbf{u}} : \mathbf{u} \in \mathcal{R}(\mathbf{f}_0)\}.$$

Therefore the set $\{\mathbf{f} \in U : \ Card\ \mathcal{R}(\mathbf{f}_0) < Card\ \mathcal{R}(\mathbf{f})\}$ is finite and consequently we can take a neighborhood sufficiently small U such that $\forall \mathbf{f} \in U$, $Card\ \mathcal{R}(\mathbf{f}) = Card\ \mathcal{R}(\mathbf{f}_0)$. Now, we are going to show that if $\mathbf{f} \longrightarrow \mathbf{f}_0$ in X, then $\mathcal{R}(f) \longrightarrow \mathcal{R}(\mathbf{f}_0)$ in the Hausdorff metric over X. Let $\epsilon > 0$, then there exists $\mathcal{U}_{\mathbf{f}_0}$ such that $\mathbf{u}_{\mathbf{f}} \in B(\mathbf{u}, \epsilon)$ for any $\mathbf{f} \in \mathcal{U}_{\mathbf{f}_0}$ hence $\mathbf{u}_{\mathbf{f}} \in \{\mathbf{u}\} + B(0, \epsilon) \subset \mathcal{R}(\mathbf{f}_0) + B(0, \epsilon)$ and consequently

$$\mathcal{R}(\mathbf{f}) \subset \{\mathbf{u}_{\mathbf{f}} : \mathbf{u} \in \mathcal{R}(\mathbf{f}_0)\} \subset \mathcal{R}(\mathbf{f}_0) + B(0, \epsilon).$$

As $\mathbf{u}_{\mathbf{f}} \in B(\mathbf{u}, \epsilon)$, then $\mathbf{u} \in B(\mathbf{u}_{\mathbf{f}}, \epsilon) = \{\mathbf{u}_{\mathbf{f}}\} + B(0, \epsilon) \subset \mathcal{R}(\mathbf{f}) + B(0, \epsilon)$, hence $\mathcal{R}(\mathbf{f}_0) \subset \mathcal{R}(\mathbf{f}) + B(0, \epsilon)$. Remark that Theorem 1.2 implies that the application

$$\mathcal{O} \ni f \mapsto \mathcal{R}(\mathbf{f}) \in D(A)$$

is continuous if we consider the Hausdorff metric over X. The function

$$\mathcal{O} \ni \mathbf{f} \mapsto Card\ \mathcal{R}(\mathbf{f}) \in \mathbb{Z}$$

is constant on every connected component of the set $\mathcal{O}$. (cf. Theorem1.1).

4 Applications.

As an application of Theorem 1.1 we shall consider first a model of Micropolar Fluids. A steady in time flow of micropolar fluid filling the domain Ω is described by the following equations in Ω [6].

$$-(\nu + \nu_r)\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 2\nu_r \text{rot } \mathbf{w} + \mathbf{f}_1, \tag{10}$$
$$-\sigma\Delta\mathbf{w} + (\mathbf{u} \cdot \nabla)\mathbf{w} - \beta\nabla \text{div } \mathbf{w} + 4\nu_r\mathbf{w} = 2\nu_r \text{rot } \mathbf{u} + \mathbf{f}_2, \tag{11}$$
$$\text{div } \mathbf{u} = 0. \tag{12}$$

Equations (10)-(12) are the conservation laws of momentum, momentum angular and mass, respectively. $\mathbf{u}$ is the velocity, p is the pressure and $\mathbf{w}$ is the angular velocity of rotation of particles. Moreover, $\mathbf{f}_1$ and $\mathbf{f}_2$ represent external fields, $\nu, \nu_r, \sigma, \beta$ are positive constants (ν is the usual Newtonian viscosity, ν_r is the microrotation viscosity, σ and β are constants that depend of new viscosities connected with the asymmetry of the stress tensor). The density of fluid is considered equal to one. We assume that Ω is a bounded set in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Let $\mathbf{u} = 0,\ \mathbf{w} = 0$ in $\partial\Omega$.

By X we denote the Hilbert space $H \times L^2(\Omega)^3$ where H is the closure of the set

$$\mathcal{V} = \{\mathbf{u} \in C_0^\infty(\Omega) : \text{div } \mathbf{u} = 0\}$$

in the norm of $L^2(\Omega)$. The norm in X is denoted by $|\cdot|$.
We introduced the following operators:

$$\begin{aligned} A(U) &= (-(\nu+\nu_r)P\Delta\mathbf{u}_1, -\sigma\Delta\mathbf{w}_1 - \beta\nabla\text{div } \mathbf{w}_1) \\ &\equiv (A_1\mathbf{u}_1, A_2\mathbf{w}_1), \\ B(U,V) &= (P((\mathbf{u}_1\cdot\nabla)\mathbf{u}_2), (\mathbf{u}_1\cdot\nabla)\mathbf{w}_2), \\ B_1(U) &= (0,0), \\ B_2(U) &= (-2\nu_r\text{rot } \mathbf{w}_1, -2\nu_r\text{rot } \mathbf{u}_1 + 4\nu_r\mathbf{w}_1), \end{aligned}$$

for $U = (\mathbf{u}_1, \mathbf{w}_1) \in D(A),\ V = (\mathbf{u}_2, \mathbf{w}_2) \in D(A)$. The operator P above is the orthogonal projection of $L^2(\Omega)^3$ on the subspace H. With this notation the system of equations (10)-(12) takes the form (1) with $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2)$.
We check the assumptions of Theorem 1.1. The operator A is self-adjoint, positive, with domain

$$D(A) = (W^{2,2}(\Omega)^3 \cap V) \times (W^{2,2}(\Omega)^3 \cap W_0^{2,2}(\Omega)^3)$$

where $W^{2,2}(\Omega)$ and $W_0^{1,2}(\Omega)$ are the usual Sobolev spaces, and V is the adherence of $\mathcal{V}$ in the norm of $W_0^{1,2}(\Omega)$; indeed, V can be characterized by $V = \{\mathbf{u} \in W_0^{1,2}(\Omega) : \text{div}\,\mathbf{u} = 0\}$. We denoted by $X^\alpha = D(A^{\alpha/2})$ and $X_i^\alpha = D(A_i^\alpha),\ i = 1,2$. The $X_2^\alpha = D(\Delta^\alpha)$. From the properties of the Laplace and Stokes operator [9], we conclude that for $\mathbf{u} \in X_1^1, \mathbf{v} \in X_2^2$,

$$\begin{aligned} |(\mathbf{u}\cdot\nabla)\mathbf{v}| &\le c|\mathbf{u}|_{L^6}|\nabla\mathbf{v}|_{L^3} \le c|\mathbf{u}|_{L^6}|\mathbf{v}|_{L^6}^{1/2}\|\mathbf{v}\|_{W^{2,2}}^{1/2} \\ &\le C|A_1^{1/2}\mathbf{u}|_{L^2}|A_i^{1/2}\mathbf{v}|_{L^2}^{1/2}|A_i\mathbf{v}|_{L^2}^{1/2}. \end{aligned}$$

Indeed the norms $|AU|$ and $\|U\|_{W^{2,2}}$ are equivalent on $D(A)$ and the norms $|A^{/2}U|$ and $\|U\|_{W^{1,2}}$ are equivalent on $D(A^{/2})$.
We can show that

$$|B(U,V)| \le c|A^{1/2}U||A^{1/2}V|^{1/2}|AV|^{1/2},$$

for $U \in X^1, V \in X^2$. The condition (2) is verifies thanks to properties of the bilinear form $b(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j} u_i \frac{\partial v_j}{\partial x_i} w_j$ to $\mathbf{u} \in D(A_1^{/2}), \mathbf{v}, \mathbf{w} \in W_0^{1,2}(\Omega)$. [8].
We also can verifies that $|(B_2(U),V)| \le c|A^{1/2}U||V|,\ U \in D(A^{1/2}), V \in X$. From Theorem (1.1) follows that for all $\mathbf{f} \in \mathcal{O}$, the set of solutions of (10)-(12) is finite and add number. Moreover the number of solutions on each connected component of $\mathcal{O}$ is constant. The consequences of Theorem 1.2 are also verifies.

As Another application of Theorem 1 and Theorem 2 we consider the stationary Navier-Stokes equations

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = f, \text{ in } \Omega \tag{13}$$

$$\text{div } \mathbf{u} = 0, \text{ in } \Omega \tag{14}$$

$$\mathbf{u} = \phi \text{ in } \partial\Omega, \tag{15}$$

where $\mathbf{f} \in H, \phi \in H^{1/2}(\partial\Omega)$ (The space of traces of function in $W^{1,2}(\Omega))^3$ with $\int_{\partial\Omega} \phi \cdot \eta ds = 0,\ \eta$ the unit outward normal on $\partial\Omega$. A study of the properties give by the Theorem 1.1, Theorem 1.2, for this system is encountered in [3]. The Theorem 1.1, Theorem 1.2 also apply in the model magneto-microplar fluid. In this model, moreover of unknown $\mathbf{u}, \mathbf{v}$ there exists the unknown $\mathbf{h}$ correspondent the magnetic field [7].

References

[1] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, (1980).

[2] E. Fernández-Cara, F. Guillén- González, M.A. Rojas-Medar, Una Formulación Abstracta para Algunas Ecuaciones de la Mecánica de los Fluidos Incompresibles, Minicurso 58 Seminário Brasileiro de Análise, Universidade Estadual de Campinas, (2003).

[3] C. Foias, R. Temam, Structure of the set of stationary solutions of the Navier-Stokes equations. Comm. Pure Appl. Math. 30, (1977), 149-164.

[4] K. Holly, Some application of the implicit function theorem to stationary Navier- Stokes equations, Ann. Polon. Math. 54,(2) (1991), 93-100.

[5] S. Smale, An infinite- dimensional version of Sards theorem, Amer. J. Math. 87, (1965), 861-866.

[6] G. Lukaszewicz, Micropolar Fluids. Theory and Applications, Modelling and Simulation in Science, Engineering and Technology, Birkäuser, Boston, Basel, Berlin(1999).

[7] M. Rojas-Medar, Magneto-Micropolar Fluid motion: existence and uniqueness of Strong Solution, Math. Nach 188, (1997), 301-319.

[8] R. Temam, Navier Stokes Equations and Nonlinear Functional Analysis, SIAM, Philadelphia, Pensylvania, (1995).

[9] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, 2nd Edition, Springer, New York (1997).